A Variation on a Theorem of Galvin and Hajnal

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ABSTRACT. Let $\aleph_\eta$ be a singular cardinal of regular uncountable cofinality κ. Let $\{\eta(\xi) : \xi < \kappa\}$ be a continuous increasing sequence with limit η, and let $\kappa_\xi = \aleph_{\eta(\xi)+\varphi(\xi)}$, $\xi < \kappa$, be regular cardinals.

Let I be a normal ideal on κ, and assume that the reduced product $\prod_{\xi<\kappa} \kappa_\xi / I$ admits a cofinal λ-scale of ordinal functions. Then $\lambda \le \aleph_{\eta+\tau}$, where $\tau = \|\varphi\|_I$ is the I-norm of φ.

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§1. Introduction.

In their paper [2] on the singular cardinal problem, Galvin and Hajnal proved the following inequality: Let κ be a regular uncountable cardinal and let $\aleph_\eta$ be a strong limit singular cardinal of cofinality κ. Then $2^{\aleph_\eta} < \aleph_{(2^{|\eta|})^+}$. To establish this inequality, they introduced the *norm* $\|\varphi\|$ of ordinal functions φ on κ, and proved the following: If $\{\eta(\xi) : \xi < \kappa\}$ is a continuous increasing sequence with limit η and if F is an almost disjoint family of ordinal functions in $\prod_{\xi<\kappa} \aleph_{\eta(\xi)+\varphi(\xi)}$, then F has size at most $\aleph_{\eta+\|\varphi\|}$.

The theorem stated in the abstract is an analog of the Galvin-Hajnal theorem and holds without the assumption that $\aleph_\eta$ is a strong limit cardinal. The proof employs techniques introduced by Shelah in his work on the singular cardinal problem; in fact, the special case (when $\varphi = 1$) is instrumental in Shelah's proof of a bound on $2^{\aleph_\omega}$.

Let A be an infinite set, and let I be an ideal on A. Let I^+ denote the set of all $X \subseteq A$ that are not in I (we call X a *positive* set). For ordinal functions f, g on A we define

$$\begin{aligned} f &=_I g && \text{if} && \{a \in A : f(a) \neq g(a)\} \in I \\ f &\leq_I g && \text{if} && \{a \in A : f(a) > g(a)\} \in I \\ f &\lneqq_I g && \text{if} && f \leq_I g \ \text{ and } \ \{a \in A : f(a) < g(a)\} \in I^+ \\ f &<_I g && \text{if} && \{a \in A : f(a) \geq g(a)\} \in I\,. \end{aligned}$$

The relation $\leq_I$ is a partial ordering (of equivalence classes). If S is a set of ordinal functions on A then g is an *upper bound* of S if $f \leq_I g$ for all $f \in S$, and g is a *least upper bound* of S if it is an upper bound and if $g \leq_I h$ for every upper bound h.

A transfinite sequence $\{f_\alpha : \alpha < \vartheta\}$ is *increasing* if $f_\alpha \leq_I f_\beta$ whenever $\alpha < \beta$, it is *strictly increasing* if $f_\alpha <_I f_\beta$ for $\alpha < \beta$. Similarly for *decreasing*.

If f is an upper bound of a set S then S is *cofinal* in f if for every $g <_I f$ there exists some $h \in S$ such that $g <_I h$. A *cofinal λ-scale* in $\prod_{a\in A} \kappa_a$ is a strictly increasing λ-sequence cofinal in the function $\langle \kappa_a : a \in A \rangle$.

If $X \subseteq A$ and $X \notin I$ then we relativize the concepts introduced above to X: We let $I{\restriction}X$ be the ideal generated by $I \cup \{A - X\}$ and then "$f \leq_I g$ on X", "$f <_I g$ on X", "an upper bound on X" etc. refer to the ideal $I{\restriction}X$.

Now let κ be a regular uncountable cardinal, and let I be a normal ideal on κ. In particular, NS denotes the ideal of all nonstationary subsets of κ. The relation $<_I$ is well founded and so each ordinal function φ on κ has a rank in this relation, which we call the *I-norm* of φ and denote $\|\varphi\|_I$. In particular, $\|\varphi\| = \|\varphi\|_{\mathrm{NS}}$ is the Galvin-Hajnal norm.

MAIN THEOREM. *Let κ be a regular uncountable cardinal and let I be a normal ideal on κ. Let $\aleph_\eta$ be a singular cardinal of cofinality κ and let $\{\eta(\xi) : \xi < \kappa\}$ be a continuous increasing sequence with limit η. Let $\{\kappa_\xi : \xi < \kappa\}$ be regular cardinals such that for each $\xi < \kappa$, $\kappa_\xi = \aleph_{\eta(\xi)+\varphi(\xi)}$, where φ is an ordinal function on κ. Let λ be a regular cardinal and assume that there is a cofinal λ-scale in $\prod_{\xi<\kappa} \kappa_\xi$, in the ordering $<_I$. Then $\lambda \leq \aleph_{\eta+\tau}$ where $\tau = \|\varphi\|_I$.*

REMARKS.

1. The theorem is not vacuously true, as cofinal scales exist: The following theorem is a consequence of Shelah's structure theory of possible cofinalities (for direct proof see Corollary 2.4 below):

Given κ and $\{\kappa_\xi : \xi < \kappa\}$ as above, there exist a maximal collection W of stationary subsets of κ such that $A \cap B$ is nonstationary for any two distinct $A, B \in W$, and for each $A \in W$ a cofinal λ_A-scale $\{f^A_\alpha : \alpha < \lambda_A\}$ in $\prod_{\xi\in A} \kappa_\xi$ (in $<_{\mathrm{NS}|A}$) for some regular cardinal λ_A.

[We don't claim that the λ_A's are all distinct.]

2. If $\kappa_\xi = \aleph_{\eta(\xi)+1}$ for all ξ, then $\prod_{\xi<\kappa} \aleph_{\eta(\xi)+1}/NS$ admits a cofinal $\aleph_{\eta+1}$-scale. This is a generalization of Silver's theorem [7] and is an important step in Shelah's proof of $\aleph_\omega^{\aleph_0} \leq 2^{\aleph_0} \cdot \aleph_{\omega_4}$ in [5] (Theorem 3.1 in [3]).

3. If $n < \omega$ and $\kappa_\xi = \aleph_{\eta(\xi)+n}$ for all ξ, then for any ultrafilter D that extends the club filter on κ, the ultrapower $\prod_{\xi<\kappa} \aleph_{\eta(\xi)+n}/D$ has cofinality at most $\aleph_{\eta+n}$. This appears as Lemma 6.3 in [1] (see also Lemma 8 in [4]).

4. Remark 3 does not extend beyond $\aleph_{\eta+\omega}$, and the following (if consistent) would be a counterexample: Let $\kappa_\xi = \aleph_{\eta(\xi)+\omega+1}$, and assume that κ is the union of disjoint stationary sets S_n such that for every n, $\prod_{\xi\in S_n} \kappa_\xi$ has a cofinal scale of length $\aleph_{\eta+n}$ (mod NS). Furthermore, assume that for some ultrafilter U on ω, the ultraproduct $\prod_{n<\omega} \aleph_{\eta+n}/U$ has cofinality $\aleph_{\eta+\omega+2}$. If D is any ultrafilter extending the filter $\{X \subseteq \kappa : \{n < \omega : S_n - X \in \mathrm{NS}\} \in U\}$, then D extends the club filter and $\mathrm{cof}\,(\prod_{\xi<\kappa} \kappa_\xi/D) = \aleph_{\eta+\omega+2}$.

5. The theorem has been known to Saharon Shelah and is related to Shelah's covering number *prc* introduced in [6].

§2. Ordinal functions modulo an ideal.

Throughout this section, I is an ideal on an infinite set A.

If $\{f_\alpha : \alpha < \lambda\}$ is a strictly increasing sequence of ordinal functions (mod I) and if λ is regular and greater than $2^{|A|}$ then the sequence $\{f_\alpha\}_\alpha$ has a least upper bound f and is cofinal in f. Here we do not assume that $\lambda > 2^{|A|}$, and use methods introduced by Shelah to obtain cofinal sequences.

LEMMA 2.1. *Let f be an ordinal function on A such that $\mathrm{cf}\, f(a) > |A|$ for all $a \in A$. Let $\{f_\alpha : \alpha < \lambda\}$ be a sequence of functions, where λ is regular and $\lambda > |A|^+$. There exists a function $g < f$ with the property that if $h < f$ is any function such that $g \leq h$ (pointwise), then for λ many α's,*

$$\{a \in A : g(a) < f_\alpha(a)\} = \{a \in A : h(a) < f_\alpha(a)\}\ .$$

Proof. Assume the contrary: for every $g < f$ there exist $\alpha < \lambda$ and $h \geq g$, $h < f$, such that for every $\beta \geq \alpha$, $B(h, f_\beta) \subsetneqq B(g, f_\beta)$, where $B(h, f_\beta) = \{a : h(a) < f_\beta(a)\}$. By induction, we construct a sequence $g_0 \leq g_1 \leq \ldots \leq g_i \leq \ldots$, $i < |A|^+$, as follows:

Let $g_0 < f$ be arbitrary. Given g_i, let g_{i+1} and α_i be such that $g_i \leq g_{i+1} < f$ and that for all $\alpha \geq \alpha_i$, $B(g_{i+1}, f_\alpha) \subsetneqq B(g_i, f_\alpha)$. If i is a limit ordinal, let g_i be the (pointwise) supremum of $\{g_j : j < i\}$; since $\operatorname{cf} f(a) > |A|$, we have $g_i < f$.

As $\lambda > |A|^+$, there is $\alpha < \lambda$ such that $\alpha \geq \alpha_i$ for all i. Then $\{B(g_i, f_\alpha) : i < |A|^+\}$ is a (descending) chain of length $|A|^+$ of subsets of A, a contradiction. □

Corollary 2.2. *Let $\{f_\alpha : \alpha < \lambda\}$ be an increasing sequence of ordinal functions on A (mod I), with $\lambda > |A|^+$ and regular, and let f be an upper bound of $\{f_\alpha\}_\alpha$ such that $\operatorname{cf} f(a) > |A|$ for every $a \in A$. If $\{f_\alpha\}_\alpha$ is not bounded below f (i.e. if there is no upper bound $g < f$), then there exists a positive set $B \subseteq A$ such that $\{f_\alpha\}_\alpha$ is cofinal in f on B.*

Proof. Let $g < f$ be as in Lemma 2.1. Since g is not an upper bound of $\{f_\alpha\}_\alpha$, there is an α such that $B(g, f_\alpha) = \{a : g(a) < f_\alpha(a)\}$ is positive, and let $B = B(g, f_\alpha)$. Now if $h < f$, we claim that $h <_I f_\beta$ on B for some β: We may assume that $g \leq h$, and by Lemma 2.1 there is some $\beta \geq \alpha$ such that $B(h, f_\beta) = B(g, f_\beta)$. But $B - B(g, f_\beta) \in I$ because $f_\alpha \leq_I f_\beta$. □

Corollary 2.3. *Let f be an ordinal function on A such that $\operatorname{cf} f(a) > |A|^+$ for all $a \in A$. There exist a positive set $B \subseteq A$, a regular cardinal $\lambda > |A|^+$ and a strictly increasing (mod I) sequence $\{f_\alpha : \alpha < \lambda\}$ of functions on B, cofinal in f.*

Proof. Let $\{f_\alpha : \alpha < \lambda\}$ be a maximal strictly increasing sequence of functions below f. We may assume that λ is a regular cardinal, and it is clear that $\lambda > |A|^+$ (because $\operatorname{cf} f(a) > |A|^+$ for all a). As $\{f_\alpha\}_\alpha$ is maximal, it is not bounded below f, and Corollary 2.2 applies. □

Corollary 2.4. *Let $\{\kappa_a : a \in A\}$ be regular cardinals such that $\kappa_a > |A|^+$ for all $a \in A$. There exist a maximal collection W of positive subsets of A such that $Y_1 \cap Y_2 \in I$ for any two distinct $Y_1, Y_2 \in W$, and for each $Y \in W$ a cofinal λ_Y-scale in $\prod_{a \in Y} \kappa_a$ (mod $I{\restriction}Y$) for some regular cardinal λ_Y.*

Proof. For every $X \in I^+$ apply Corollary 2.3. to $I{\restriction}X$ to get a positive $Y \subseteq X$ and a sequence $\{f_\alpha : \alpha < \lambda_Y\}$ cofinal in $\langle \kappa_a : a \in Y \rangle$. □

Definition 2.5. Let $\{f_\alpha : \alpha < \lambda\}$ be a strictly increasing (mod I) sequence of ordinal functions on A with λ regular, and let $\gamma < \lambda$ be a regular uncountable cardinal. The sequence $\{f_\alpha\}_\alpha$ is γ-*rapid* if for every $\beta < \lambda$ of cofinality γ there exists a closed unbounded subset $C \subset \beta$ such that for every limit ordinal $\alpha < \lambda$, $f_\alpha >_I s_{C \cap \alpha}$ where $s_{C\cap\alpha}$ is the (pointwise) supremum of $\{f_\nu : \nu \in C \cap \alpha\}$ (i.e. $s_{C\cap\alpha}(a) = \sup\{f_\nu(a) : \nu \in C \cap \alpha\}$ for all $a \in A$).

Lemma 2.6. *Let $\gamma < \lambda$ be regular uncountable cardinals such that $|A| < \gamma$ and let $\{f_\alpha : \alpha < \lambda\}$ be a γ-rapid strictly increasing (mod I) sequence of ordinal functions on A.*

For each $a \in A$, let S_a be a set of ordinals such that $|S_a| < \gamma$. There exists an $\alpha < \lambda$ with the property that for every $h \in \prod_{a \in A} S_a$, if $h >_I f_\alpha$ then h is an upper bound of $\{f_\alpha\}_\alpha$.

PROOF. Assume the contrary: for every α there exists an $h \in \prod_{a \in A} S_a$ such that $h >_I f_\alpha$ but h is not an upper bound. By induction, we construct a continuous increasing sequence α_i, $i < \gamma$, and functions $h_i \in \prod_{a \in A} S_a$ such that for every i, $f_{\alpha_i} <_I h_i$ and $f_{\alpha_{i+1}} \not\leq_I h_i$. Let $\beta = \lim_{i \to \gamma} \alpha_i$.

As $\{f_\alpha\}_\alpha$ is γ-rapid, there exists a club $C \subset \beta$ such that for every $\alpha \in C$, $f_\alpha >_I s_{C \cap \alpha}$. We may assume that $\alpha_i \in C$ for every $i < \gamma$ (otherwise replace $\{\alpha_i : i < \gamma\}$ by the intersection of C and $\{\alpha_i : i < \gamma\}$.

For each $i < \gamma$ we have $s_{C \cap \alpha_i} <_I f_{\alpha_i} <_I h_i \not\geq_I f_{\alpha_{i+1}}$, and so there exists some $a_i \in A$ such that

$$s_{C \cap \alpha_i}(a_i) < f_{\alpha_i}(a_i) < h_i(a_i) < f_{\alpha_{i+1}}(a_i) .$$

As $\gamma > |A|$, there exist a set $Z \subset \gamma$ of size γ and some $a \in A$ such that $a_i = a_j = a$ for all $i, j \in Z$. Now if $i, j \in Z$ are such that $i + 1 < j$, then $\alpha_{i+1} \in C \cap \alpha_j$ and we have

$$h_i(a) < f_{\alpha_{i+1}}(a) \leq s_{C \cap \alpha_j}(a) < h_j(a) .$$

This is a contradiction because $|S_a| < \gamma$ while $|Z| = \gamma$. □

COROLLARY 2.7. *If $|A| < \gamma < \lambda$ and $\{f_\alpha : \alpha < \lambda\}$ is γ-rapid, and if f is the least upper bound of $\{f_\alpha\}_\alpha$, then $\operatorname{cf} f(a) \geq \gamma$ for almost all $a \in A$ (mod I).*

PROOF. Let f be an upper bound of $\{f_\alpha\}_\alpha$, and assume that the set $B = \{a \in A : \operatorname{cf} f(a) < \gamma\}$ is positive. We will show that f is not the least upper bound; we shall find an upper bound h such that $h \lneqq_I f$.

For $a \in B$, let S_a be a cofinal subset of $f(a)$, of size $< \gamma$. For $a \in A - B$, let $S_a = \{f(a)\}$. By Lemma 2.6. there is an $\alpha < \lambda$ such that for every $h \in \prod_{a \in A} S_a$, $h >_I f_\alpha$ implies that h is an upper bound. Given this α, we consider a function $h \in \prod_{a \in A} S_a$ as follows: if $a \in B$ and $f_\alpha(a) < f(a)$, let $h(a) \in S_a$ be such that $f_\alpha(a) < h(a) < f(a)$; for $a \in A - B$ let $h(a) = f(a)$. By Lemma 2.6., h is an upper bound of $\{f_\alpha\}_\alpha$, and clearly $h \lneqq_I f$. □

THEOREM 2.8. *Let $\{f_\alpha : \alpha < \lambda\}$ be a γ-rapid strictly increasing sequence (mod I) on A, with γ and λ regular, $|A|^+ \leq \gamma < \lambda$. Then $\{f_\alpha\}_\alpha$ has a least upper bound.*

PROOF. Assume the contrary: for every upper bound f there exists an upper bound g such that $g \lneqq_I f$. By induction, we construct a decreasing sequence of upper bounds $h_0 \gneqq_I h_1 \gneqq_I \ldots \gneqq h_\xi \gneqq \ldots$, $\xi < |A|^+$. Let h_0 be any function such that $h_0(a) > f_\alpha(a)$ for all $a \in A$. Given h_ξ, let $h_{\xi+1}$ be any upper bound such that $h_{\xi+1} \lneqq_I h_\xi$.

If $\eta < |A|^+$ is a limit ordinal, consider the sets

$$S_a^\eta = \{h_\xi(a) : \xi < \eta\} \qquad\qquad (a \in A) .$$

By Lemma 2.6 there is some $\alpha < \lambda$ such that for all $h \in \prod_a S_a^\eta$, if $h >_I f_\alpha$ then h is an upper bound. Let $\alpha = \alpha_\eta$ be least such. Let, for every $a \in A$,

$$h_\eta(a) = \text{the least } \beta \in S_a^\eta \text{ such that } \beta > f_\alpha(a)\,.$$

Clearly, $h_\eta >_I f_\alpha$, and so h_η is an upper bound. If $\xi < \eta$ then for almost all a, $h_\xi(a) > f_\alpha(a)$ and $h_\xi(a) \in S_a^\eta$, and therefore $h_\xi \geq_I h_\eta$.

Now let $\alpha = \sup\{\alpha_\eta : \eta < |A|^+\}$, and for each $a \in A$ let

$$S_a = \{h_\xi(a) : \xi < |A|^+\} = \bigcup_\eta S_a^\eta\,.$$

Let, for every $a \in A$,

$$h(a) = \text{the least } \beta \in S_a \text{ such that } \beta > f_\alpha(a)\,.$$

If $\xi < |A|^+$ then for almost all a, $h_\xi(a) > f_\alpha(a)$ and $h_\xi(a) \in S_a$, and so $h_\xi \geq_I h$.

For each $a \in A$ there exists $\eta(a)$ such that $h(a) \in S_a^{\eta(a)}$; if we let $\eta = \sup\{\eta(a) : a \in A\}$, we have $h(a) \in S_a^\eta$ for all $a \in A$. Now $h(a) > f_\alpha(a) \geq f_{\alpha_\eta}(a)$ for almost all a, and therefore $h \geq_I h_\eta$. A contradiction. □

§3. Proof of Main Theorem.

Let κ be a regular uncountable cardinal, and let I be a normal ideal on κ. Let $\aleph_\eta$ be a singular cardinal of cofinality κ and let $\{\eta(\xi) : \xi < \kappa\}$ be a continuous increasing sequence with limit η. We shall prove, (by induction on $\|\varphi\|_I$) that if $\kappa_\xi = \aleph_{\eta(\xi)+\varphi(\xi)}$, $\xi < \kappa$, are regular cardinals then there is no cofinal λ-scale in $\prod_{\xi<\kappa} \kappa_\xi$ (mod I), for any regular $\lambda > \aleph_{\eta+\|\varphi\|_I}$.

Assume that the theorem fails, and let τ be the least ordinal such that for some φ with $\|\varphi\|_I = \tau$ the cardinals $\kappa_\xi = \aleph_{\eta(\xi)+\varphi(\xi)}$ are regular and there is a regular $\lambda > \aleph_{\eta+\tau}$, and a λ-scale cofinal in $\prod_{\xi<\kappa} \kappa_\xi$ mod I. Let $\mu = \aleph_{\eta+\tau}$.

We shall prove that μ is regular, and find regular cardinals $\lambda_\xi = \aleph_{\eta(\xi)+\psi(\xi)}$ such that $\psi <_I \varphi$, and a μ-scale cofinal in $\prod_{\xi<\kappa} \lambda_\xi$ mod I. As $\sigma = \|\psi\|_I < \tau$ and $\mu > \aleph_{\eta+\sigma}$, we will have a contradiction.

Lemma 3.1. *μ is a successor cardinal (hence regular).*

Proof. Let A be the set of all $\xi < \kappa$ such that $\varphi(\xi)$ is a limit ordinal, and let $B = \kappa - A$. If $A \in I$, then almost all $\varphi(\xi)$ are successor ordinals and so $\|\varphi\|_I$ is a successor ordinal, and so μ is a successor cardinal.

Thus assume that A is positive. We shall prove that $\|\varphi\|_{I|A} > \|\varphi\|_I$, and therefore $\|\varphi\|_I = \|\varphi\|_{I|B}$, and again $\|\varphi\|_I$ is a successor. For each $\xi \in A$, $\aleph_{\eta(\xi)+\varphi(\xi)}$ is regular, and therefore inaccessible, and it follows that $\varphi(\xi) = \kappa_\xi$. Hence $\prod_{\xi\in A} \varphi(\xi)$ has a λ-scale, and so $\|\varphi\|_{I|A} \geq \lambda$. Since $\lambda > \mu \geq \tau = \|\varphi\|_I$, the claim follows. □

Lemma 3.2. *There is a strictly increasing sequence $\{f_\alpha : \alpha < \mu\}$ below $\langle \kappa_\xi : \xi < \kappa\rangle$ that is γ-rapid for every regular uncountable $\gamma < \aleph_\eta$.*

Proof. Since there is a cofinal λ-scale below $\langle\kappa_\xi\rangle_\xi$, it follows that every set of functions in $\prod_\xi \kappa_\xi$ of size less than λ has an upper bound (mod I) in $\prod_\xi \kappa_\xi$ (namely some function in the scale). We construct a rapid sequence $f_\alpha,\ \alpha < \mu$ in $\prod_\xi \kappa_\xi$ as follows: For each limit ordinal $\beta < \mu$ such that $\omega < \operatorname{cf}\beta < \aleph_\eta$ choose a closed unbounded $C_\beta \subset \beta$ of order type $\operatorname{cf}\beta$.

Let $f_0 \in \prod_\xi \kappa_\xi$ be arbitrary, and given f_α, let $f_{\alpha+1} \in \prod_\xi \kappa_\xi$ be such that $f_\alpha <_I f_{\alpha+1}$. Now let $\alpha < \mu$ be a limit ordinal and let $\{f_\nu : \nu < \alpha\}$ be the functions constructed so far. For every $\beta < \mu$ with $\omega < \operatorname{cf}\beta < \aleph_\eta$, let s_β be the pointwise supremum of $\{f_\nu : \nu \in C_\beta \cap \alpha\}$. Since $|C_\beta \cap \alpha| < \operatorname{cf}\beta$, we have $s_\beta(\xi) < \kappa_\xi$ whenever $\kappa_\xi \geq \operatorname{cf}\beta$, therefore for almost all ξ (mod I). We let $f_\alpha \in \prod_\xi \kappa_\xi$ be an upper bound (mod I) of the set (of size $\leq \mu$) $\{f_\nu : \nu < \alpha\} \cup \{s_\beta : \beta < \mu\}$. □

Lemma 3.3. *The sequence $\{f_\alpha : \alpha < \mu\}$ has a least upper bound f such that $f <_I \langle\kappa_\xi : \xi < \kappa\rangle$ and for every ξ, $\operatorname{cf} f(\xi) > \kappa$ and $\operatorname{cf} f(\xi) > \aleph_{\eta(\xi)}$.*

Proof. A least upper bound f exists by Theorem 2.8, as $\{f_\alpha\}_\alpha$ is κ^+-rapid. The set $\{f_\alpha\}_\alpha$ has size $< \lambda$ and so it has an upper bound in $\prod_\xi \kappa_\xi$; therefore $f <_I \langle\kappa_\xi\rangle_\xi$. By Corollary 2.7, $\operatorname{cf} f(\xi) \geq \kappa^+$ for almost all ξ.

As $\aleph_\eta$ is singular, there are closed unbounded many ξ such that $\aleph_{\eta(\xi)}$ is singular. Thus if $\operatorname{cf} f(\xi) \leq \aleph_{\eta(\xi)}$ on a positive set, there exists, by normality of I, some regular $\gamma < \aleph_\eta$ such that $\operatorname{cf} f(\xi) < \gamma$ on a positive set. This contradicts Corollary 2.7, as $\{f_\alpha\}_\alpha$ is γ-rapid. □

For each $\xi < \kappa$, let $\lambda_\xi = \operatorname{cf} f(\xi)$. For every ξ, λ_ξ is regular, $\lambda_\xi > \kappa$, and $\lambda_\xi > \aleph_{\eta(\xi)}$, and for almost all ξ, $\lambda_\xi < \kappa_\xi$. Thus for some $\psi <_I \varphi$, $\lambda_\xi = \aleph_{\eta(\xi)+\psi(\xi)}$ for all ξ. We shall finish the proof by finding a cofinal μ-scale in $\prod_{\xi<\kappa} \lambda_\xi \bmod I$.

Lemma 3.4. *There is an increasing sequence $\{g_\alpha : \alpha < \mu\}$ in $\prod_{\xi<\kappa} \lambda_\xi / I$ such that for every positive $X \subseteq \kappa$ there is a positive $Y \subseteq X$ on which $\{g_\alpha\}_\alpha$ is cofinal in $\langle\lambda_\xi : \xi < \kappa\rangle$.*

Proof. For each $\xi < \kappa$, let $\beta_\xi(i),\ i < \lambda_\xi$, be an increasing sequence with limit $f(\xi)$. For every $\alpha < \mu$ let

$$g_\alpha(\xi) = \text{least } i < \lambda_\xi \text{ such that } \beta_\xi(i) \geq f_\alpha(\xi)\,.$$

If $\alpha < \beta$ then $g_\alpha \leq_I g_\beta$. If $X \subseteq \kappa$ is positive, then $\{g_\alpha : \alpha < \mu\}$ is not bounded (mod I) on X: if it were, then $\{f_\alpha : \alpha < \mu\}$ would be bounded on X below f, and f would not be the least upper bound. By Corollary 2.2 there is a positive $Y \subseteq X$ such that $\{g_\alpha\}_\alpha$ is cofinal in $\prod_{\xi\in Y} \lambda_\xi \bmod I$. □

Let A be a set of regular cardinals. Following Shelah, we define $\prod A = \prod_{\nu\in A} \nu$, and

$$\operatorname{pcf} A = \{\operatorname{cf}(\prod A/D) : D \text{ an ultrafilter on } A\}\,.$$

The fundamental structural theorem on pcf A is as follows:

THEOREM 3.5. (Shelah [5]; see also [1]) *If A is a set of regular cardinals such that $|A|^+ < \min A$, then there are sets $B_\nu \subseteq A$, $\nu \in \operatorname{pcf} A$, (called generators), such that for every $\nu \in \operatorname{pcf} A$,*

a) $\nu = \max \operatorname{pcf} B_\nu$

b) $\nu \notin \operatorname{pcf}(A - B_\nu)$

c) $\prod B_\nu$ *has a cofinal ν-scale $\{f^\nu_\alpha : \alpha < \nu\}$ mod J_ν, where J_ν is the ideal on B_ν generated by the sets B_γ, $\gamma < \nu$.*

Let A be the set $\{\lambda_\xi : \xi < \kappa\}$; also, for $X \subseteq \kappa$ let $A_X = \{\lambda_\xi : \xi \in X\}$. For every function $k \in \prod A$, let $h = H(k)$ be the function in $\prod_{\xi<\kappa} \lambda_\xi$ defined by $h(\xi) = k(\lambda_\xi)$. Let $H = \{H(k) : k \in \prod A\}$.

LEMMA 3.6. *The set H is cofinal in $\prod_{\xi<\kappa} \lambda_\xi$.*

PROOF. For every $g \in \prod_\xi \lambda_\xi$, define k on A as follows: $k(\lambda_\xi) = \sup\{g(\zeta) : \lambda_\zeta = \lambda_\xi\}$. As $\lambda_\xi > \kappa$, we have $k \in \prod A$, and clearly $g \leq h(k)$. □

LEMMA 3.7. *If $X \subseteq \kappa$ is positive, then $\mu \in \operatorname{pcf} A_X$.*

PROOF. Let X be positive. Let $\{g_\alpha : \alpha < \mu\}$ be the sequence in $\prod_{\xi<\kappa} \lambda_\xi$ given by Lemma 3.4.; there is a positive $Y \subseteq X$ such that $\{g_\alpha\}_\alpha$ is cofinal on Y mod I. It follows that every set of functions in $\prod_{\xi\in Y} \lambda_\xi$ of size $< \mu$ has an upper bound in $\prod_\xi \lambda_\xi$ mod $I{\restriction}Y$. As H is a cofinal set, we can find a sequence $\{k_\alpha : \alpha < \mu\}$ in $\prod A$ such that $\{H(k_\alpha) : \alpha < \mu\}$ is a cofinal μ-scale in $\prod_\xi \lambda_\xi$ mod $I{\restriction}Y$.

Now let J be the ideal on A_Y defined by

$$Z \in J \quad \text{if} \quad \{\xi \in Y : \lambda_\xi \in Z\} \in I \qquad\qquad (Z \subseteq A_Y)\,.$$

It follows that $\{k_\alpha : \alpha < \mu\}$ is a cofinal μ-scale in $\prod A_Y$ mod J. So if D is any ultrafilter on Y extending the dual of J, we have $\operatorname{cf}(\prod A_Y/D) = \mu$, and therefore $\mu \in \operatorname{pcf} A_X$. □

COROLLARY 3.8. *Let B_ν, $\nu \in \operatorname{pcf} A$, be the generators of* pcf A *given by Theorem 3.5. Then*

a) *For almost all ξ (mod I), $\lambda_\xi \in B_\mu$.*

b) *For every $\nu < \mu$, for almost all ξ, $\lambda_\xi \notin B_\nu$.*

PROOF. Otherwise, there is a positive set X such that either A_X is disjoint from B_μ, or $A_X \subseteq B_\nu$ for some $\nu < \mu$. Then we have (by Lemma 3.7) either $\mu \in \operatorname{pcf}(A - B_\mu)$ or $\mu \in \operatorname{pcf} B_\nu$ for $\nu < \mu$, contradicting either a) or b) of Theorem 3.5. □

We shall now construct a cofinal μ-scale on $\prod_{\xi<\kappa}\lambda_\xi$. Let $\{f^\mu_\alpha : \alpha < \mu\}$ be a cofinal μ-scale on $\prod B_\mu$ mod J_μ, where J_μ is the ideal on B_μ generated by $\{B_\nu : \nu < \mu\}$ (by Theorem 3.5 c)).

For each $\alpha < \mu$ let $h_\alpha = H(f^\mu_\alpha)$; by Corollary 3.8 a), h_α is defined almost everywhere (mod I). If $\alpha < \beta$ then $f^\mu_\alpha <_{J_\mu} f^\mu_\beta$, and by 3.8 b), $h_\alpha <_I h_\beta$; thus $\{h_\alpha : \alpha < \mu\}$ is strictly increasing. Finally, if $g \in \prod_{\xi<\kappa}\lambda_\xi$ is arbitrary, there exists (by Lemma 3.6) a function $k \in \prod A$ such that $g < H(k)$, so $k <_{J_\mu} f^\mu_\alpha$ for some α, and by 3.8 b) again, $g <_I h_\alpha$; thus $\{h_\alpha : \alpha < \mu\}$ is cofinal. The proof is now complete. □

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