Contents

I. Stationary Sets

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1. The closed unbounded filter

1.1. Closed unbounded sets

Stationary sets play a fundamental role in modern set theory. This chapter attempts to explain this role and to describe the structure of stationary sets of ordinals and their generalization.

The concept of stationary sets first appeared in the 1950's; the definition is due to G. Bloch [16], and the fundamental theorem on stationary sets was proved by G. Fodor in [24]. However, the concept of a stationary set is implicit in the work of P. Mahlo [71].

The precursor of Fodor's Theorem is the 1929 result of P. Alexandroff and P. Urysohn [2]: if $f(\alpha) < \alpha$ for all α such that $0 < \alpha < \omega_1$, then f is constant on an uncountable set.

Let us call an ordinal function f *regressive* if $f(\alpha) < \alpha$ whenever $\alpha > 0$. Fodor's Theorem (Theorem 1.5) states that every regressive function on a stationary set is constant on a stationary set. As a consequence, a set $S \subseteq \omega_1$ is stationary if and only if every regressive function on S is constant on an uncountable set.

In this section we develop the theory of closed unbounded and stationary subsets of a regular uncountable cardinal.

If X is a set of ordinals, then α is a *limit point* of X if $\alpha > 0$ and $\sup(X \cap \alpha) = \alpha$. A set $X \subseteq \kappa$ is *closed* (in the order topology on κ) if and only if X includes $\mathrm{Lim}(X)$, the set of all limit points of X less than κ.

1.1 Definition. Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is *closed unbounded* (or *club* for short) if it is closed and also an unbounded subset of κ. A set $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every closed unbounded $C \subseteq \kappa$.

It is easily seen that the intersection of any number of closed sets is closed. The basic observation is that if C_1 and C_2 are both closed unbounded, then $C_1 \cap C_2$ is also closed unbounded. This leads to the following basic property.

1.2 Proposition. *The intersection of less than κ closed unbounded subsets of κ is closed unbounded.*

Consequently, the closed unbounded sets generate a κ-complete filter on κ called the *closed unbounded* filter. The dual ideal (which is κ-complete and contains all singletons) consists of all sets that are disjoint from some closed unbounded sets – the nonstationary sets, and is thus called the *nonstationary* ideal, denoted I_{NS}.

If I is any nontrivial ideal on κ, then I^+ denotes the set $P(\kappa) - I$ of all *I-positive* sets. Thus stationary subsets of κ are exactly those that are I_{NS}-positive.

1.3 Definition. Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a κ-sequence of subsets of κ. Its diagonal intersection is the set

$$\Delta_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \textstyle\bigcap_{\alpha<\xi} X_\alpha\} ;$$

its diagonal union is

$$\Sigma_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \textstyle\bigcup_{\alpha<\xi} X_\alpha\} .$$

The following lemma states that the closed unbounded filter is closed under diagonal intersections (or dually, that the nonstationary ideal is closed under diagonal unions):

1.4 Lemma. *If $\langle C_\alpha : \alpha < \kappa \rangle$ is a sequence of closed unbounded subsets of κ, then its diagonal intersection is closed unbounded.*

This immediately implies Fodor's Theorem:

1.5 Theorem (Fodor [24]). *If S is a stationary subset of κ and if f is a regressive function on S, then there exists some $\gamma < \kappa$ such that $f(\alpha) = \gamma$ on a stationary subset of S.*

Proof. Let us assume that for each $\gamma < \kappa$ there exists a closed unbounded set C_γ such that $f(\alpha) \neq \gamma$ for each $\alpha \in S \cap C_\gamma$. Let $C = \Delta_{\gamma<\kappa} C_\gamma$. As C is closed unbounded, there exists an $\alpha > 0$ in $S \cap C$. By the definition of C it follows that $f(\alpha) \geq \alpha$, a contradiction. $\dashv$

A nontrivial κ-complete ideal I on κ is called *normal* (and so is its dual filter) if I is closed under diagonal unions; equivalently, if for every $A \in I^+$, every regressive function on A is constant on some I-positive set. Thus Fodor's Theorem (or Lemma 1.4) states that the nonstationary ideal (and the club filter) is normal. In fact, the nonstationary ideal is the smallest normal κ-complete ideal on κ:

1.6 Proposition. *If F is a normal κ-complete filter on κ, then F contains all closed unbounded sets.*

Proof. If C is a club subset of κ, let $\langle a_\alpha : \alpha < \kappa \rangle$ be the increasing enumeration of C. Then

$$C \supseteq \Delta_{\alpha<\kappa}\{\xi : a_{\alpha+1} < \xi < \kappa\} \in F,$$

because F contains all final segments (being nontrivial and κ-complete). $\dashv$

In other words, if I is normal, then every I-positive set is stationary.

The quotient algebra $B = P(\kappa)/I_{NS}$ is a κ-complete Boolean algebra, where the Boolean operations $\sum_{\alpha<\gamma}$ and $\prod_{\alpha<\gamma}$ for $\gamma < \kappa$ are induced by $\bigcup_{\alpha<\gamma}$ and $\bigcap_{\alpha<\gamma}$. Fodor's Theorem implies that B is in fact κ^+-complete: if $\{X_\alpha : \alpha < \kappa\}$ is a collection of subsets of κ, then $\Delta_{\alpha<\kappa}X_\alpha$ and $\Sigma_{\alpha<\kappa}X_\alpha$ are, respectively, the greatest lower bound and the least upper bound of the equivalence classes $X_\alpha/I_{NS} \in B$. This observation also shows that if $\langle X_\alpha : \alpha < \kappa\rangle$ and $\langle Y_\alpha : \alpha < \kappa\rangle$ are two enumerations of the same collection, then $\Delta_\alpha X_\alpha$ and $\Delta_\alpha Y_\alpha$ differ only by a nonstationary set.

The following characterization of the club filter is often useful, in particular when used in its generalized form (see Section 6). Let $F : [\kappa]^{<\omega} \to \kappa$; an ordinal $\gamma < \kappa$ is a *closure point* of F if $F(\alpha_1, \ldots, \alpha_n) < \gamma$ whenever $\alpha_1, \ldots, \alpha_n < \gamma$. It is easy to see that the set Cl_F of all closure points of F is a club. Conversely, if C is a club, define $F : [\kappa]^{<\omega} \to \kappa$ by letting $F(e)$ be the least element of C greater than $\max(e)$. It is clear that $Cl_F = \mathrm{Lim}(C)$. Thus every club contains Cl_F for some F, and we have this characterization of the club filter:

1.7 Proposition. *The club filter is generated by the sets Cl_F, for all $F : [\kappa]^{<\omega} \to \kappa$. A set $S \subseteq \kappa$ is stationary if and only if for every $F : [\kappa]^{<\omega} \to \kappa$, S contains a closure point of F.*

1.2. Splitting stationary sets

It is not immediately obvious that the club filter is not an ultrafilter, that is that there exist stationary sets that are *co-stationary*, i.e. whose complement is stationary. The basic result is the following theorem of Solovay:

1.8 Theorem (Solovay [85]). *Let κ be a regular uncountable cardinal. Then every stationary subset of κ can be partitioned into κ disjoint stationary sets.*

Solovay's proof of this basic result of combinatorial set theory uses methods of forcing and large cardinals, and we shall describe it later in this section. For an elementary proof, see e.g. [49], p. 434.

To illustrate the combinatorics involved, let us prove a special case of Solovay's theorem.

1.9 Proposition. *There exist $\aleph_1$ pairwise disjoint stationary subsets of ω_1.*

Proof. For every limit ordinal $\alpha < \omega_1$, choose an increasing sequence $\{a_n^\alpha\}_{n=0}^\infty$ with limit α. We claim that there is an n such that for all $\eta < \omega_1$, there are stationary many α such that $a_n^\alpha \geq \eta$: Otherwise there exists, for each n, some η_n such that $a_n^\alpha \geq \eta_n$ for only a nonstationary set of α's. By ω_1-completeness, for all but a nonstationary set of α's the sequences $\{a_n^\alpha\}_n$ are bounded by $\sup_n \eta_n$. A contradiction.

Thus let n be such that for all η, the set $S_\eta = \{\alpha : a_n^\alpha \geq \eta\}$ is stationary. The function $f(\alpha) = a_n^\alpha$ is regressive and so by Fodor's Theorem, there is some $\gamma_\eta \geq \eta$ such that $T_\eta = \{\alpha : a_n^\alpha = \gamma_\eta\}$ is stationary. Clearly, there are $\aleph_1$ distinct values of γ_η and therefore $\aleph_1$ mutually disjoint sets T_η. $\dashv$

Let κ be a regular uncountable cardinal, and let $\lambda < \kappa$ be regular. Let

$$\mathbf{E}_\lambda^\kappa = \{\alpha < \kappa : \mathrm{cf}\ \alpha = \lambda\}.$$

For each λ, $\mathbf{E}_\lambda^\kappa$ is a stationary set. An easy modification of the proof of 1.9 above shows that for every regular $\lambda < \kappa$, every stationary subset of $\mathbf{E}_\lambda^\kappa$ can be split into κ disjoint stationary sets.

The union $\bigcup_\lambda \mathbf{E}_\lambda^\kappa$ is the set of all singular limit ordinals. Its complement is the set Reg of all regular cardinals $\alpha < \kappa$. The set Reg is stationary just in case κ is a Mahlo cardinal.

1.3. Generic ultrapowers

Let M be a transitive model of ZFC, and let κ be a cardinal in M. Let U be an *M-ultrafilter*, i.e. an ultrafilter on the set algebra $P(\kappa) \cap M$. Using functions $f \in M$ on κ, one can form an ultrapower $N = Ult_U(M)$, which is a model of ZFC but not necessarily well-founded:

$$f =^* g \iff \{\alpha : f(\alpha) = g(\alpha)\} \in U\,,$$
$$f \in^* g \iff \{\alpha : f(\alpha) \in g(\alpha)\} \in U\,.$$

The (equivalence classes of) constant functions $c_x(\alpha) = x$ provide an elementary embedding $j : (M, \epsilon) \to (N, \in^*)$, where $j(x) = c_x$, for all $x \in M$.

An M-ultrafilter U is *M-κ-complete* if it is closed under intersections of families $\{X_\alpha : \alpha < \gamma\} \in M$, for all $\gamma < \kappa$; U is *normal* if every regressive $f \in M$ is constant on a set in U.

1.10 Proposition. *Let U be a nonprincipal M-κ-complete, normal M-ultrafilter on κ. Then the ordinals of N have a well-ordered initial segment of order type at least $\kappa + 1$, $j(\gamma) = \gamma$ for all $\gamma < \kappa$, and κ is represented in N by the diagonal function $d(\alpha) = \alpha$.*

Now let κ be a regular uncountable cardinal and consider the forcing notion $(P, <)$ where P is the collection of all stationary subsets of κ, and the ordering is by inclusion. Let B be the complete Boolean algebra $B = B(P)$, the completion of $(P, <)$. Equivalently, B is the completion of the Boolean algebra $P(\kappa)/I_{NS}$. Let us consider the generic extension $V[G]$ given by a generic $G \subseteq P$. It is rather clear that G is a nonprincipal V-κ-complete normal ultrafilter on κ. Thus Proposition 1.10 applies, where $N = Ult_G(V)$. The model $Ult_G(V)$ is called a *generic ultrapower.*

There is more on generic ultrapowers in Foreman's chapter in this volume; here we use them to present the original argument of Solovay's [85]. First we prove a lemma (that will be generalized in Section 2):

1.11 Lemma. *Let κ be a regular uncountable cardinal, and let S be a stationary set. Then the set*

$$T = \{\alpha \in S : \textit{either } \alpha \notin \textit{Reg or } S \cap \alpha \textit{ is not a stationary subset of } \alpha\}$$

is stationary.

Proof. Let C be a club and let us show that $T \cap C$ is nonempty. Let α be the least element of the nonempty set $S \cap C'$ where $C' = \mathrm{Lim}(C - \omega)$. If α is not regular, then $\alpha \in T \cap C$ and we are done, so assume that $\alpha \in$ Reg. Now $C' \cap \alpha$ is a club subset of α disjoint from $S \cap \alpha$, and so $\alpha \in T$. $\dashv$

We shall now outline the proof of Solovay's Theorem:

Proof. (Theorem 1.8.) Let S be a stationary subset of κ that cannot be partitioned into κ disjoint stationary sets. By 1.9 and the remarks following its proof, we have $S \subseteq \text{Reg}$. Let $I = I_{NS} \restriction S$, i.e. $I = \{X \subseteq \kappa : X \cap S \in I_{NS}\}$. The ideal I is *κ-saturated*, i.e. every disjoint family $W \subset I^+$ has size less than κ; equivalently, $B = P(\kappa)/I$ has the κ-chain condition. I is also κ-complete and normal.

Let $G \subset I^+$ be generic, and let $N = Ult_G(V)$ be the generic ultrapower. As I is κ-saturated, N is well-founded (this is proved by showing that every name $\dot{f}$ for a function in V on κ can be replaced by an actual function on κ). Thus we have (in $V[G]$) an elementary embedding $j : V \to N$ where N is a transitive class, $j(\gamma) = \gamma$ for all $\gamma < \kappa$, and κ is represented in N by the diagonal function $d(\alpha) = \alpha$. Note that if $A \subseteq \kappa$ is any set (in V), then $A \in N$: this is because $A = j(A) \cap \kappa$; in fact A is represented by the function $f(\alpha) = A \cap \alpha$.

Now we use the fact that κ-c.c. forcing preserves stationarity (cf. Theorem 1.13 below). Thus S is stationary in $V[G]$, and because $N \subset V[G]$, S is a stationary set in the model N. By the ultrapower theorem we have

$$V[G] \vDash S \cap \alpha \text{ is stationary for } G\text{-almost all } \alpha.$$

This, translated into forcing, gives

$$\{\alpha \in S : S \cap \alpha \text{ is not stationary}\} \in I$$

but that contradicts Lemma 1.11. $\dashv$

Another major application of generic ultrapowers is Silver's Theorem:

1.12 Theorem (Silver [84]). *Let λ be a singular cardinal of uncountable cofinality. If $2^\alpha = \alpha^+$ for all cardinals $\alpha < \lambda$, then $2^\lambda = \lambda^+$.*

Silver's Theorem is actually stronger than this. It assumes only that $2^\alpha = \alpha^+$ for a stationary set of α's (see Section 2 for the definition of "stationary" when λ is not regular). The proof uses a generic ultrapower. Even though $Ult_G(V)$ is not necessarily well founded, the method of generic ultrapowers enables one to conclude that $2^\lambda = \lambda^+$ when $2^\alpha = \alpha^+$ holds almost everywhere.

Silver's Theorem can be proved by purely combinatorial methods [10, 11]. In [30], Galvin and Hajnal used combinatorial properties of stationary sets to prove a substantial generalization of Silver's Theorem (superseded only by Shelah's powerful pcf theory). For further generalizations using stationary sets and generic ultrapowers, see [51] and [52].

One of the concepts introduced in [30] is the *Galvin-Hajnal norm* of an ordinal function. If f and g are ordinal functions on a regular uncountable cardinal κ, let $f < g$ if $\{\alpha < \kappa : f(\alpha) < g(\alpha)\}$ contains a club. The relation $<$ is a well-founded partial order, and the norm $\|f\|$ is the rank of f in the relation $<$.

We remark that if $f < g$, then in the generic ultrapower (by I_{NS}), the ordinal represented by f is smaller than the ordinal represented by g.

By induction on η one can easily show that for each $\eta < \kappa^+$ there exists a *canonical* function $f_\eta : \kappa \to \kappa$ of norm η, i.e. $\|f_\eta\| = \eta$ and whenever $\|h\| = \eta$, then $\{\alpha : f_\eta(\alpha) \leq h(\alpha)\}$ contains a club. (Proof: Let $f_0(\alpha) = 0$, $f_{\eta+1}(\alpha) = f_\eta(\alpha) + 1$. If $\eta < \kappa^+$ is a limit ordinal, let $\lambda = \operatorname{cf} \eta$ and let $\eta = \lim_{\xi \to \lambda} \eta_\xi$. If $\lambda < \kappa$, let $f_\eta(\alpha) = \sup_{\xi<\lambda} f_{\eta_\xi}(\alpha)$ and if $\lambda = \kappa$, let $f_\eta(\alpha) = \sup_{\xi<\alpha} f_{\eta_\xi}(\alpha)$.)

A canonical function of norm κ^+ may or may not exist, but is consistent with ZFC (cf. [53]). The existence of canonical function f_η for all η is equiconsistent with a measurable cardinal [50].

1.4. Stationary sets in generic extensions

Let M and N be transitive models and let $M \subseteq N$. Let κ be a regular uncountable cardinal and let $S \in M$ be a subset of κ. Clearly, if S is stationary in the model N, then S is stationary in M; the converse is not necessarily true, and κ may even not be regular or uncountable in N. It is important to know which forcing extensions preserve stationarity and we shall return to the general case in Section 5. For now, we state two important special cases:

1.13 Theorem. *Let κ be a regular uncountable cardinal and let P be a notion of forcing.*

(a) If P satisfies the κ-chain condition, then every club $C \in V[G]$ has a club subset D in the ground model. Hence every stationary S remains stationary in $V[G]$.

(b) If P is λ-closed for every $\lambda < \kappa$, then every stationary S remains stationary in $V[G]$.

Proof. (outline) (a) This follows from this basic fact on forcing: if P is κ-c.c., then every unbounded $A \subset \kappa$ in $V[G]$ has an unbounded subset in V.

(b) Let $p \Vdash \dot{C}$ is a club; we find a $\gamma \in S$ and a $q \leq p$ such that $q \Vdash \gamma \in \dot{C}$ as follows: we construct an increasing continuous ordinal sequence $\{\gamma_\alpha\}_{\alpha<\kappa}$ and a decreasing sequence $\{p_\alpha\}$ of conditions such that $p_{\alpha+1} \Vdash \gamma_{\alpha+1} \in \dot{C}$, and if α is a limit ordinal, then $\gamma_\alpha = \lim_{\xi<\alpha} \gamma_\xi$ and p_α is a lower bound of $\{p_\xi\}_{\xi<\alpha}$. There is some limit ordinal α such that $\gamma_\alpha \in S$. It follows that $p_\alpha \Vdash \gamma_\alpha \in \dot{C}$. ⊣

We shall now describe the standard way of controlling stationary sets in generic extensions, so called *shooting a club*. First we deal with the simplest case when $\kappa = \aleph_1$. Let S be a stationary subset of ω_1, and consider the following forcing P_S (cf. [9]): The forcing conditions are all bounded closed sets p of countable ordinals such that $p \subset S$. A condition q is stronger than p if q end-extends p, i.e. $p = q \cap \alpha$ for some α.

It is clear that this forcing produces ("shoots") a closed unbounded subset of S in the generic extension, thus the complement of S becomes nonstationary. The main point of [9] is that ω_1 is preserved and in fact $V[G]$ adds no new countable sets. Also, every stationary subset of S remains stationary.

The forcing P_S has the obvious generalization to $\kappa > \aleph_1$, but more care is required to guarantee that no new small sets of ordinals are added. For instance, this is the case when S contains the set Sing of all singular ordinals $< \kappa$. For a more detailed discussion of this problem see [1].

1.5. Some combinatorial principles

There has been a proliferation of combinatorial principles involving closed unbounded and stationary sets. Most can be traced back to Jensen's investigation of the fine structure of L [59] and generalize either Jensen's *diamond* ($\diamondsuit$) or *square* ($\square$). These principles are discussed elsewhere in this volume; we conclude this section by briefly mentioning diamond and *club-guessing*, and only their typical special cases.

1.14 Theorem ($\diamondsuit(\aleph_1)$, Jensen [59]). *Assume $V = L$. There exists a sequence $\langle a_\alpha : \alpha < \omega_1\rangle$ with each $a_\alpha \subseteq \alpha$, such that for every $A \subseteq \omega_1$, the set $\{\alpha < \omega_1 : A \cap \alpha = a_\alpha\}$ is stationary.*

(Note that every $A \subseteq \omega$ is equal to some a_α, and so $\diamondsuit(\aleph_1)$ implies $2^{\aleph_0} = \aleph_1$.)

1.15 Theorem ($\diamondsuit(\mathbf{E}^{\aleph_2}_{\aleph_0})$, Gregory [40]). *Assume* GCH. *There exists a sequence $\langle a_\alpha : \alpha \in \mathbf{E}^{\aleph_2}_{\aleph_0}\rangle$ with each $a_\alpha \subseteq \alpha$, such that for every $A \subseteq \omega_2$, the set $\{\alpha < \omega_2 : A \cap \alpha = a_\alpha\}$ is stationary.*

1.16 Theorem (Club-guessing, Shelah [82]). *There exists a sequence $\langle c_\alpha : \alpha \in \mathbf{E}^{\aleph_3}_{\aleph_1}\rangle$, where each c_α is a closed unbounded subset of α, such that for every club $C \subseteq \omega_3$, the set $\{\alpha : c_\alpha \subset C\}$ is stationary.*

Unlike most generalizations of square and diamond, Theorem 1.16 is a theorem of ZFC but we note that the gap (between $\aleph_1$ and $\aleph_3$) is essential.

2. Reflection

2.1. Reflecting stationary sets

An important property of stationary sets is *reflection.* It is used in several applications, and provides a structure among stationary sets – it induces a well founded hierarchy. Natural questions about reflection and the hierarchy are closely related to large cardinal properties.

We start with a generalization of stationary sets. Let α be a limit ordinal of uncountable cofinality, say cf $\alpha = \kappa > \aleph_0$. A set $S \subseteq \alpha$ is *stationary* if it meets every closed unbounded subset of α. The closed unbounded subsets of α generate a κ-complete filter, and Fodor's Theorem 1.5 yields this:

2.1 Lemma. *If f is a regressive function on a stationary set $S \subseteq \alpha$, then there exists a $\gamma < \alpha$ such that $f(\xi) < \gamma$ on a stationary subset of S.*

If S is a set of ordinals and α is a limit ordinal such that cf $\alpha > \omega$, we say that S is *stationary in* α if $S \cap \alpha$ is a stationary subset of α.

2.2 Definition. Let κ be a regular uncountable cardinal and let S be a stationary subset of κ. If $\alpha < \kappa$ and cf $\alpha > \omega$, S *reflects at* α if S is stationary in α. S *reflects* if it reflects at some $\alpha < \kappa$.

It is implicit in the definition that $\kappa > \aleph_1$.

For our first observation, let $\alpha < \kappa$ be such that cf $\alpha > \omega$. There is a club $C \subseteq \alpha$ of order type cf α such that every element of C has cofinality $<$ cf α. Thus if $S \subseteq \kappa$ is such that every $\beta \in S$ has cofinality $\geq$ cf α, then S does not reflect at α. In particular, if $\kappa = \lambda^+$ where λ is regular, then the stationary set $\mathbf{E}^\kappa_\lambda$ does not reflect.

On the other hand, if $\lambda < \kappa$ is regular and $\lambda^+ < \kappa$, then $\mathbf{E}^\kappa_\lambda$ reflects at every $\alpha < \kappa$ such that cf $\alpha > \lambda$.

To investigate reflection systematically, let us first look at the simplest case, when $\kappa = \aleph_2$. Let $E_0 = \mathbf{E}^{\aleph_2}_{\aleph_0}$ and $E_1 = \mathbf{E}^{\aleph_2}_{\aleph_1}$. The set E_1 does not reflect; can every stationary $S \subseteq E_0$ reflect?

Let us recall Jensen's Square Principle [59]:

($\square_\kappa$) There exists a sequence $\langle C_\alpha : \alpha \in \text{Lim }(\kappa^+)\rangle$ such that

(i) C_α is club in α,

(ii) if $\beta \in \mathrm{Lim}(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$,

(iii) if cf $\alpha < \kappa$, then $|C_\alpha| < \kappa$.

Now assume that $\square_{\omega_1}$ holds and let $\langle C_\alpha : \alpha \in \mathrm{Lim}\ (\omega_2)\rangle$ be a square sequence. Note that for each $\alpha \in E_1$, the order type of C_α is ω_1. It follows that there exists a countable limit ordinal η such that the set $S = \{\gamma \in E_0 : \gamma \text{ is the } \eta^{\text{th}} \text{ element of some } C_\alpha\}$ is stationary. But for every $\alpha \in E_1$, S has at most one element in common with C_α, and so S does not reflect.

Thus $\square_{\omega_1}$ implies that there is a nonreflecting stationary subset of $\mathbf{E}^{\aleph_2}_{\aleph_0}$. Since $\square_{\omega_1}$ holds unless $\aleph_2$ is Mahlo in L, the consistency strength of "every $S \subseteq \mathbf{E}^{\aleph_2}_{\aleph_0}$ reflects" is at least a Mahlo cardinal. This is in fact the exact strength:

2.3 Theorem (Harrington-Shelah [41]). *The following are equiconsistent:*

(i) the existence of a Mahlo cardinal.

(ii) every stationary set $S \subseteq \mathbf{E}^{\aleph_2}_{\aleph_0}$ reflects.

Theorem 2.3 improves a previous result of Baumgartner [6] who proved the consistency of (ii) from a weakly compact cardinal. Note that (ii) implies that every stationary set $S \subseteq \mathbf{E}^{\aleph_2}_{\aleph_0}$ reflects at stationary many $\alpha \in \mathbf{E}^{\aleph_2}_{\aleph_1}$.

A related result of Magidor (to which we return later in this section) gives this equiconsistency:

2.4 Theorem (Magidor [70]). *The following are equiconsistent:*

(i) the existence of a weakly compact cardinal,

(ii) every stationary set $S \subseteq \mathbf{E}^{\aleph_2}_{\aleph_0}$ reflects at almost all $\alpha \in \mathbf{E}^{\aleph_2}_{\aleph_1}$.

Here, "almost all" means all but a nonstationary set.

Let us now address the question whether it is possible that *every* stationary subset of κ reflects. We have seen that this is not the case when κ is the successor of a regular cardinal. Thus κ must be either inaccessible or $\kappa = \lambda^+$ where λ is singular.

Note that because a weakly compact cardinal is Π^1_1 indescribable, every stationary subset of it reflects. In [68], Kunen showed that it is consistent that every stationary $S \subseteq \kappa$ reflects while κ is not weakly compact. In [76] it is shown that the consistency strength of "every stationary subset of κ reflects" is strictly between greatly Mahlo and weakly compact. (For definition of greatly Mahlo, see Section 2.2.)

If, in addition, we require that κ be a successor cardinal, then much stronger assumptions are necessary. The argument we gave above using $\square_{\omega_1}$ works for any κ:

2.5 Proposition (Jensen). *If $\square_\lambda$ holds, then there is a nonreflecting stationary subset of $\mathbf{E}^{\lambda^+}_{\aleph_0}$.*

As the consistency strength of $\neg\square_\lambda$ for singular λ is at least a strong cardinal (as shown by Jensen), one needs at least that for the consistency of "every stationary $S \subseteq \lambda^+$ reflects". In [70], Magidor proved the consistency of "every stationary subset of $\aleph_{\omega+1}$ reflects" from the existence of infinitely many supercompact cardinals.

We mention the following applications of nonreflecting stationary sets:

2.6 Theorem (Mekler-Shelah [76]). *The following are equiconsistent:*

(i) every stationary $S \subseteq \kappa$ reflects,

(ii) every κ-free abelian group is κ^+-free.

2.7 Theorem (Tryba [90]). *If a regular cardinal κ is Jónsson, then every stationary $S \subseteq \kappa$ reflects.*

2.8 Theorem (Todorčević [88]). *If Rado's Conjecture holds, then for every regular $\kappa > \aleph_1$, every stationary $S \subseteq \mathbf{E}^{\kappa}_{\aleph_0}$ reflects.*

2.2. A hierarchy of stationary sets

Consider the following operation (the *Mahlo operation*) on stationary sets. For a stationary set $S \subseteq \kappa$, the *trace* of S is the set of all α at which S reflects:

$$Tr(S) = \{\alpha < \kappa : \text{cf } \alpha > \omega \text{ and } S \cap \alpha \text{ is stationary}\}.$$

The following basic properties of trace are easily verified.

2.9 Lemma. *(a) If $S \subseteq T$, then $Tr(S) \subseteq Tr(T)$,*

(b) $Tr(S \cup T) = Tr(S) \cup Tr(T)$,

(c) $Tr(Tr(S)) \subseteq Tr(S)$,

(d) if $S \simeq T$ mod I_{NS}, then $Tr(S) \simeq Tr(T)$ mod I_{NS}.

Property (d) shows that the Mahlo operation may be considered as an operation on the Boolean algebra $P(\kappa)/I_{NS}$.

If $\lambda < \kappa$ is regular, let $\mathbf{M}^\kappa_\lambda = \{\alpha < \kappa : \operatorname{cf} \alpha \geq \lambda\}$, and note that $Tr(\mathbf{E}^\kappa_\lambda) = Tr(\mathbf{M}^\kappa_\lambda) = \mathbf{M}^\kappa_{\lambda^+}$.

The Mahlo operation on $P(\kappa)/I_{NS}$ can be iterated α times, for $\alpha < \kappa^+$. Let

$$\begin{aligned} M_0 &= \kappa \\ M_{\alpha+1} &= Tr(M_\alpha) \\ M_\alpha &= \Delta_{\xi<\kappa} M_{\alpha_\xi} \qquad (\alpha \text{ limit}, \alpha = \{\alpha_\xi : \xi < \kappa\}). \end{aligned}$$

The sets M_α are defined mod I_{NS} (the limit stages depend on the enumeration of α). The sequence $\{M_\alpha\}_{\alpha<\kappa^+}$ is decreasing mod I_{NS}, and when $\alpha < \kappa$, then $M_\alpha = \mathbf{M}^\kappa_\lambda$ where λ is the α^{th} regular cardinal. Note that κ is (weakly) Mahlo just in case $M_\kappa = Reg$ is stationary, and that by Lemma 1.11, $\{M_\alpha\}_\alpha$ is strictly decreasing (mod I_{NS}, as long as M_α is stationary). Following [13], κ is called *greatly Mahlo* if M_α is stationary for every $\alpha < \kappa^+$.

We shall now consider the following relation between stationary subsets of κ.

2.10 Definition (Jech [47]).

$$S < T \text{ iff } S \cap \alpha \text{ is stationary for almost all } \alpha \in T.$$

In other words, $S < T$ iff $Tr(S) \supseteq T$ mod I_{NS}. As an example, if $\lambda < \mu$ are regular, then $\mathbf{E}^\kappa_\lambda < \mathbf{E}^\kappa_\mu$. Note also that the language of generic ultrapowers gives this description of $<$:

2.11 Proposition. *$S < T$ iff $T \Vdash S$ is stationary in $Ult_G(V)$.*

The following lemma states the basic properties of $<$.

2.12 Lemma. *(a) $A < Tr(A)$,*

(b) if $A < B$ and $B < C$ than $A < C$,

(c) if $A \simeq A'$ and $B \simeq B'$ mod I_{NS}, and if $A < B$, then $A' < B'$.

By (c), $<$ can be considered a relation on $P(\kappa)/I_{NS}$. By Proposition 1.11, $<$ is irreflexive and so it is a partial ordering. The next theorem shows that the partial ordering $<$ is well founded.

2.13 Theorem (Jech [47]). *The relation $<$ is well founded.*

Proof. Assume to the contrary that there are stationary sets such that $A_1 > A_2 > A_3 > \cdots$. Therefore there are clubs C_n such that $A_n \cap C_n \subseteq Tr(A_{n+1})$

for $n = 1, 2, \ldots$. For each n, let

$$B_n = A_n \cap C_n \cap \mathrm{Lim}(C_{n+1}) \cap \mathrm{Lim}(\mathrm{Lim}(C_{n+2})) \cap \cdots$$

Each B_n is stationary, and for every n, $B_n \subseteq Tr(B_{n+1})$. Let $\alpha_n = \min(B_n)$. Since $B_{n+1} \cap \alpha_n$ is stationary, we have $\alpha_{n+1} < \alpha_n$, and therefore, a decreasing sequence $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$. A contradiction. $\dashv$

As $<$ is well founded, we can define the *order* of stationary sets $A \subseteq \kappa$, and of the cardinal κ:

$$o(A) = \sup\{o(X) + 1 : X < A\}\,,$$

$$o(\kappa) = \sup\{o(A) + 1 : A \subseteq \kappa \text{ stationary}\}.$$

We also define $o(\aleph_0) = 0$, and $o(\alpha) = o(\mathrm{cf}\ (\alpha))$ for every limit ordinal α.

Note that $o(\mathbf{E}^\kappa_{\aleph_0}) = 0$, and in general $o(\mathbf{E}^\kappa_\lambda) = o(\mathbf{M}^\kappa_\lambda) = \alpha$, if λ is the α^{th} regular cardinal. Also, $o(\aleph_n) = n$, $o(\kappa) \geq \kappa + 1$ iff κ is Mahlo, and $o(\kappa) \geq \kappa^+$ iff κ is greatly Mahlo.

2.3. Canonical stationary sets

If λ is the α^{th} regular cardinal, then $\mathbf{E}^\kappa_\lambda$ has order α; moreover, the set is canonical, in the sense explained below. In fact, canonical stationary sets exist for all orders $\alpha < \kappa^+$.

Let E be a stationary set of order α. If $X \subseteq E$ is stationary, then $o(X) \geq o(E)$. We call E *canonical of order* α if (i) every stationary $X \subseteq E$ has order α, and (ii) E meets every set of order α.

Clearly, a canonical set of order α is unique (mod I_{NS}), and two canonical sets of different orders are disjoint (mod I_{NS}). In the following proposition, "maximal" and $\simeq$ is meant mod I_{NS}.

2.14 Proposition (Jech [47]). *A canonical set E of order α exists iff there exists a maximal set M of order α. Then (mod I_{NS})*

$$E \simeq M - Tr(M)\,,\ M \simeq E \cup Tr(E)\,,\quad \textit{and}\ Tr(E) \simeq Tr(M)\,.$$

One can show that the sets M_α obtained by iterating the Mahlo operation are maximal (as long as they are stationary). Thus when we let $E_\alpha = M_\alpha - Tr(M_\alpha)$, we get canonical stationary sets, of all orders $\alpha < \kappa^+$ (for $\alpha < o(\kappa)$).

The canonical stationary sets E_α and the canonical function f_α (of Galvin-Hajnal norm α) are closely related:

2.15 Proposition (Jech [47]). *For every $\alpha < \kappa^+$, $\alpha < o(\kappa)$,*

$$E_\alpha \simeq \{\xi < \kappa : f_\alpha(\xi) = o(\xi)\}.$$

2.4. Full reflection

Let us address the question of what is the largest possible amount of reflection, for stationary subsets of a given κ. As $A < B$ means that A reflects at almost all points of B, we would like to maximize the relation $<$. But $A < B$ implies that $o(A) < o(B)$, so we might ask whether it is possible that $A < B$ for any two stationary sets such that $o(A) < o(B)$.

By Magidor's Theorem 2.4 it is consistent that $S < \mathbf{E}^{\aleph_2}_{\aleph_1}$, and therefore $S < T$ for every S of order 0 and every T of order 1. However, this does not generalize, as the following lemma shows that when $\kappa \geq \aleph_3$, then there exist S and T with $o(S) = 0$ and $o(T) = 1$ such that $S \not< T$.

2.16 Lemma (Jech-Shelah [54]). *If $\kappa \geq \aleph_3$, then there exist stationary sets $S \subseteq \mathbf{E}^\kappa_{\aleph_0}$ and $T \subseteq \mathbf{E}^\kappa_{\aleph_1}$ such that S does not reflect at any $\alpha \in T$.*

Proof. Let S_γ, $\gamma < \omega_2$, be pairwise disjoint stationary subsets of $\mathbf{E}^\kappa_{\aleph_0}$, and let C_α, $\alpha \in \mathbf{E}^\kappa_{\aleph_1}$, be such that for every α, C_α is a club subset of α, of order type ω_1. Because at most $\aleph_1$ of the sets S_γ meet each C_α, there exists for each α some $\gamma(\alpha)$ such that $C_\alpha \cap S_{\gamma(\alpha)} = \emptyset$.

There exists some γ such that the set $T = \{\alpha : \gamma(\alpha) = \gamma\}$ is stationary; let $S = S_\gamma$. For every $\alpha \in T$, $S \cap C_\alpha = \emptyset$ and so S does not reflect at α. $\dashv$

This lemma illustrates some of the difficulties involved when dealing with reflection at singular ordinals. This problem is investigated in detail in [54], where the best possible consistency result is proved for stationary subsets of the $\aleph_n$, $n < \omega$.

Let us say that a stationary set $S \subseteq \kappa$ *reflects fully at regular cardinals* if for any stationary set T of regular cardinals $o(S) < o(T)$ implies $S < T$, and let us call *Full Reflection* the statement that every stationary subset of κ reflects fully at regular cardinals.

Full Reflection is of course nontrivial only if κ is a Mahlo cardinal. A modification of Theorem 2.4 shows that Full Reflection for a Mahlo cardinal is equiconsistent with weak compactness. The following theorem establishes the consistency strength of Full Reflection for cardinals in the Mahlo hierarchy:

2.17 Theorem (Jech-Shelah [55]). *The following are equiconsistent, for every $\alpha \le \kappa^+$:*

(i) κ *is* Π^1_α*-indescribable,*

(ii) κ *is* α*-Mahlo and Full Reflection holds.*

(A regular cardinal κ is α-Mahlo if $o(\kappa) \ge \kappa + \alpha$; κ is Π^1_1-indescribable iff it is weakly compact.)

Full Reflection is also consistent with large cardinals. The paper [57] proves the consistency of Full Reflection with the existence of a measurable cardinal. This has been improved and further generalized in [38].

Finally, the paper [91] shows that any well-founded partial order of size $\le \kappa^+$ can be realized by the reflection ordering $<$ on stationary subsets of κ, in some generic extension (using $P^2\kappa$-strong κ in the ground model).

3. Saturation

3.1. κ^+−saturation

By Solovay's 1.8 every stationary subset of κ can be split into κ disjoint stationary sets. In other words, for every stationary $S \subseteq \kappa$, the ideal $I_{NS} \restriction S$ is not κ-saturated. A natural question is if the nonstationary ideal can be κ^+-saturated.

An ideal I on κ is κ^+-*saturated* if the Boolean algebra $P(\kappa)/I$ has the κ^+-chain condition. Thus $I_{NS} \restriction S$ is κ^+-saturated when there do not exist κ^+ stationary subsets of S such that the intersection of any two of them is nonstationary. The existence and properties of κ^+-saturated ideals have been thoroughly studied since their introduction in [85], and involve large cardinals. The reader will find more details in Foreman's chapter in this volume. We shall concentrate on the special case when I is the nonstationary ideal.

The main question, whether the nonstationary ideal can be κ^+-saturated, has been answered. But a number of related questions are still open.

3.1 Theorem (Gitik-Shelah [37]). *The nonstationary ideal on* κ *is not* κ^+*-saturated, for every regular cardinal* $\kappa \ge \aleph_2$.

3.2 Theorem (Shelah). *It is consistent, relative to the existence of a Woodin cardinal, that the nonstationary ideal on* $\aleph_1$ *is* $\aleph_2$*-saturated.*

The consistency result in Theorem 3.2 was first proved in [87] using a strong determinacy assumption. That hypothesis was reduced in [92] to AD, while in [27], the assumption was the existence of a supercompact cardinal. Shelah's result (announced in [81]) is close to optimal: by Steel [86], the saturation of I_{NS} plus the existence of a measurable cardinal imply the existence of an inner model with a Woodin cardinal.

All the models mentioned in the preceding paragraph satisfy $2^{\aleph_0} > \aleph_1$. This may not be accidental, and it has been conjectured that the saturation of I_{NS} on $\aleph_1$ implies that $2^{\aleph_0} > \aleph_1$. In fact, Woodin proved this [94] under the addional assumption that there exists a measurable cardinal. We note in passing that by [27], $2^{\aleph_0} = \aleph_1$ is consistent with $I_{NS} \restriction S$ being saturated for some stationary S.

Woodin's construction [94] yields a model (starting from AD) in which the ideal I_{NS} is $\aleph_1$-*dense*, i.e. the algebra $P(\omega_1)/I_{NS}$ has a dense set of size $\aleph_1$. This, and Woodin's more recent work using Steel's inner model theory, gives the following equiconsistency.

3.3 Theorem (Woodin). *The following are equiconsistent:*

(i) ZF + AD*,*

(ii) there are infinitely many Woodin cardinals,

(iii) the nonstationary ideal on $\aleph_1$ is $\aleph_1$-dense.

As for the continuum hypothesis, Shelah proved in [80] that if I_{NS} is $\aleph_1$-dense, then $2^{\aleph_0} = 2^{\aleph_1}$.

We remark that the mere existence of a saturated ideal affects cardinal arithmetic, cf. [63] and [52].

Let us now return to Theorem 3.1. The general result proved in [37] is this:

3.4 Theorem (Gitik-Shelah [37]). *If ν is a regular cardinal and $\nu^+ < \kappa$, then $I_{NS} \restriction \mathbf{E}^\kappa_\nu$ is not κ^+-saturated.*

The proof of 3.4 combines an earlier result of Shelah (Theorem 3.7 below) with an application of the method of guessing clubs (as in 1.16). The earlier result uses generic ultrapowers and states that if $\kappa = \lambda^+$ and $\nu \neq \operatorname{cf} \lambda$ is regular, then no ideal concentrating on $\mathbf{E}^\kappa_\nu$ is κ^+-saturated.

The method of generic ultrapowers is well suited for κ^+-saturated ideals. Forcing with $P(\kappa)/I$ where I is a normal κ-complete κ^+-saturated ideal makes the generic ultrapower $N = Ult_G(V)$ well founded, preserves the cardinal κ^+, and satisfies $P^N(\kappa) = P^{V[G]}(\kappa)$. It follows that all cardinals

$< \kappa$ are preserved in $V[G]$, and it is obvious that if $\mathbf{E}^\kappa_\nu \in G$, then N (and therefore $V[G]$ as well) satisfies $\mathrm{cf}\,\kappa = \nu$.

Shelah's Theorem 3.7 below follows from a simple combinatorial lemma. Let λ be a cardinal and let $\alpha < \lambda^+$ be a limit ordinal. Let us call a family $\{X_\xi : \xi < \lambda^+\}$ a *strongly almost disjoint* (s.a.d.) family of subsets of α if every $X_\xi \subseteq \alpha$ is unbounded, and if for every $\vartheta < \lambda^+$ there exist ordinals $\delta_\xi < \alpha$, for $\xi < \vartheta$, such that the sets $X_\xi - \delta_\xi$, $\xi < \vartheta$, are pairwise disjoint. Note that if κ is a regular cardinal than there is a s.a.d. family $\{X_\xi : \xi < \kappa^+\}$ of subsets of κ.

3.5 Lemma. *If $\alpha < \lambda^+$ and $\mathrm{cf}\,\alpha \neq \mathrm{cf}\,\lambda$, then there exists no strongly almost disjoint family of subsets of α.*

Proof. Assume to the contrary that $\{X_\xi : \xi < \lambda^+\}$ is a s.a.d. family of subsets of α. We may assume that each X_ξ has order type $\mathrm{cf}\,\alpha$. Let f be a function mapping λ onto α. Since $\mathrm{cf}\,\lambda \neq \mathrm{cf}\,\alpha$ there exists for each ξ some $\gamma_\xi < \lambda$ such that $X_\xi \cap f``\gamma_\xi$ is cofinal in α. There is some γ and a set $W \subset \lambda^+$ of size λ such that $\gamma_\xi = \gamma$ for all $\xi \in W$. Let $\vartheta > \sup W$. By the assumption on the X_ξ there exist ordinals $\delta_\xi < \alpha$, $\xi < \vartheta$, such that the $X_\xi - \delta_\xi$ are pairwise disjoint. Thus $f^{-1}(X_\xi - \delta_\xi)$, $\xi \in W$, are λ pairwise disjoint subsets of γ. A contradiction. $\dashv$

3.6 Corollary (Shelah [79]). *If κ is a regular cardinal and if a forcing P makes $\mathrm{cf}\,\kappa \neq \mathrm{cf}\,|\kappa|$, then P collapses κ^+.*

Proof. Assume that κ^+ is not collapsed; thus in $V[G]$, $(\kappa^+)^V = \lambda^+$ where $\lambda = |\kappa|$. In V there is a s.a.d. family $\{X_\xi : \xi < (\kappa^+)^V\}$, and it remains a s.a.d. family in $V[G]$, of size λ^+. Since $\mathrm{cf}\,\kappa \neq \mathrm{cf}\,\lambda$, in $V[G]$, this contradicts Lemma 3.5. $\dashv$

3.7 Theorem (Shelah). *If $\kappa = \lambda^+$, if $\nu \neq \mathrm{cf}\,\lambda$ is regular and if I is a normal κ-complete κ^+-saturated ideal on κ, then $\mathbf{E}^\kappa_\nu \in I$.*

Proof. If not, then forcing with I-positive subsets of $\mathbf{E}^\kappa_\nu$ preserves κ^+ as well as $\mathrm{cf}\,\lambda$, and makes $\mathrm{cf}\,\kappa = \nu$; a contradicton. $\dashv$

Theorem 3.4 leaves open the following problem: If λ is a regular cardinal, can $I_{NS} \restriction \mathbf{E}^{\lambda^+}_\lambda$ be λ^{++}-saturated? (For instance can $I_{NS} \restriction \mathbf{E}^{\aleph_2}_{\aleph_1}$ be $\aleph_3$-saturated?) Let us also mention that for all regular ν and κ not excluded by Corollary 3.7, it is consistent that $I_{NS} \restriction S$ is κ^+-saturated for some $S \subset \mathbf{E}^\kappa_\nu$ (see [33]).

If κ is a large cardinal, then $I_{NS} \restriction Reg$ can be κ^+-saturated, as the following theorem shows. Of course, κ cannot be too large: if κ is greatly Mahlo, then the canonical stationary sets E_α $\kappa \le \alpha < \kappa^+$ witness nonsaturation.

3.8 Theorem (Jech-Woodin [58]). *For any $\alpha < \kappa^+$, the following are equiconsistent:*

(i) κ is measurable of order α,

(ii) κ is α-Mahlo and the ideal $I_{NS} \restriction Reg$ on κ is κ^+-saturated.

3.2. Precipitousness

An important property of saturated ideals is that the generic ultrapower is well-founded. It has been recognized that this property is important enough to single out and study the class of ideals that have it. The ideals for which the generic ultrapower is well founded are called *precipitous.* They are described in detail in Foreman's chapter in this volume; here we address the question of when the nonstationary ideal is precipitous.

Precipitous ideals were introduced by Jech and Prikry in [51]. There are several equivalent formulations of precipitousness. Let I be an ideal on some set E. An *I-partition* is a maximal family of I-positive sets such that the intersection of any two of them is in I. Let $\mathcal{G}_I$ denote the infinite game of two players who alternately pick I-positive sets S_n such that $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$. The first player wins if $\bigcap_{n=1}^{\infty} S_n = \emptyset$.

3.9 Theorem (Jech-Prikry [51, 45, 46, 29]). *Let I be an ideal on a set E. The following are equivalent:*

(i) forcing with $P(E)/I$ makes the generic ultrapower well-founded,

(ii) for every sequence $\{W_n\}_{n=1}^{\infty}$ of I-partitions there exists a sequence $\{X_n\}_{n=1}^{\infty}$ such that $X_n \in W_n$ for each n, and $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$,

(iii) the first player does not have a winning strategy in the game $\mathcal{G}_I$.

The problem of whether the nonstationary ideal on κ can be precipitous involves large cardinals. For $\kappa = \aleph_1$ the exact consistency strength is the existence of a measurable cardinal:

3.10 Theorem (Jech-Magidor-Mitchell-Prikry [50]). *The following are equiconsistent:*

(i) there exists a measurable cardinal,

(ii) the nonstationary ideal on $\aleph_1$ is precipitous.

For $\kappa \geq \aleph_2$, stronger large cardinal assumptions are involved. For $\kappa = \aleph_2$, the consistency strength is a measurable of order 2:

3.11 Theorem (Gitik [31]). *The following are equiconsistent:*

(i) there exists a measurable cardinal of order 2,

(ii) I_{NS} on $\aleph_2$ is precipitous.

For the general case, the paper [47] provided lower bounds for the consistency strength of "I_{NS} is precipitous," in terms of the Mitchell order, while models with I_{NS} precipitous for $\kappa > \aleph_2$ were constructed in [33] and [27] from strong assumptions. In [35] and [36], Gitik established the exact consistency strength of "I_{NS} on κ is precipitous" when κ is the successor of a regular cardinal λ (the existence of an $(\omega, \lambda+1)$-repeat point), as well as nearly optimal lower and upper consistency bounds for κ inaccessible. Additional lower bounds for the case $\kappa = \lambda^+$ where λ is a large cardinal appear in [95].

The problem of whether the nonstationary ideal on κ can be precipitous while κ is measurable was first addressed by Kakuda in [61] who proved that many measurables are necessary. This lower bound was improved to having Mitchell order $\kappa^+ + 1$ in [47], and to a repeat point in [34]. Gitik's paper also shows that the existence of a supercompact cardinal suffices for the consistency of the nonstationary ideal on a supercompact cardinal being precipitous.

4. The closed unbounded filter on $P_\kappa\lambda$

4.1. Closed unbounded sets in $P_\kappa A$

One of the useful tools of combinatorial set theory is a generalization of the concepts of closed unbounded set and stationary set. This generalization, introduced in [43] and [44], replaces $\langle \kappa, < \rangle$ with $\langle P_\kappa\lambda, \subset \rangle$, and is justified by the fact that the crucial Theorem 1.5 remains true under the generalization.

Let κ be a regular uncountable cardinal and let A be a set of cardinality at least κ. Let $P_\kappa A$ denote the set $\{x : x \subset A \text{ and } |x| < \kappa\}$. Furthermore, we let $[X]^\nu = \{x \subseteq X : |x| = \nu\}$ whenever $|X| \geq \nu$ and ν an infinite cardinal.

4.1 Definition (Jech [44]). Let κ be a regular uncountable cardinal and let $|A| \geq \kappa$.

A set $X \subseteq P_\kappa A$ is *unbounded* (in $P_\kappa A$) if for every $x \in P_\kappa A$ there is a $y \supset x$ such that $y \in X$.

A set $X \subseteq P_\kappa A$ is *closed* (in $P_\kappa A$) if for any chain $x_0 \subseteq x_1 \subseteq \cdots \subseteq x_\xi \subseteq \cdots$, $\xi < \alpha$, of sets in X, with $\alpha < \kappa$, the union $\bigcup_{\xi<\alpha} x_\xi$ is in X.

A set $C \subseteq P_\kappa A$ is *closed unbounded* if it is closed and unbounded.

A set $S \subseteq P_\kappa A$ is *stationary* (in $P_\kappa A$) if $S \cap C \neq \emptyset$ for every closed unbounded $C \subseteq P_\kappa A$.

The *closed unbounded filter* on $P_\kappa A$ is the filter generated by the closed unbounded sets. We remark that when $A = \kappa$, then the set $\kappa \subset P_\kappa \kappa$ is closed unbounded, and the club filter on κ is the restriction to κ of the club filter on $P_\kappa \kappa$. As before, the basic observation is that the intersection of two clubs is a club, and we have again:

4.2 Proposition. *The club filter on $P_\kappa A$ is κ-complete.*

For the generalization of Theorem 1.5, let us first define the *diagonal intersection.*

$$\Delta_{a\in A}\, X_a = \{x \in P_\kappa A : x \in \bigcap_{a \in x} X_a\}.$$

The generalization of Lemma 1.4 is this:

4.3 Lemma. *If $\langle C_a : a \in A\rangle$ is a sequence of closed unbounded sets in $P_\kappa A$, then its diagonal intersection is closed unbounded.*

Again, this lemma immediately implies the appropriate generalization of Theorem 1.5:

4.4 Theorem (Jech [44]). *If S is a stationary set in $P_\kappa A$ and if f is a function on S such that $f(x) \in x$ for every $x \in S - \{\emptyset\}$, then there exists some $a \in A$ such that $f(x) = a$ on a stationary subset of S.*

In Proposition 1.6 we showed that the club filter is the smallest normal filter on κ. We shall now do the same for $P_\kappa A$. A $\kappa-$complete filter F on $P_\kappa A$ is *normal* if for every $a \in A$, $\{x \in P_\kappa A : a \in x\} \in F$, and if F is closed under diagonal intersections.

The following fact (proved by induction on $|D|$) is quite useful; D is $\subseteq$-*directed* if for any $x, y \in D$ there is a $z \in D$ such that $x \cup y \subseteq z$.

4.5 Proposition. *If X is a closed set in $P_\kappa A$, then for any $\subseteq$-directed D with $|D| < \kappa$, $\bigcup D \in X$.*

Let $f : [A]^{<\omega} \to P_\kappa A$; a set $x \in P_\kappa A$ is a *closure point* of f is $f(e) \subseteq x$ whenever $e \subseteq x$. The set Cl_f of all closure points $x \in P_\kappa A$ is easily seen to be a club. More importantly, the sets Cl_f generate the club filter:

4.6 Proposition (Menas [77]). *For every closed unbounded set in $P_\kappa A$ there is an $f : [A]^{<\omega} \to P_\kappa A$ such that $Cl_f \subseteq C$.*

Proof. By induction on $|e|$ we find an infinite set $f(e) \in C$ such that $e \subset f(e)$ and that $f(e') \subseteq f(e)$ whenever $e' \subset e$. To see that $Cl_f \subseteq C$, let x be a closure point of f. As $x = \bigcup\{f(e) : e \in [x]^{<\omega}\}$ is the union of a small $\subseteq$-directed subset of C, we have $x \in C$. $\dashv$

4.7 Corollary (Carr [18]). *If F is a normal $\kappa-$complete filter on $P_\kappa A$, then F contains all closed unbounded sets.*

Proof. Let F^+ denote the F-positive sets, those whose complement is not in F. A consequence of normality is that if $X \in F^+$ and g is a function on X such that $g(x) \in [x]^{<\omega}$ for all $x \in X$, then g is constant on a set in F^+.

Now assume that there is a club not in F. Thus there is an $f : A \to P_\kappa A$ such that the complement X of Cl_f is F-positive. For each $x \in X$ there is some $e = g(x) \in [x]^{<\omega}$ such that $f(e) \nsubseteq x$. Therefore there is some e such that $\{x : f(e) \nsubseteq x\} \in F^+$. This is a contradiction, because $\{x : f(e) \subseteq x\} \in F$. $\dashv$

As another consequence of Proposition 4.6 we consider *projections* and *liftings* of stationary sets. Let $A \subseteq B$ (and $|A| \geq \kappa$). For $X \in P_\kappa B$, the *projection* of X is the set

$$X \restriction A = \{x \cap A : x \in X\};$$

for $Y \in P_\kappa A$, the *lifting* of Y is

$$Y^B = \{x \in P_\kappa B : x \cap A \in Y\}.$$

4.8 Proposition (Menas [77]). *Let $A \subseteq B$.*

(i) If S is stationary in $P_\kappa B$, then $S \restriction A$ is stationary in $P_\kappa A$.

(ii) If S is stationary in $P_\kappa A$, then S^B is stationary in $P_\kappa B$.

Proof. (i) is easy and holds because if C is a club in $P_\kappa A$, then C^B is a club in $P_\kappa B$. For (ii), it suffices to prove that if C is a club in $P_\kappa B$, then $C \restriction A$ contains a club in $P_\kappa A$.

If $C \subseteq P_\kappa B$ is a club, by Proposition 4.6 there is an $f : [B]^{<\omega} \to P_\kappa B$ such that $Cl_f \subseteq C$. Let $g : [A]^{<\omega} \to P_\kappa A$ be as follows: let $g(E) =$ (the f-closure of e) $\cap A$. Since $Cl_f \restriction A = Cl_g$, we have $Cl_g \subseteq C \restriction A$. $\dashv$

When $\kappa = \aleph_1$, Proposition 4.6 can be improved by replacing f by a function with values in A, i.e. an operation on A. For $f : [A]^{<\omega} \to A$, let Cl_f denote the set $\{x : f(e) \in x$ whenever $e \subseteq x\}$. The following characterization of the club filter on $P_{\omega_1}A$ was given in [66]; this and [67] used $P_{\omega_1}A$ in the study of model theory.

4.9 Theorem (Kueker [66]). *The club filter on $P_{\omega_1}A$ is generated by the sets Cl_F where $F : [A]^{<\omega} \to A$.*

When $\kappa > \aleph_1$, then the clubs Cl_F where $F : [A]^{<\omega} \to A$, do not generate the club filter: every F has countable closure points while the set of all uncountable $x \in P_\kappa A$ is closed unbounded. However, a slight modification of Theorem 4.9 works, namely Proposition 4.10 below. Let us call the club Cl_F for $F : [A]^{<\omega} \to A$ *strongly closed unbounded*.

Let us consider $P_\kappa\lambda$ where $\lambda \geq \kappa$. We note that the set

$$\{x \in P_\kappa\lambda : x \cap \kappa \in \kappa\}$$

is closed unbounded. It turns out that the club filter is generated by adding this set to the filter generated by the strongly club sets.

4.10 Proposition ([27]). *For every club C in $P_\kappa\lambda$ there exists a function $F : [\lambda]^{<\omega} \to \lambda$ such that*

$$\{x \in P_\kappa\lambda : x \cap \kappa \in \kappa \text{ and } F``[x]^{<\omega} \subseteq x\} \subseteq C.$$

Now let us consider $P_\kappa\lambda$ for $\kappa = \nu^+$ where ν is uncountable. As the set $[\lambda]^\nu$ is closed unbounded in $P_{\nu^+}\lambda$, let us consider the restriction of the club filter to $[\lambda]^\nu$. We say that a set $S \subseteq [\lambda]^\nu$ is *weakly stationary* if it meets every strongly club set. It turns out that the question whether weakly stationary sets are stationary involves large cardinals. By Proposition 4.10, this question depends on whether the set $\{x \in [\lambda]^\nu : x \cap \nu^+ \in \nu^+\}$ is in the strongly club filter. The following reformulation, implicit in [27], establishes the relation to large cardinals:

4.11 Proposition. *There exists a weakly stationary nonstationary set in $[\lambda]^\nu$ if and only if the (nonstationary) set $\{x \in [\lambda]^\nu : x \not\supseteq \nu\}$ is weakly stationary.*

4.12 Corollary. *The following are equivalent:*

(i) The club filter on $[\omega_2]^{\aleph_1}$ is not generated by strongly club sets,

(ii) Chang's Conjecture.

4.2. Splitting stationary sets

Let us now address the question whether stationary sets can be split into a large number of disjoint stationary sets. In particular, does Theorem 1.8 generalize to $P_\kappa A$? As only the size of A matters, and the club filter on $P_\kappa\kappa$ is basically just the club filter on κ, we shall consider subsets of $P_\kappa\lambda$ where λ is a cardinal and $\lambda > \kappa$.

We have $|P_\kappa\lambda| = \lambda^{<\kappa}$ and so the maximal possible size of a disjoint family of subsets of $P_\kappa\lambda$ is $\lambda^{<\kappa}$. While it is consistent that every stationary set splits into $\lambda^{<\kappa}$ disjoint stationary subsets (see Corollary 4.18), this is not provable in ZFC. The reason is that there may exist closed unbounded sets in $P_\kappa\lambda$ whose size is less than $\lambda^{<\kappa}$. For instance, [8] shows that there exists a club in $P_{\omega_3}\omega_4$ of size $\aleph_4^{\aleph_1}$; thus if $2^{\aleph_2} > 2^{\aleph_1}\cdot\aleph_4$, then $P_{\omega_3}\omega_4$ is not the union of $\aleph_4^{<\aleph_3}$ disjoint stationary sets. An earlier result [12] proved the consistency of a stationary set $S \subseteq P_{\omega_1}\omega_2$ such that $I_{NS} \restriction S$ is $2^{\aleph_0}$-saturated.

A modification of Solovay's proof, using the generic ultrapower by I_{NS}, gives this:

4.13 Theorem (Gitik [32]). *Every stationary subset of $P_\kappa\lambda$ can be partitioned into κ disjoint stationary sets.*

The question of splitting stationary sets has been more or less completely solved for splitting into λ sets. Let us first observe that the nonexistence of λ disjoint stationary sets is equivalent to λ-saturation:

4.14 Lemma. *If X_α, $\alpha < \lambda$, are stationary sets in $P_\kappa\lambda$ such that $X_\alpha \cap X_\beta \in I_{NS}$ for all $\alpha \neq \beta$, then there exist pairwise disjoint stationary sets Y_α with $Y_\alpha \subseteq X_\alpha$ for all $\alpha < \lambda$.*

Proof. Let $Y_\alpha = X_\alpha \cap \{x : \alpha \in x \text{ and } \forall\beta \in x(\beta \neq \alpha \to x \notin X_\beta)\}$. $\dashv$

A long succession of results by Jech [44], Menas [77], Baumgartner, DiPrisco and Marek [19], Matsubara [72], [73] established the following:

4.15 Theorem. *(i) $P_\kappa\lambda$ can be partitioned into λ disjoint stationary sets.*

(ii) If κ is a successor cardinal, than every stationary subset of $P_\kappa\lambda$ can be partitioned into λ disjoint stationary sets.

(iii) If $0^\#$ does not exist, then every stationary subset of $P_\kappa\lambda$ can be partitioned into λ disjoint stationary sets.

A complete proof of this theorem can be found in Kanamori's book [62]. The results (ii) and (iii) are best possible, in the following sense:

4.16 Theorem (Gitik [32]). *It is consistent, relative to a supercompact cardinal, that κ is inaccessible, $\lambda > \kappa$, and some stationary set $S \subset P_\kappa\lambda$ cannot be partitioned into κ^+ disjoint stationary subsets.*

(For a simplification of Gitik's proof, as well as further results, see [83].)

The proof of Theorem 4.15 involves the following set, which clearly is stationary:

$$S_0 = \{x \in P_\kappa\lambda : |x \cap \kappa| = |x|\}.$$

This stationary set can be partitioned into λ disjoint stationary sets, and if either $\kappa = \nu^+$ or $0^\#$ does not exist, then S_0 contains a club (cf. [62]).

Clearly, in Gitik's model the set S_0 does not contain a club. Thus the statement that for some κ and λ,

$$\{x \in P_\kappa\lambda : |x \cap \kappa| < |x|\}$$

is stationary, is a consistent large cardinal statement. Its exact consistency strength (between $0^\#$ and Ramsey) is pinned down in [5] and [20].

As for splitting into $\lambda^{<\kappa}$ sets, the following result of Matsubara together with Theorem 4.15 proves the consistency result mentioned earlier:

4.17 Proposition (Matsubara [74]). *Assume* GCH. *If cf $\lambda < \kappa$, then every stationary subset of $P_\kappa\lambda$ can be partitioned into λ^+ disjoint stationary sets.*

4.18 Corollary. *Assume* GCH *and that $0^\#$ does not exist. Then every stationary subset of $P_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

4.3. Saturation

By Theorem 4.15, the nonstationary ideal on $P_\kappa\lambda$ is not λ-saturated (even though $I_{NS} \restriction S$ can be κ^+-saturated for some S). The next question is whether it can be λ^+-saturated, and the answer is again no.

4.19 Theorem (Foreman-Magidor [26]). *For every regular uncountable cardinal κ and every cardinal $\lambda > \kappa$, the nonstationary ideal on $P_\kappa\lambda$ is not λ^+-saturated.*

We note that special cases of this theorem have been proved earlier, cf . [4], [73], [60] and [17].

When dealing with λ^+-saturation, we naturally employ generic ultrapowers and use the fact that the ultrapower is well founded; a normal λ^+-saturated ideal on $P_\kappa\lambda$ is precipitous. The nonstationary ideal on $P_\kappa\lambda$ (for regular λ) can be precipitous. The consistency, a result of Goldring [39], is relative to a Woodin cardinal, and strengthens an earlier result in [27]. On the other hand, the paper [75] gives instances of κ and λ for which I_{NS} on $P_\kappa\lambda$ cannot be precipitous.

5. Proper forcing and other applications

5.1. Proper forcing

One of the most fruitful applications of the club filter on $P_{\omega_1}A$ is Shelah's concept of *proper forcing*. As proper forcing is discussed in detail in Abraham's chapter in this volume, I shall only give a brief account in this section. The rest of Section 5 deals with applications of the club filter on $P_{\omega_1}A$ in the theory of Boolean algebras.

When dealing with closed unbounded sets in $P_{\omega_1}A$ we may as well restrict ourselves to infinite sets, and thus consider the space $[A]^{\aleph_0}$ (where A is an uncountable set). By Kueker's Theorem 4.9, a set $X \subseteq [A]^{\aleph_0}$ is in the club filter just in case it contains the set Cl_F of all closure points of some operation on A. Equivalently, X contains all elementary submodels of some model with universe A. A useful modification of this is the following consequence of Proposition 4.8.

5.1 Proposition. *A set $X \subseteq [A]^{\aleph_0}$ is in the club filter if for some sufficiently large λ, X contains $M \cap A$, for all countable $M \prec \mathcal{H}_\lambda$ such that $A \in M$.*

Here $\mathcal{H}_\lambda = \langle H_\lambda, \in\rangle$ where H_λ is the set of all sets hereditarily of power $< \lambda$; "sufficiently large" means $2^{|TC(A)|} < \lambda$.

Let us now turn to proper forcing. First we remark that the preservation theorem 1.13 generalizes to $[A]^{\aleph_0}$:

5.2 Theorem. *(a) If P satisfies ccc, then every club $C \in V[G]$ in $[A]^{\aleph_0}$ has a club subset in the ground model. Hence every stationary subset of $[A]^{\aleph_0}$ remains stationary in $V[G]$.*

(b) If P is countably closed, then every stationary subset of $[A]^{\aleph_0}$ remains stationary in $V[G]$.

For a proof, we refer the reader to [7], Theorem 2.3, or [48] p. 87. This leads to the important definition, cf. [79]:

5.3 Definition. A notion of forcing P is *proper* if for every uncountable set A, every stationary subset of $[A]^{\aleph_0}$ remains stationary in $V[G]$.

There are several equivalent definitions of properness, most using the club filter on $[A]^{\aleph_0}$. Let me state one of them (see [79], p. 77, [48], p. 97):

5.4 Proposition. *A complete Boolean algebra B is proper if and only if for every nonzero $a \in B$ for every uncountable λ and every collection $\{a_{\alpha\beta} : \alpha, \beta < \lambda\}$ such that $\sum_{\beta<\lambda} a_{\alpha\beta} = a$ for every $\alpha < \lambda$, there exists a club $C \subseteq [\lambda]^{\aleph_0}$ such that $\prod_{\alpha\in x} \sum_{\beta\in x} a_{\alpha\beta} \neq 0$ for all $x \in C$.*

5.2. Projective and Cohen Boolean algebras

The club filter on $[A]^{\aleph_0}$ turns out to be a useful tool in the study of Boolean algebras. Here we present a uniform approach to the investigation of two related concepts, projective and Cohen Boolean algebras. For simplicity, we consider only atomless Boolean algebras of uniform density.

5.5 Definition. (a) A Boolean algebra B is *projective* if for some Boolean algebra C, the free product $B \oplus C$ is a free Boolean algebra.

(b) A Boolean algebra B is a *Cohen algebra* if its completion is isomorphic to the completion of a free Boolean algebra.

Projective algebras have several other (equivalent) definitions and are projective in the sense of universal algebra; we refer to [64] for details. Forcing with Cohen algebras adds Cohen reals. In the present context, it is the following equivalences that make these two classes interesting: Let A be a subalgebra of a Boolean algebra B. A is a *relatively complete* subalgebra of B, $A \leq_{rc} B$, if for each $b \in B$ there is a smallest element $a \in A$ such that $b \leq a$. A is a *regular subalgebra* of B, $A \leq_{reg} B$, if every maximal antichain in A is maximal in B. Let $\langle A_1 \cup A_2\rangle$ denote the subalgebra generated by $A_1 \cup A_2$.

5.6 Theorem. (a) (Ščepin [78]) *A Boolean algebra B is projective if and only if the set $\{A \in [B]^{\aleph_0} : A \leq_{rc} B\}$ contains a club C with the property that for all $A_1, A_2 \in C$, $\langle A_1 \cup A_2\rangle \in C$.*

(b) (Koppelberg [65], Balcar-Jech-Zapletal [3]) *A Boolean algebra B is Cohen if and only if the set $\{A \in [B]^{\aleph_0} : A \leq_{reg} B\}$ contains a club with the property that for all $A_1, A_2 \in C$, $\langle A_1 \cup A_2\rangle \in C$.*

This leads naturally to the following concepts:

5.7 Definition. (a) (Ščepin) A Boolean algebra B is *openly generated* if the set $\{A \in [B]^{\aleph_0} : A \leq_{rc} B\}$ contains a club.

(b) (Fuchino-Jech) A Boolean algebra B is *semi-Cohen* if the set $\{A \in [B]^{\aleph_0} : A \leq_{reg} B\}$ contains a club.

Openly generated (also called rc-filtered) and semi-Cohen Boolean algebras are investigated systematically in [42] and [3], respectively.

Our first observation is that every projective algebra is openly generated and every Cohen algebra is semi-Cohen; and if $|B| = \aleph_1$ and B is openly generated (or semi-Cohen), then B is projective (or Cohen). Because a σ-closed forcing preserves stationary sets in $[B]^{\aleph_0}$ (by Theorem 5.2), we have:

5.8 Corollary. *B is openly generated (resp. semi-Cohen) iff $V^P \vDash B$ is projective (resp. Cohen), where P is the σ-closed collapse of $|B|$ onto $\aleph_1$.*

An immediate consequence is that the completion of a semi-Cohen algebra is semi-Cohen.

Using some simple algebra and Proposition 4.8, one can show that every rc-subalgebra of an openly generated algebra is openly generated and every regular subalgebra of a semi-Cohen algebra is semi-Cohen. Consequently, we have:

5.9 Corollary. (a) (Ščepin) *If B is projective and $A \leq_{rc} B$ has size $\aleph_1$, then A is projective.*

(b) (Koppelberg) *If B is a Cohen algebra and $A \leq_{reg} B$ has size $\aleph_1$, then A is Cohen.*

Finally, the use of the club filter yields a simple proof of the following theorem:

5.10 Theorem. (a) (Ščepin [78], Fuchino [28]) *The union of any continuous $\leq_{rc}$-chain of openly generated algebras is openly generated.*

(b) (Balcar-Jech-Zapletal [3]) *The union of any continuous $\leq_{reg}$-chain of semi-Cohen algebras is semi-Cohen.*

Proof. Let B be the union and let λ be sufficiently large; by Proposition 5.1 it suffices to show that for every countable $M \prec \mathcal{H}_\lambda$ such that $B \in M$, $B \cap M$ is a relatively complete (resp. regular) subalgebra of B. It is not very difficult to prove this. $\dashv$

6. Reflection

In Section 2 we introduced the important concept of reflection. One can expect that its generalization to $P_\kappa\lambda$ will be equally important. This is indeed the case, and in particular, reflection of stationary sets in $[\lambda]^{\aleph_0}$ at sets of cardinality $\aleph_1$ plays a significant role in applications of Martin's Maximum.

Let us begin with a generalization of reflection of which very little is known (see [56] for a consistency result): Let κ be inaccessible, and let $\lambda > \kappa$. For each $x \in P_\kappa\lambda$, let $\kappa_x = x \cap \kappa$; note that for almost all x, κ_x is a cardinal. When κ_a is regular uncountable, we say that a stationary S reflects at a if $S \cap P_{\kappa_a} a$ is a stationary subset of $P_{\kappa_a} a$.

The following argument shows that there are limitations to reflection: Let $S \subseteq \mathbf{E}^\lambda_{\aleph_0}$ and $T \subseteq \mathbf{E}^\lambda_{\aleph_1}$ be such that S does not reflect at any $\alpha \in T$ (see Lemma 2.16). Let $\widehat{S} = \{x \in P_\kappa\lambda : \sup x \in S\}$ and $\widehat{T} = \{a \in P_\kappa\lambda : \sup a \in T\}$. Then $\widehat{S}$ does not reflect at any $a \in \widehat{T}$.

A similar generalization leads to significant results in the large cardinal theory and we shall now investigate this generalization.

6.1. Reflection principles

In [27], Foreman, Magidor and Shelah introduced Martin's Maximum and proved a number of consequences. Let us recall that *Martin's Maximum* (MM) states that whenever P is a notion of forcing that preserves stationary subsets of $\aleph_1$, and D is a family of $\aleph_1$ dense subsets of P, then there exists a D-generic filter on P. By [27] Martin's Maximum is consistent relative to a supercompact cardinal.

Among the consequences of MM proved in [27] are the following:

- The nonstationary ideal on $\aleph_1$ is $\aleph_2$-saturated
- For every regular $\kappa \geq \aleph_2$, every stationary set $S \subseteq \mathbf{E}^\kappa_{\aleph_0}$ contains a closed set of order type ω_1.
- $2^{\aleph_0} = \aleph_2$.
- For every regular $\kappa \geq \aleph_2$, $\kappa^{\aleph_0} = \kappa$.

The authors of [27] introduced the following *Reflection Principle* and proved that it follows from MM.

If S is a stationary subset of $[\lambda]^{\aleph_0}$ and $X \in [\lambda]^{\aleph_1}$ we say that S *reflects at* X if $S \cap [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$.

6.1 Definition (**Reflection Principle**, Foreman-Magidor-Shelah [27]). For every regular $\lambda \geq \aleph_2$, every stationary set $S \subseteq [\lambda]^{\aleph_0}$ reflects at some $X \in [\lambda]^{\aleph_1}$ such that $X \supseteq \omega_1$.

For a given regular λ, let us call the property in Definition 6.1 *Reflection Principle at* λ. As for the extra condition $X \supseteq \omega_1$, this is not just an ad hoc requirement. Its role is clarified in the following two propositions. (Compare this with the remark following Theorem 2.3.)

6.2 Proposition (Feng-Jech [21]). *Let $\lambda \geq \aleph_2$ be a regular cardinal.*

(a) Reflection Principle at λ holds if and only if for every stationary set $S \subseteq [\lambda]^{\aleph_0}$, the set $\{X \in [\lambda]^{\aleph_1} : S$ reflects at $X\}$ is stationary in $[\lambda]^{\aleph_1}$.

(b) Every stationary $S \subseteq [\lambda]^{\aleph_0}$ reflects at some $X \in [\lambda]^{\aleph_1}$ if and only if for every stationary set $\subseteq [\lambda]^{\aleph_0}$, the set $\{X \in [\lambda]^{\aleph_1} : S$ reflects at $X\}$ is weakly stationary in $[\lambda]^{\aleph_1}$.

For $\lambda = \aleph_2$ the assumption $X \supseteq \omega_1$ can be dropped; it is unknown if the same is true in general:

6.3 Proposition (Feng-Jech [21]). *Reflection Principle at $\aleph_2$ holds if and only if every stationary $S \subseteq [\omega_2]^{\aleph_0}$ reflects at some $X \in [\omega_2]^{\aleph_1}$.*

The significance of this and related reflection principles is illustrated by the fact that they imply the major consequences of MM. Firstly, Reflection Principle at $\aleph_2$ implies that the continuum is at most $\aleph_2$:

6.4 Theorem (Shelah [80], Todorčević [89]). *If every stationary $S \subseteq [\omega_2]^{\aleph_0}$ reflects at some $X \in [\omega_2]^{\aleph_1}$, then $2^{\aleph_0} \leq \aleph_2$.*

Proof. For each uncountable $\alpha < \omega_2$, let $C_\alpha \subseteq [\alpha]^{\aleph_0}$ be a club of cardinality $\aleph_1$, and let $D = \bigcup_{\omega_1 \leq \alpha < \omega_2} C_\alpha$. By Propositions 6.3 and 6.2(a), the set D contains a club, and we have $|D| = \aleph_2$. However, it is proved in [12] that every club in $[\omega_2]^{\aleph_0}$ has cardinality $\aleph_2^{\aleph_0}$; hence $2^{\aleph_0} \leq \aleph_2$. $\dashv$

Reflection Principle at $\aleph_2$ is not particularly strong; it is equiconsistent with the existence of a weakly compact cardinal. A modification of Magidor's construction [70] gives a model in which every stationary $S \subseteq [\omega_2]^{\aleph_0}$ reflects at $[\alpha]^{\aleph_0}$ for almost all $\alpha \in \mathbf{E}^{\aleph_2}_{\aleph_1}$.

The general Reflection Principle, for all regular $\lambda \geq \aleph_2$, is a stronger large cardinal property. A modification of the proof of Theorem 25 in [27] shows that the Reflection Principle implies that the nonstationary ideal on ω_1 is presaturated (i.e. precipitous, and forcing with $P(\omega_1)/I_{NS}$ preserves ω_2). This has strong large cardinal consequences.

The Reflection Principle follows from MM and in fact from a weaker forcing axiom MA^+ (σ-closed). (This latter axiom is known to be strictly weaker than MM). In fact, MA^+ (σ-closed) implies (cf. [14]) for every regular $\lambda \geq \aleph_2$, for every stationary set $\cdot S \subseteq [\lambda]^{\aleph_0}$, the set $\{X \in [\lambda]^{\aleph_1} : S \text{ reflects at } X\}$ meets every ω_1-closed unbounded set C in $[\lambda]^{\aleph_1}$. This reflection principle was introduced in [23].

Todorčević formulated a strengthening of the Reflection Principle and proved that his *Strong Reflection Principle* (SRP) implies that the nonstationary ideal on ω_1 is $\aleph_2$-saturated, that every stationary subset of $\mathbf{E}^{\kappa}_{\aleph_0}$ contains a closed copy of ω_1 and that for every regular $\kappa \geq \aleph_2$, $\kappa^{\aleph_0} = \kappa$ (cf. [15]). In [22], another reflection principle is introduced, called *Projective Stationary Reflection* (PSR), and proved to be equivalent to the Strong Reflection Principle (SRP=PSR).

6.5 Definition ([22]). A stationary set $S \subseteq [A]^{\aleph_0}$ where $A \supseteq \omega_1$, is *projective stationary* if for every club $C \subseteq [A]^{\aleph_0}$, the projection $(S \cap C) \restriction \omega_1 = \{x \cap \omega_1 : x \in S \cap C\}$ to ω_1 contains a club.

6.6 Definition (**Projective Stationary Reflection** (PSR), Feng-Jech [22]). For every regular $\lambda \geq \aleph_2$, every projective stationary set $S \subseteq [H_\lambda]^{\aleph_0}$ contains an increasing continuous $\in$-chain $\{N_\alpha : \alpha < \omega_1\}$ of elementary submodels of H_λ.

If $S \subseteq [H_\lambda]^{\aleph_0}$ is stationary, let P_S be the forcing notion consisting of countable increasing continuous $\in$-chains $\{N_\alpha : \alpha < \gamma\} \subseteq S$ of elementary submodels of H_λ. The set S is projective stationary just in case P_S preserves stationary subsets of ω_1. Thus PSR follows from Martin's Maximum. It is also proved in [22] that PSR implies the Reflection Principle.

6.7 Theorem (Feng-Jech [22]). *Assume PSR. (a) For every regular $\kappa \geq \aleph_2$, every stationary set $S \subseteq \mathbf{E}^{\kappa}_{\aleph_0}$ contains a closed set of order type ω_1.*

(b) The nonstationary ideal on ω_1 is $\aleph_2$-saturated.

Proof. (a) This is proved by applying PSR to the projective stationary set

$$\{N \in [H_\kappa]^{\aleph_0} : S \in N \prec H_\kappa \text{ and } \sup(N \cap \kappa) \in S\},$$

where S is a given stationary subset of $\mathbf{E}^{\kappa}_{\aleph_0}$.

(b) Let A be a maximal antichain of stationary subsets of ω_1. Then the set

$$X = \{N \in [H_{\omega_2}]^{\aleph_0} : A \in N \prec H_{\omega_2} \text{ and } N \cap \omega_1 \in S \text{ for some } S \in A \cap N\}$$

is projective stationary. By PSR, there exists an $\in$-chain $\{N_\alpha : \alpha < \omega_1\} \subset X$, and we let $N = \bigcup_{\alpha<\omega_1} N_\alpha$. One can verify that $A \subset N$, and therefore $|A| \leq \aleph_1$. $\dashv$

Finally, recent work of Woodin shows that Strong Reflection implies that $2^{\aleph_0} = \aleph_2$, in fact $\boldsymbol{\delta}^1_2 = \omega_2$:

6.8 Theorem (Woodin [94]). *Assume SRP. Then the set $\{N \in [H_{\omega_3}]^{\aleph_1} : N \prec H_{\omega_3}$ and the order type of $N \cap \omega_3$ is $\omega_1\}$ is weakly stationary. This together with the saturation of the nonstationary ideal, implies that $\boldsymbol{\delta}^1_2 = \omega_2$.*

6.2. Nonreflecting stationary sets

The results about reflecting stationary sets in $[\lambda]^{\aleph_0}$ at sets of size $\aleph_1$ do not generalize to $[\lambda]^\kappa$ for $\kappa \geq \aleph_1$. For instance, the analog of the Reflection Principle is false:

6.9 Proposition. *If λ is sufficiently large, then it is not the case that every stationary $S \subseteq [\lambda]^{\aleph_1}$ reflects at some $X \in [\lambda]^{\aleph_2}$ such that $X \supseteq \omega_2$.*

In Section 6.1 we mentioned that the Reflection Principle implies that I_{NS} on ω_1 is presaturated. To prove Proposition 6.9, one first shows that the generalization of the Reflection Principle would yield presaturation of I_{NS} on ω_2, thus (as in Shelah's Corollary 3.7) a forcing notion that changes the cofinality of ω_2 to ω while preserving $\aleph_1$ and $\aleph_3$. But that is impossible.

Specific examples of nonreflecting stationary subsets of $[\lambda]^{\aleph_1}$ are given in [25]. That paper also explains why the consistency proof of Reflection Principle does not generalize. A model of MA^+ (σ-closed) is obtained by Lévy collapsing (to $\aleph_1$) cardinals below a supercompact. A crucial fact is that the collapse preserves stationary sets in $[\lambda]^{\aleph_0}$ (Theorem 5.2(a)). Unfortunately, the analog of this is false in general, as $< \kappa$-closed forcing can destroy stationary sets in $P_\kappa\lambda$.

Following [27], a model $N \prec H_\lambda$ is *internally approachable* (IA) if there exists a chain $\langle N_\alpha : \alpha < \gamma\rangle$ whose initial segments belong to N, with $N = \bigcup_{\alpha<\gamma} N_\alpha$. Let κ be a regular uncountable cardinal and let $\lambda \geq \kappa$ be regular. The set $P_\kappa H_\lambda \cap IA$ is stationary and its projection to κ contains a club. Moreover, every countable N is internally approachable and so $[H_\lambda]^{\aleph_0} \cap IA$ contains a club.

It is proved in [25] that every $< \kappa$-closed forcing preserves stationary subsets of $P_\kappa H_\lambda \cap IA$, and that the $< \kappa$-closed collapse of H_λ shoots a club through $P_\kappa H_\lambda \cap IA$. As for a generalization of Reflection Principle, they prove that if cardinals between κ and a supercompact are collapsed to κ, then in the resulting model, every stationary set $S \subseteq P_\kappa H_\lambda \cap IA$ reflects at a set of size κ.

7. Stationary tower forcing

In this last section we give a brief description of the stationary tower forcing, introduced by Woodin in [93]. See also [69].

Let δ be an inaccessible cardinal. Let Q and P the following notions of forcing (the *stationary tower forcing*):

A forcing condition in Q is a pair (A, S) where $A \in V_\delta$ and S is a stationary subset of $[A]^{\aleph_0}$; $(A, S) < (B, T)$ if $A \supseteq B$ and $S \restriction B \subseteq T$.

A forcing condition in P is a pair (A, S) where $A \in V_\delta$ and S is a weakly stationary subset of $P_{|A|}A$; $(A, S) < (B, T)$ if $A \supseteq B$ and $S \restriction B \subseteq T$.

In fact, the stationary tower forcing is somewhat more general than these two examples, and uses the following generalization of stationary sets (considered e.g. in [20]). A set S is *stationary* in $P(A)$ if $S \subseteq P(A)$ and if for every $F : [A]^{<\omega} \to A$, S contains a closure point of F, i.e. a set $X \subseteq A$ such that $F(e) \in X$ for all $e \in [X]^{<\omega}$. As in Proposition 4.8, projections and liftings of stationary sets are stationary. Also, the analog of Theorem 4.4 holds. Note that the sets $S \restriction P_\kappa A$ are exactly the weakly stationary sets in $P_\kappa A$, and $S \restriction \{X \in P_\kappa \lambda : X \cap \kappa \in \kappa\}$ are the stationary sets in $P_\kappa \lambda$.

The general version of stationary tower forcing uses conditions (A, S) where S is stationary in $P(A)$.

If G is a generic filter on Q, then for each $A \in V_\delta$, the set $G_A = \{S : (A, S) \in G\}$ is a V-ultrafilter on $([A]^{\aleph_0})^V$; similarly for P. Moreover, if $A \subseteq B$, then G_B projects to G_A. In $V[G]$ we form a limit ultrapower $M = Ult_G(V)$ by the G_A, $A \in V_\delta$. The elements of M are represented by functions (in V) whose domain is some $A \in V_\delta$. Let $j : V \to M$ be the *generic embedding*, i.e. the elementary embedding from V into the limit ultrapower.

The ultrapower has a well founded initial segment up to at least δ: each ordinal $\alpha \leq \delta$ is represented by the function $f_\alpha(x) = x \cap \alpha$. The identity function $id(x) = x$ represents the set $j``V_\delta$. Woodin's main tool is the following:

7.1 Theorem (Woodin [93]). *Suppose δ is a Woodin cardinal. If G is a generic on either Q or P, then the generic ultrapower $Ult_G(V)$ is well founded, and the model M is closed under sequences of length $< \delta$.*

When forcing with Q, one has $crit(j) = \omega_1$ and $j(\omega_1) = \delta$. For applications, see [93].

Forcing with P gives more flexibility and yields various strong forcing results. We conclude this section with a typical application. Assume that $\aleph_\omega$ is strong limit. Let

$$S = \{X \in [V_{\aleph_{\omega+1}}]^{\aleph_\omega} : X \cap \aleph_{\omega+1} \in \aleph_{\omega+1} \text{ and } \mathrm{cf}\,(X \cap \aleph_{\omega+1}) = \aleph_3\}$$

and let G be a generic on P such that $S \in G$. Then $crit(j) = \aleph_{\omega+1}$ and $\mathrm{cf}\,^M\aleph_{\omega+1} = \aleph_3$. As $P^{V[G]}(\omega_n) = P^M(\omega_n) = P^V(\omega_n)$ for all n, we conclude that forcing with P (below $(V_{\aleph_{\omega+1}}, S)$) changes the cofinality of $\aleph_{\omega+1}$ to $\aleph_3$ while preserving $\aleph_\omega$.

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