

# FULL REFLECTION OF STATIONARY SETS AT REGULAR CARDINALS

THOMAS JECH AND SAHARON SHELAH

The Pennsylvania State University  
The Hebrew University

## 0. Introduction.

A stationary subset  $S$  of a regular uncountable cardinal  $\kappa$  *reflects fully at regular cardinals* if for every stationary set  $T \subseteq \kappa$  of higher order consisting of regular cardinals there exists an  $\alpha \in T$  such that  $S \cap \alpha$  is a stationary subset of  $\alpha$ . We prove that the Axiom of Full Reflection which states that every stationary set reflects fully at regular cardinals, together with the existence of  $n$ -Mahlo cardinals is equiconsistent with the existence of  $\Pi_n^1$ -indescribable cardinals. We also state the appropriate generalization for greatly Mahlo cardinals.

## 1. Results.

It has been proved [7], [3] that reflection of stationary sets is a large cardinal property. We address the question of what is the largest possible amount of reflection. Due to complications that arise at singular ordinals, we deal in this paper exclusively with reflection at regular cardinals. (And so we deal with stationary subsets of cardinals that are at least Mahlo cardinals.)<sup>1</sup>

If  $S$  is a stationary subset of a regular uncountable cardinal  $\kappa$ , then the *trace* of  $S$  is the set

$$Tr(S) = \{\alpha < \kappa : S \cap \alpha \text{ is stationary in } \alpha\}$$

(and we say that  $S$  *reflects at*  $\alpha$ ). If  $S$  and  $T$  are both stationary, we define

$$S < T \text{ if for almost all } \alpha \in T, \alpha \in Tr(S)$$

and say that  $S$  *reflects fully* in  $T$ . (Throughout the paper, “for almost all” means “except for a nonstationary set of points”). As proved in [4],  $<$  is a well founded relation; the *order*  $o(S)$  of a stationary set is the rank of  $S$  in this relation.

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1980 *Mathematics Subject Classification* (1985 *Revision*). 03E.

*Key words and phrases.* stationary sets, full reflection, Mahlo cardinals, indescribable cardinals, iterated forcing.

The first author was supported by NSF and by a Fulbright grant. He wishes to express his gratitude to the Hebrew University for its hospitality.

The second author wishes to thank the BSF. This paper is No. 383 on his publication list.

<sup>1</sup>If  $\kappa \geq \aleph_3$  then there exist stationary sets  $S \subseteq \{\alpha < \kappa : \text{cf } \alpha = \aleph_0\}$  and  $T \subseteq \{\beta < \kappa : \text{cf } \beta = \aleph_1\}$ , such that  $S$  does not reflect at any  $\beta \in T$ .

If the trace of  $S$  is stationary, then clearly  $o(S) < o(Tr(S))$ . We say that  $S$  *reflects fully at regular cardinals* if its trace meets every stationary set  $T$  of regular cardinals such that  $o(S) < o(T)$ . In other words, if for all stationary sets  $T$  of regular cardinals,

$$o(S) < o(T) \text{ implies } S < T.$$

**Axiom of Full Reflection for  $\kappa$ .** *Every stationary subset of  $\kappa$  reflects fully at regular cardinals.*

In this paper we investigate full reflection together with the existence of cardinals in the Mahlo hierarchy. Let  $Reg$  be the set of all regular limit cardinals  $\alpha < \kappa$ , and for each  $\eta < \kappa^+$  let

$$E_\eta = Tr^\eta(Reg) - Tr^{\eta+1}(Reg)$$

(cf. [2]), and call  $\kappa$   $\eta$ -Mahlo where  $\eta \leq \kappa^+$  is the least  $\eta$  such that  $E_\eta$  is nonstationary. In particular,

$E_0 =$  inaccessible non Mahlo cardinals

$E_1 =$  1-Mahlo cardinals, etc.

We also denote

$E_{-1} = Sing =$  the set of all singular ordinals  $\alpha < \kappa$ .

It is well known [4] that each  $E_\eta$ , the  $\eta$ th *canonical* stationary set is equal (up to the equivalence almost everywhere) to the set

$$\{\alpha < \kappa : \alpha \text{ is } f_\eta(\alpha)\text{-Mahlo}\}$$

where  $f_\eta$  is the *canonical*  $\eta$ th function. A  $\kappa^+$ -Mahlo cardinal  $\kappa$  is called *greatly Mahlo* [2].

If  $\kappa$  is less than greatly Mahlo (or if it is greatly Mahlo and the canonical stationary sets form a maximal antichain) then Full Reflection for  $\kappa$  is equivalent to the statement

*For every  $\eta \geq -1$ , every stationary  $S \subseteq E_\eta$  reflects almost everywhere in every  $E_\nu$ ,  $\nu > \eta$ .*

The simplest case of full reflection is when  $\kappa$  is 1-Mahlo; then full reflection states that every stationary  $S \subseteq Sing$  reflects at almost every  $\alpha \in E_0$ . We will show that this is equiconsistent with the existence of a weakly compact cardinal. More generally, we shall prove that full reflection together with the existence of  $n$ -Mahlo cardinals is equiconsistent with the existence of  $\Pi_n^1$ -indescribable cardinals.

To state the general theorem for cardinals higher up in the Mahlo hierarchy, we first give some definitions. We assume that the reader is familiar with  $\Pi_n^1$ -indescribability. A “formula” means a formula of second order logic for  $\langle V_\kappa, \in \rangle$ .

**Definition.** (a) A formula is  $\Pi_{\eta+1}^1$  if it is of the form  $\forall X \neg \varphi$  where  $\varphi$  is a  $\Pi_\eta^1$  formula.

(b) If  $\eta < \kappa^+$  is a limit ordinal, a formula is  $\Pi_\eta^1$  if it is of the form  $\exists \nu < \eta \varphi(\nu, \cdot)$  where  $\varphi(\nu, \cdot)$  is a  $\Pi_\nu^1$  formula.

For  $\alpha \leq \kappa$  and  $\eta < \kappa^+$  we define the satisfaction relation  $\langle V_\alpha, \in \rangle \models \varphi$  for  $\Pi_\eta^1$  formulas in the obvious way, the only difficulty arising for limit  $\eta$ , which is handled as follows:

$$\langle V_\alpha, \in \rangle \models \exists \nu < \eta \varphi(\nu, \cdot) \quad \text{if} \quad \exists \nu < f_\eta(\alpha) \langle V_\alpha, \in \rangle \models \varphi(\nu, \cdot)$$

where  $f_\eta$  is the  $\eta$ th canonical function.

**Definition.**  $\kappa$  is  $\Pi_\eta^1$ -*indescribable* ( $\eta < \kappa^+$ ) if for every  $\Pi_\eta^1$  formula  $\varphi$  and every  $Y \subseteq V_\kappa$ , if  $\langle V_\kappa, \in \rangle \models \varphi(Y)$  then there exists some  $\alpha < \kappa$  such that  $\langle V_\alpha, E \rangle \models \varphi(Y \cap V_\alpha)$ .  
 $\kappa$  is  $\Pi_{\kappa^+}^1$ -*indescribable* if it is  $\Pi_\eta^1$ -indescribable for all  $\eta < \kappa^+$ .

**Theorem A.** *Assuming the Axiom of Full Reflection for  $\kappa$ , we have for every  $\eta \leq (\kappa^+)^L$ : Every  $\eta$ -Mahlo cardinal is  $\Pi_\eta^1$ -indescribable in  $L$ .*

**Theorem B.** *Assume that the ground model satisfies  $V = L$ . There is a generic extension  $V[G]$  that preserves cardinals and cofinalities (and satisfies GCH) such that for every cardinal  $\kappa$  in  $V$  and every  $\eta \leq \kappa^+$ :*

- (1) (a) *If  $\kappa$  is  $\Pi_\eta^1$ -indescribable in  $V$  then  $\kappa$  is  $\eta$ -Mahlo in  $V[G]$ .*
- (2) (b)  *$V[G]$  satisfies the Axiom of Full Reflection.*

## 2. Proof of Theorem A.

Throughout this section we assume full reflection. The theorem is proved by induction on  $\kappa$ . We shall give the proof for the finite case of the Mahlo hierarchy; the general case requires only minor modifications.

Let  $F_0^\kappa$  denote the club filter on  $\kappa$  in  $L$ , and for  $n > 0$ , let  $F_n^\kappa$  denote the  $\Pi_n^1$  filter on  $\kappa$  in  $L$ , i.e. the filter on  $P(\kappa) \cap L$  generated by the sets  $\{\alpha < \kappa : L_\alpha \models \varphi\}$  where  $\varphi$  is a  $\Pi_n^1$  formula true in  $L_\kappa$ . If  $\kappa$  is  $\Pi_n^1$ -indescribable then  $F_n^\kappa$  is a proper filter. The  $\Pi_n^1$  ideal on  $\kappa$  is the dual of  $F_n^\kappa$ .

By induction on  $n$  we prove the following lemma which implies the theorem.

**Lemma 2.1.** *Let  $A \in L$  be a subset of  $\kappa$  that is in the  $\Pi_n^1$  ideal. Then  $A \cap E_{n-1}$  is nonstationary.*

To see that the Lemma implies Theorem A, let  $n \geq 1$ , and letting  $A = \kappa$ , we have the implication

$$\kappa \text{ is in the } \Pi_n^1 \text{ ideal in } L \Rightarrow E_{n-1} \text{ is nonstationary,}$$

and so

$$\kappa \text{ is not } \Pi_n^1\text{-indescribable in } L \Rightarrow \kappa \text{ is not } n\text{-Mahlo.}$$

*Proof.* The case  $n = 0$  is trivial (if  $A$  is nonstationary in  $L$  then  $A \cap \text{Sing}$  is nonstationary). Thus assume that the statement is true for  $n$ , for all  $\lambda \leq \kappa$ , and let us prove it for  $n + 1$  for  $\kappa$ . Let  $A$  be a subset of  $\kappa$ ,  $A \in L$ , and let  $\varphi$  be a  $\Pi_n^1$  formula such that for all  $\alpha \in A$  there is some  $X_\alpha \in L$ ,  $X_\alpha \subseteq \alpha$ , such that  $L_\alpha \models \varphi(X_\alpha)$ . Assuming that  $A \cap E_n$  is stationary, we shall find an  $X \in L$ ,  $X \subseteq \kappa$ , such that  $L_\kappa \models \varphi(X)$ . Let  $B \supseteq A$  be the set

$$B = \{\alpha < \kappa : \exists X \in L \ L_\alpha \models \varphi(X)\},$$

and for each  $\alpha \in B$  let  $X_\alpha$  be the least such  $X$  (in  $L$ ). For each  $\alpha \in B$ ,  $X_\alpha \in L_\beta$  where  $\beta < \alpha^+$ , and so let  $\beta$  be the least such  $\beta$ . Let  $Z_\alpha \in \{0, 1\}^\alpha \cap L$  be such that  $Z_\alpha$  codes  $\langle L_\beta, \in, X_\alpha \rangle$  (we include in  $Z_\alpha$  the elementary diagram of the structure  $\langle L_\beta, \in, X_\alpha \rangle$ ).

For every  $\lambda \in E_n \cap B$ , let

$$B_\lambda = \{\alpha < \lambda : \alpha \in B \text{ and } Z_\alpha = Z_\lambda|_\alpha\}.$$

We have

$$\begin{aligned} B_\lambda &\supseteq \{\alpha < \lambda : Z_\lambda|_\alpha \text{ codes } \langle L_\beta, \in, X \rangle \text{ where } \beta \text{ is the least } \beta \\ &\quad \text{and } X \text{ is the least } X \text{ such that } L_\alpha \models \varphi(X) \text{ and } X = X_\lambda \cap \alpha\} \\ &= \{\alpha < \lambda : L_\alpha \models \psi(Z_\lambda|_\alpha, X_\lambda \cap \alpha)\} \end{aligned}$$

where  $\psi$  is a  $\Pi_n^1 \wedge \Sigma_n^1$  statement, and hence  $B_\lambda$  belongs to the filter  $F_n^\lambda$ . By the induction hypothesis there is a club  $C_\lambda \subseteq \lambda$  such that  $B \cap E_{n-1} \supseteq B_\lambda \cap E_{n-1} \supseteq C_\lambda \cap E_{n-1}$ .

**Lemma 2.2.** *There is a club  $C \subseteq \kappa$  such that  $B \cap E_{n-1} \supseteq C \cap E_{n-1}$ .*

*Proof.* If not then  $E_{n-1} - B$  is stationary. This set reflects at almost all  $\lambda \in E_n$ , and since  $B \cap E_n$  is stationary, there is  $\lambda \in B \cap E_n$  such that  $(E_{n-1} - B) \cap \lambda$  is stationary in  $\lambda$ . But  $B \cap E_{n-1} \supseteq C_\lambda \cap E_{n-1}$ , a contradiction.  $\square$

**Definition 2.3.** For each  $t \in L \cap \{0, 1\}^{<\kappa}$ , let

$$S_t = \{\alpha \in E_{n-1} : t \subset Z_\alpha\}.$$

Since  $B \cap E_{n-1}$  is almost all of  $E_{n-1}$ , there is for each  $\gamma < \kappa$  some  $t \in \{0, 1\}^\gamma$  such that  $S_t$  is stationary.

**Lemma 2.4.** *If  $t, u \in \{0, 1\}^{<\kappa}$  are such that both  $S_t$  and  $S_u$  are stationary then  $t \subseteq u$  or  $u \subseteq t$ .*

*Proof.* Let  $\lambda \in B \cap E_n$  be such that both  $S_t \cap \lambda$  and  $S_u \cap \lambda$  are stationary in  $\lambda$ . Let  $\alpha, \beta \in C_\lambda$  be such that  $\alpha \in S_t$  and  $\beta \in S_u$ . Since we have  $t \subset Z_\alpha \subset Z_\lambda$  and  $u \subset Z_\beta \subset Z_\lambda$ , it follows that  $t \subseteq u$  or  $u \subseteq t$ .  $\square$

**Corollary 2.5.** *For each  $\gamma < \kappa$  there is  $t_\gamma \in \{0, 1\}^\gamma$  such that  $S_{t_\gamma}$  is almost all of  $E_{n-1}$ .*

**Corollary 2.6.** *There is a club  $D \subseteq \kappa$  such that for all  $\alpha \in D$ , if  $\alpha \in E_{n-1}$  then  $\alpha \in B$  and  $t_\alpha \subset Z_\alpha$ .*

*Proof.* Let  $D$  be the intersection of  $C$  with the diagonal intersection of the witnesses for the  $S_{t_\gamma}$ .  $\square$

**Definition.**  $Z = \bigcup \{t_\gamma : \gamma < \kappa\}$ .

**Lemma 2.7.** *For almost all  $\alpha \in E_{n-1}$ ,  $Z \cap \alpha = Z_\alpha$ .*

*Proof.* By Corollary 2.6, if  $\alpha \in D \cap E_{n-1}$  then  $Z_\alpha = t_\alpha$ .  $\square$

Now we can finish the proof of Lemma 2.1: The set  $Z$  codes a set  $X \subseteq \kappa$  and witnesses that  $X \in L$ . We claim that  $L_\kappa \models \varphi(X)$ . If not, then the set  $\{\alpha < \kappa : L_\alpha \models \neg\varphi(X \cap \alpha)\}$  is in the filter  $F_n^\kappa$  (because  $\neg\varphi$  is  $\Sigma_n^1$ ). By the induction hypothesis,  $L_\alpha \models \neg\varphi(X \cap \alpha)$  for almost all  $\alpha \in E_{n-1}$ . On the other hand, for almost all  $\alpha \in E_{n-1}$  we have  $L_\alpha \models \varphi(X_\alpha)$  and by Lemma 2.7, for almost all  $\alpha \in E_{n-1}$ ,  $X \cap \alpha = X_\alpha$ ; a contradiction.  $\square$

### 3. Proof of Theorem B: Cases 0 and 1.

The model is constructed by iterated forcing. (We refer to [5] for unexplained notation and terminology). Iterating with Easton support, we do a nontrivial construction only at stage  $\kappa$  where  $\kappa$  is a Mahlo cardinal.

Assume that we have constructed the forcing below  $\kappa$ , and denote it  $Q$ , and denote the model  $V(Q)$ ; if  $\lambda < \kappa$  then  $Q|\lambda$  is the forcing below  $\lambda$  and  $Q_\lambda \in V(Q|\lambda)$  is the forcing at  $\lambda$ . The rest of the proof will be to describe  $Q_\kappa$ . The forcing below  $\kappa$  has size  $\kappa$  and satisfies the  $\kappa$ -chain condition; the forcing at  $\kappa$  will be essentially  $< \kappa$ -closed (for every  $\lambda < \kappa$  has a  $\lambda$ -closed dense set) and will satisfy the  $\kappa^+$ -chain condition. Thus cardinals and cofinalities are preserved, and stationary subsets of  $\kappa$  can only be made nonstationary by forcing at  $\kappa$ , not below  $\kappa$  and not after stage  $\kappa$ ; after stage  $\kappa$  no subsets of  $\kappa$  are added.

By induction, we assume that Full Reflection holds in  $V(Q)$  for subsets of all  $\lambda < \kappa$ . We also assume this for every  $\lambda < \kappa$ :

- (a) If  $\lambda$  is inaccessible but not weakly compact in  $V$  then  $\lambda$  is non Mahlo in  $V[Q]$ .
- (b) If  $\lambda$  is  $\Pi_1^1$ -indescribable but not  $\Pi_2^1$ -indescribable in  $V$ , then  $\lambda$  is 1-Mahlo in  $V[Q]$ .
- (c) And so on accordingly.

Let  $E_0, E_1, E_2$ , etc. denote the subsets of  $\kappa$  consisting of all inaccessible non Mahlo, 1-Mahlo, 2-Mahlo etc. cardinals in  $V[Q]$ .

The forcing  $Q_\kappa$  will guarantee Full Reflection for subsets of  $\kappa$  and make  $\kappa$  into a cardinal of the appropriate Mahlo class, depending on its indescribability in  $V$ . (For instance, if  $\kappa$  is  $\Pi_2^1$ -indescribable but not  $\Pi_3^1$ -indescribable, it will be 2-Mahlo in  $V(Q * Q_\kappa)$ .)

The forcing  $Q_\kappa$  is an iteration of length  $\kappa^+$  with  $< \kappa$ -support of forcing notions that shoot a club through a given set. We recall ([1], [7], [6]) how one shoots a club through a single set, and how such forcing iterates: Given a set  $B \subseteq \kappa$ , the conditions for shooting a club through  $B$  are closed bounded sets  $p$  of ordinals such that  $p \subseteq B$ , ordered by end-extension. In our iteration, the  $B$  will always include the set  $\text{Sing}$  of all singular ordinals below  $\kappa$ , which guarantees that the forcing is essentially  $< \kappa$ -closed. One consequence of this is that at stage  $\alpha$  of the iteration, when shooting a club through (a name for) a set  $B \in V(Q * Q_\kappa|\alpha)$ , the conditions can be taken to be sets in  $V(Q)$  rather than (names for) sets in  $V(Q * Q_\kappa|\alpha)$ .

We use the standard device of iterated forcing: as  $Q_\kappa$  satisfies the  $\kappa^+$ -chain condition, it is possible to enumerate all names for subsets of  $\kappa$  such that the  $\beta$ th name belongs to  $V(Q * Q_\kappa|\beta)$ , and such that each name appears cofinally often in the enumeration. We call this a *canonical enumeration*.

We use the following two facts about the forcing:

**Lemma 3.1.** *If we shoot a club through  $B$ , then every stationary subset of  $B$  remains stationary.*

*Proof.* See [5], Lemma 7.38.  $\square$

**Lemma 3.2.** *If  $B$  contains a club, then shooting a club through  $B$  has a dense set that is a  $< \kappa$ -closed forcing (and so preserves all stationary sets).*

*Proof.* Let  $C \subseteq B$  be a club, and let  $D = \{p : \max(p) \in C\}$ .  $\square$

*Remark.* There is a unique forcing of size  $\kappa$  that is  $< \kappa$ -closed (and nontrivial), namely the one adding a Cohen subset of  $\kappa$ . We shall henceforth call every forcing that has such forcing as a dense subset *the Cohen forcing* for  $\kappa$ .

We shall describe the construction of  $Q_\kappa$  for the cases when  $\kappa$  is respectively inaccessible, weakly compact and  $\Pi_2^1$ -indescribable, and then outline the general case. Some details in the three low cases have to be handled separately from the general case.

**Case 0.**  $Q_\gamma$  for  $\gamma$  which is Mahlo but not weakly compact.

We assume that we have constructed  $Q|\gamma$ , and construct  $Q_\gamma$  in  $V(Q|\gamma)$ . To construct  $Q_\gamma$ , we first shoot a club through the set  $Sing$  and then do an iteration of length  $\gamma^+$  (with  $< \gamma$ -support), where at the stage  $\alpha$  we shoot a club through  $B_\alpha$  where  $\{B_\alpha : \alpha < \gamma^+\}$  is a canonical enumeration of all potential subsets of  $\gamma$  such that  $B_\alpha \supseteq Sing$ . As  $Sing$  contains a club,  $\gamma$  is in  $V(Q * P)$  non-Mahlo. As  $Q_\gamma$  is essentially  $< \kappa$ -closed,  $\kappa$  remains inaccessible.

In this case, Full Reflection for subsets of  $\gamma$  is (vacuously) true.

This completes the proof of Case 0. We shall now introduce some machinery that (as well as its generalization) we need later.

**Definition 3.3.** Let  $\gamma$  be an inaccessible cardinal. An *iteration of order 0* (for  $\gamma$ ) is an iteration of length  $< \gamma^+$  such that at each stage  $\alpha$  we shoot a club through some  $B_\alpha$  with the property that  $B_\alpha \supseteq Sing$ .

**Lemma 3.4.** (a) If  $P$  and  $R$  are iterations, and  $P$  is of order 0 then  $P \Vdash (R \text{ is of order } 0)$  if and only if  $R$  is of order 0.

(b) If  $\dot{R}$  is a  $P$ -name then  $P * \dot{R}$  is an iteration of order 0 if and only if  $P$  is an iteration of order 0 and  $P \Vdash (\dot{R} \text{ is an iteration of order } 0)$ .

(c) If  $A \subseteq Sing$  is stationary and  $P$  is an iteration of order 0 then  $P \Vdash A$  is stationary.

*Proof.* (a) and (b) are obvious, and (c) is proved as follows: Consider the forcing  $R$  that shoots a club through  $Sing$ .  $R$  is an iteration (of length 1) of order 0, and  $R * P \Vdash A$  is stationary, because  $R$  preserves  $A$  by Lemma 3.1, and forces that  $P$  is the iterated Cohen forcing (by Lemma 3.2). Since  $R$  commutes with  $P$ , we note that  $A$  is stationary in some extension of the forcing extension by  $P$ , and so  $P \Vdash A$  is stationary.  $\square$

We stated Lemma 3.4 in order to prepare ground for the (less trivial) generalization. We remark that “ $P$  is an iteration of order 0” is a first order property over  $V_\gamma$  (using a subset of  $V_\gamma$  to code the length of the iteration). The following lemma, that does not have an analog at higher cases, simplifies somewhat the handling of Case 1.

**Lemma 3.5.** If  $\gamma$  has a  $\Pi_1^1$  property  $\varphi$  and  $P$  is a  $< \gamma$ -closed forcing, then  $P \Vdash \varphi(\gamma)$ .

*Proof.* Let  $\varphi(\gamma) = \forall X \sigma(X)$ , where  $\sigma$  is a 1st order property. Toward a contradiction, let  $p_0 \in P$  and  $\dot{X}$  be such that  $p_0 \Vdash \neg \sigma(\dot{X})$ . Construct a descending  $\gamma$ -sequence of conditions  $p_0 \geq p_1 \geq \dots \geq p_\alpha \geq \dots$  and a continuous sequence  $\gamma_0 < \gamma_1 < \dots < \gamma_\alpha < \dots$  such that for each  $\alpha$ ,  $p_\alpha \Vdash \neg \sigma(\dot{X} \cap \gamma_\alpha)$ , and that  $p_\alpha$  decides  $\dot{X} \cap \gamma_\alpha$ ; say  $p_\alpha \Vdash \dot{X} \cap \gamma_\alpha = X_\alpha$ . Let  $X = \bigcup_{\alpha < \gamma} X_\alpha$ . There is a club  $C$  such that for all  $\alpha \in C$ ,  $\sigma(X \cap \alpha)$ . This is a contradiction since for some  $\alpha \in C$ ,  $\gamma_\alpha = \alpha$ .  $\square$

**Case 1.**  $\lambda$  is  $\Pi_1^1$ -indescribable but not  $\Pi_2^1$ -indescribable.

We assume that  $Q|\lambda$  has been defined, and we shall define an iteration  $Q_\lambda$  of length  $\lambda^+$ . The idea is to shoot clubs through the sets  $Sing \cup (Tr(S) \cap E_0)$ , for all stationary sets  $S \subseteq Sing$  (including those that appear at some stage of the iteration). Even though this approach would work in this case, we need to do more in order to assure that the construction will work at higher cases. For that reason we use a different approach.

At each stage of the iteration, we define a filter  $F_1$  on  $E_0$ , such that the filters all extend the  $\Pi_1^1$  filter on  $\lambda$  in  $V$ , that the filters get bigger as the iteration progresses, and that sets that are positive modulo  $F_1$  remain positive (and therefore stationary) at all later stages. The iteration consists of shooting clubs through sets  $B$  such that  $B \supseteq Sing$  and  $B \cap E_0 \in F_1$ , so that eventually every such  $B$  is taken care of. The crucial property of  $F_1$  is that whenever  $S$  is a stationary subset of  $Sing$ , then  $Tr(S) \cap E_0 \in F_1$ . Thus at the end of the iteration, every stationary subset of  $Sing$  reflects fully. Of course, we have to show that the filter  $F_1$  is nontrivial, that is that in  $V(Q|\lambda)$  the set  $E_0$  is positive mod  $F_1$ .

We now give the definition of the filter  $F_1$  on  $E_0$ . The definition is nonabsolute enough so that  $F_1$  will be different in each model  $V(Q|\lambda * Q_\lambda|\alpha)$  for different  $\alpha$ 's.

**Definition 3.6.** Let  $C_\lambda$  denote the forcing that shoots a club through  $Sing$ .

If  $\varphi$  is a  $\Pi_1^1$  formula and  $X \subseteq \lambda$ , let

$$B(\varphi, X) = \{\gamma \in E_0 : \varphi(\gamma, X \cap \gamma)\}$$

The filter  $F_1$  is generated by the sets  $B(\varphi, X)$  for those  $\varphi$  and  $X$  such that  $C_\lambda \Vdash \varphi(\lambda, X)$ . A set  $A \subseteq E_0$  is *positive* (or *1-positive*), if for every  $\Pi_1^1$  formula  $\varphi$  and every  $X \subseteq \lambda$ , if  $C_\lambda \Vdash \varphi(\lambda, X)$  then there exists a  $\gamma \in A$  such that  $\varphi(\gamma, X \cap \gamma)$ .

*Remarks.*

1. The filter  $F_1$  extends the club filter (which is generated by the sets  $B(\varphi, X)$  where  $\varphi$  is first-order). Hence every positive set is stationary.

2. The property “ $A$  is 1-positive” is  $\Pi_2^1$ .

**Lemma 3.7.** *In  $V(Q|\lambda)$ ,  $E_0$  is positive.*

*Proof.* We recall that in  $V$ ,  $\lambda$  is  $\Pi_1^1$ -indescribable, and  $E_0$  is the set of inaccessible, non-weakly-compact cardinals. Let  $Q = Q|\lambda$ . So let  $\varphi$  be a  $\Pi_1^1$  formula, let  $\dot{X}$  be a  $Q$ -name for a subset of  $\lambda$ , and assume that  $V(Q * C_\lambda) \models \varphi(\lambda, \dot{X})$ . The statement that  $Q * C_\lambda \Vdash \varphi(\lambda, \dot{X})$  is a  $\Pi_1^1$  statement (about  $Q$ ,  $C$  and  $\dot{X}$ ). By  $\Pi_1^1$ -indescribability, this reflects to some  $\gamma \in E_0$  (as  $E_0$  is positive in the  $\Pi_1^1$  filter). Since  $Q \cap V_\gamma = Q|\gamma$  and since  $Q|\gamma$  satisfies the  $\gamma$ -chain condition, the name  $\dot{X}$  reflects to the  $Q|\gamma$ -name for  $\dot{X} \cap \gamma$ . Also  $C_\lambda \cap V_\gamma = C_\gamma$ . Hence

$$Q|\gamma * C_\gamma \Vdash \varphi(\gamma, \dot{X} \cap \gamma).$$

What we want to show is that  $V(Q) \models \varphi(\gamma, \dot{X} \cap \gamma)$ . Since forcing above  $\gamma$  does not add subsets of  $\gamma$  it is enough to show that  $V(Q|\gamma * Q_\gamma) \models \varphi$ . However,  $C_\gamma$  was the first stage of  $Q_\gamma$  (see Case 0), and the rest of  $Q_\gamma$  is the iterated Cohen forcing for  $\gamma$ . By Lemma 3.5, if  $\varphi$  is true in  $V(Q|\gamma * C_\gamma)$ , then it is true in  $V(Q|\gamma * Q_\gamma)$ .  $\square$

**Lemma 3.8.** *If  $S \subseteq \text{Sing}$  is stationary, then the set  $\{\gamma \in E_0 : S \cap \gamma \text{ is stationary}\}$  is in  $F_1$ .*

*Proof.* The property  $\varphi(\lambda, S)$  which states that  $S$  is stationary is  $\Pi_1^1$ . If we show that  $C_\lambda \Vdash \varphi(\lambda, S)$ , then  $\{\gamma \in E_0 : \varphi(\gamma, S \cap \gamma)\}$  is in  $F_1$ . But forcing with  $C_\lambda$  preserves stationarity of  $S$ , by Lemma 3.1.  $\square$

**Definition 3.9.** An *iteration of order 1* (for  $\lambda$ ) is an iteration of length  $< \lambda^+$  such that at each stage  $\alpha$  we shoot a club through some  $B_\alpha$  such that  $B_\alpha \supseteq \text{Sing}$  and  $B_\alpha \cap E_0 \in F_1$ .

*Remark.* If we include the witnesses for  $B_\alpha \cap E_0 \in F_1$  as parameters in the definition, i.e.  $\varphi_\alpha, X_\alpha$  such that  $C_\lambda \Vdash \varphi_\alpha(\lambda, X_\alpha)$  and  $B_\alpha \cap E_0 \supseteq \{\gamma \in E_0 : \varphi(\gamma, X_\alpha \cap \gamma)\}$ , then the property “ $P$  is an iteration of order 1” is  $\Pi_1^1$ .

We shall now give the definition of  $Q_\lambda$ :

**Definition 3.10.**  $Q_\lambda$  is (in  $V(Q(\lambda))$ ) an iteration of length  $\lambda^+$ , such that for each  $\alpha < \lambda^+$ ,  $Q_\lambda \upharpoonright \alpha$  is an iteration of order 1, and such that each potential  $B$  is used as  $B_\beta$  at cofinally many stages  $\beta$ .

We will now show that both “ $B \in F_1$ ” and “ $A$  is positive” are preserved under iterations of order 1:

**Lemma 3.11.** *If  $B \in F_1$  and  $P$  is an iteration of order 1 then  $P \Vdash B \in F_1$ . Moreover, if  $(\varphi, X)$  is a witness for  $B \in F_1$ , then it remains a witness after forcing with  $P$ .*

*Proof.* Let  $B \supseteq B(\varphi, X)$  where  $\varphi$  is  $\Pi_1^1$  and  $C_\lambda \Vdash \varphi(\lambda, X)$ , and let  $P$  be an iteration of order 1. As  $P$  does not add bounded subsets,  $B(\varphi, X)$  remains the same, and so we have to verify that  $P \Vdash (C_\lambda \Vdash \varphi)$ . However,  $C_\lambda$  commutes with  $P$ , and moreover,  $C_\lambda$  forces that  $P$  is the Cohen forcing (because after  $C_\lambda$ ,  $P$  shoots clubs through sets that contain a club, see Lemma 3.2). By Lemma 3.5,  $C_\lambda \Vdash \varphi$  implies that  $C_\lambda \Vdash (P \Vdash \varphi)$ .  $\square$

**Lemma 3.12.** *If  $A \subseteq E_0$  is positive and  $P$  is an iteration of order 1 then  $P \Vdash A$  is positive.*

We postpone the proof of this crucial lemma for a while. We remark that the assumption under which Lemma 3.12 will be proved is that the model in which we are working contains  $V(Q|\lambda)$ ; this assumption will be satisfied in the future when the Lemma is applied.

**Lemma 3.13.** (a) *If  $P$  and  $R$  are iterations, and  $P$  is of order 1 then  $P \Vdash (R \text{ is of order 1})$  if and only if  $R$  is of order 1.*

(b) *If  $\dot{R}$  is a  $P$ -name then  $P * \dot{R}$  is an iteration of order 1 if and only if  $P$  is an iteration of order 1 and  $P \Vdash (\dot{R} \text{ is an iteration of order 1})$ .*

(c) *Every iteration of order 1 is an iteration of order 0.*

*Proof.* Both (a) and (b) are consequences of Lemma 3.12. The decision whether a particular stage of the iteration  $R$  satisfies the definition of being of order 1 depends only on whether  $B_\alpha \in F_1$ , which does not depend on  $P$ .

(c) is trivial.  $\square$

**Corollary 3.14.** *In  $V(Q|\lambda * Q_\lambda)$ ,  $E_0$  is stationary (so  $\lambda$  is 1-Mahlo), and every stationary  $S \subseteq \text{Sing}$  reflects fully in  $E_0$ .*

*Proof.* Suppose that  $E_0$  is not stationary. Then it is disjoint from some club  $C$ , which appears at some stage  $\alpha < \lambda^+$  of the iteration  $Q_\lambda$ . So  $E_0$  is nonstationary in  $V(Q|\lambda * Q_\lambda|(\alpha + 1))$ . This is a contradiction, since  $E_0$  is positive in that model, by Lemmas 3.7 and 3.12.

If  $S$  is a stationary subset of  $\text{Sing}$ , then  $S \in V(Q|\lambda * Q_\lambda|\alpha)$  for some  $\alpha$  and so by Lemma 3.8,  $B = \text{Tr}(S) \cap E_0 \in F_1$  (in that model). Hence  $B$  remains in  $F_1$  at all later stages, and eventually,  $B = B_\alpha$  is used at stage  $\alpha$ , that is we produce a club  $C$  so that  $B \supseteq C \cap E_0$ . Since  $Q_\lambda$  adds no bounded subsets of  $\lambda$ , the trace of  $S$  remains the same, and so  $S$  reflects fully in  $V(Q|\lambda * Q_\lambda)$ .  $\square$

**Proof of Lemma 3.12.**

Let  $\varphi$  be a  $\Pi_1^1$  property, and let  $\dot{X}$  be a  $P$ -name for a subset of  $\lambda$ . Let  $p \in P$  be a condition that forces that  $C_\lambda \Vdash \varphi(\lambda, \dot{X})$ . We are going to find a stronger  $q \in P$  and a  $\gamma \in A$  such that  $q$  forces  $\varphi(\gamma, \dot{X} \cap \gamma)$ .

$P$  is an iteration of order 1, of length  $\alpha$ . At stage  $\beta$  of the iteration, we have  $P|\beta$ -names  $\dot{B}_\beta$ ,  $\varphi_\beta$  and  $\dot{X}_\beta$  for a set  $\supseteq \text{Sing}$ , a  $\Pi_1^1$  formula, and a subset of  $\lambda$  such that  $P|\beta$  forces that  $C_\lambda \Vdash \varphi_\beta(\lambda, \dot{X}_\beta)$  and that  $\dot{B}_\beta \supseteq \{\gamma \in E_0 : \varphi_\beta(\gamma, \dot{X}_\beta \cap \gamma)\}$ , and we shoot a club through  $\dot{B}_\beta$ .

Let  $\psi$  be the following statement (about  $V_\lambda$  and a relation on  $V_\lambda$  that codes a model of size  $\lambda$  including the relevant parameters and satisfying enough axioms of ZFC; the relation will also insure that the model  $M$  below has the properties that we list):

*$P$  is an iteration of length  $\alpha$ , at each stage shooting a club through  $\dot{B}_\beta \supseteq \text{Sing}$ , and  $p \Vdash \varphi(\lambda, \dot{X})$  and for every  $\beta < \alpha$ ,  $P|\beta \Vdash \varphi_\beta(\lambda, \dot{X}_\beta)$ .*

First we note that  $\psi$  is a  $\Pi_1^1$  property. Secondly, we claim that  $C_\lambda \Vdash \psi$ : In the forcing extension by  $C_\lambda$ ,  $P$  is still an iteration etc., and  $p \Vdash \varphi$  and  $P|\beta \Vdash \varphi_\beta$  because in the ground model,  $p \Vdash (C_\lambda \Vdash \varphi)$  and  $P|\beta \Vdash (C_\lambda \Vdash \varphi_\beta)$ , and  $C_\lambda$  commutes with  $P$ .

Thus, since  $A$  is positive in the ground model, there exists some  $\gamma \in A$  such that  $\psi(\gamma, \text{parameters} \cap V_\gamma)$ . This gives us a model  $M$  of size  $\gamma$ , and its transitive collapse  $N = \pi(M)$ , with the following properties:

- (a)  $M \cap \lambda = \gamma$ ,
- (b)  $P, p, \dot{X} \in M$  and  $M \models P$  is an iteration given by  $\{\dot{B}_\beta : \beta < \alpha\}$ ,
- (c)  $p \Vdash \varphi(\gamma, \pi(\dot{X}))$  (the forcing  $\Vdash$  is in  $\pi(P)$ ),
- (d)  $\forall \beta < \alpha$ , if  $\beta \in M$ , then  $\pi(P|\beta) \Vdash \varphi_\beta(\gamma, \pi(\dot{X}_\beta))$ .

It follows that  $\pi(P)$  is an iteration on  $\gamma$  (or order 0), of length  $\pi(\alpha)$ , that at stage  $\pi(\beta)$  shoots a club through  $\pi(\dot{B}_\beta)$ . Also,  $p \Vdash \pi(\dot{X}) = \dot{X} \cap \gamma$  (forcing in  $P$ ).

**Sublemma 3.12.1.** *There exists an  $N$ -generic filter  $G \ni p$  on  $\pi(P)$  such that if  $X \subseteq \gamma$  denotes the  $G$ -interpretation  $\pi(\dot{X})/G$  of  $\pi(\dot{X})$ , and for each  $\beta \in M$ ,  $X_\beta = \pi(\dot{X}_\beta)/G$ , then  $\varphi(\gamma, X)$  and  $\varphi_\beta(\gamma, X_\beta)$  hold.*

*Proof.* We assume that  $V(Q|\lambda)$  is a part of our universe, and that no subsets of  $\gamma$  have been added after  $Q_\gamma$ . So it suffices to find  $G$  in  $V(Q|\gamma * Q_\gamma)$ . Note also that  $E_0 \cap \gamma$  is

nonstationary (as  $\gamma$  was made non Mahlo by  $Q_\gamma$ ). Since  $\pi(P)$  is an iteration of order 0, since  $Sing$  contains a club, and because  $\pi(P)$  has size  $\gamma$ , it is the Cohen forcing for  $\gamma$ , and therefore isomorphic to the forcing at each stage of the iteration  $Q_\gamma$  except the first one (which is  $C_\gamma$ ).

There is  $\eta < \gamma^+$  such that  $V(Q|\gamma * Q_\gamma|\eta)$  contains  $\pi(P), \pi(\dot{X})$ , all members of  $N$ , and all  $\pi(\dot{X}_\beta), \beta \in M$ . Also, the statements  $p \Vdash \varphi(\gamma, \pi(\dot{X}))$  and  $\pi(P|\beta) \Vdash \varphi_\beta(\gamma, \pi(\dot{X}_\beta))$ , being  $\Pi_1^1$  and true, are true in  $V(Q|\gamma * Q_\gamma|\eta)$ . As  $\pi(P)$  (below  $p$ ) as well as the  $\pi(P|\beta)$  are isomorphic to the  $\eta^{th}$  stage  $Q_\gamma(\eta)$  of  $Q_\gamma$ , and we do have a generic filter for  $Q_\gamma(\eta)$  over  $V(Q|\gamma * Q_\gamma|\eta)$ , we have a  $G$  that is  $N$ -generic for  $\pi(P)$  and  $\pi(P|\beta)$ . If we let  $X = \pi(\dot{X})/G$  and  $X_\beta = \pi(\dot{X}_\beta)/G$ , then in  $V(Q|\gamma * Q_\gamma|(\eta+1))$  we have  $\varphi(\gamma, X)$  and  $\varphi_\beta(\gamma, X_\beta)$ . Since the rest of the iteration  $Q_\gamma$  is the iterated Cohen forcing, we use Lemma 3.5 again to conclude that  $\varphi(\gamma, X)$  and  $\varphi_\beta(\gamma, X_\beta)$  are true in  $V(Q|\gamma * Q_\gamma)$ , hence are true.  $\square$

Now let  $H = \pi^{-1}(G)$  and for every  $\beta \in M$  let  $B_\beta = \pi(\dot{B}_\beta)/G$ . By induction on  $\beta \in M$ , we construct a condition  $q \leq p$  (with support  $\subseteq M$ ) as follows: For each  $\xi \in M$ , let  $q(\xi) = H_\xi \cup \{\lambda\}$ . This is a closed set of ordinals. At stage  $\beta$ ,  $q|\beta$  a condition by the induction hypothesis, and  $q|\beta \supseteq H|\beta$  (consequently,  $q|\beta$  forces  $\dot{X}_\beta \cap \gamma = X_\beta$  and  $\dot{B}_\beta \cap \gamma = B_\beta$ ).  $H_\beta$  is a closed set of ordinals, cofinal in  $\gamma$ , and  $H_\beta \subseteq B_\beta$ . We let  $q(\beta) = H_\beta \cup \{\gamma\}$ . In order that  $q|(\beta+1)$  is a condition it is necessary that  $q|\beta \Vdash \gamma \in \dot{B}_\beta$ . But by Sublemma 3.12.1 we have  $\varphi_\beta(\gamma, X_\beta)$ , so this is forced by  $P$  (which does not add subsets of  $\gamma$ ), and since  $q|\beta \Vdash X_\beta = \dot{X}_\beta \cap \gamma$ , we have  $q|\beta \Vdash \varphi_\beta(\gamma, \dot{X}_\beta \cap \gamma)$ . But this implies that  $q|\beta \Vdash \gamma \in \dot{B}_\beta$ . Hence  $q|(\beta+1)$  is a condition, which extends  $H|(\beta+1)$ .

Therefore  $q$  is a condition, and since  $q \supseteq H$ , we have  $q \Vdash \dot{X} \cap \gamma = X$ . But  $\varphi(\gamma, X)$  holds by Sublemma 3.12.1., so it is forced by  $q$ , and so  $q \Vdash \varphi(\gamma, \dot{X} \cap \gamma)$ , as required.  $\square$

#### 4. Case 2 and up.

Let  $\kappa$  be  $\Pi_2^1$ -indescribable but not  $\Pi_3^1$ -indescribable. Below  $\kappa$ , we have four different types of limit cardinals in  $V$ :

- $Sing$  = the singular cardinals
- $E_0$  = inaccessible not weakly compact
- $E_1 = \Pi_1^1$ - but not  $\Pi_2^1$ -indescribable
- the rest =  $\Pi_2^1$  indescribable

We shall prove a sequence of lemmas (and give a sequence of definitions), analogous to 3.6–3.14. Whenever possible, we use the same argument; however, there are some changes and additional complications.

**Definition 4.1.** A  $\Pi_2^1$  formula  $\varphi$  is *absolute* for  $\lambda \in E_1$  if for every  $\alpha < \lambda^+$  and every  $X \in V(Q|\lambda * Q_\lambda|\alpha)$ ,

- (1)  $V(Q|\lambda * Q_\lambda|\alpha) \models$  (for every iteration  $R$  of order 1,  $\varphi(\lambda, X)$  iff  $R \Vdash \varphi(\lambda, X)$ ),
- (2)  $V(Q|\lambda * Q_\lambda|\alpha) \models \varphi(\lambda, X)$  implies  $V(Q|\lambda * Q_\lambda) \models \varphi(\lambda, X)$ , and
- (3)  $V(Q|\lambda * Q_\lambda|\alpha) \models \neg\varphi(\lambda, X)$  implies  $V(Q|\lambda * Q_\lambda) \models \neg\varphi(\lambda, X)$ .

We say that  $\varphi$  is *absolute* if it is absolute for all  $\lambda \in E_1$ ,  $\lambda < \kappa$ .

**Definition 4.2.** If  $\varphi$  is a  $\Pi_2^1$  formula and  $X \subseteq \kappa$ , let

$$B(\varphi, X) = \{\lambda \in E_1 : \varphi(\lambda, X \cap \lambda)\}.$$

The filter  $F_2$  is generated by the sets  $B(\varphi, X)$  where  $\varphi$  is an absolute  $\Pi_2^1$  formula and  $X$  is such that  $R \Vdash \varphi(\kappa, X)$ , for all iterations  $R$  of order 1.

A set  $A \subseteq E_1$  is *positive* (2-positive) if for any absolute  $\Pi_2^1$  formula  $\varphi$  and every  $X \subseteq \kappa$ , if every iteration  $R$  of order 1 forces  $\varphi(\kappa, X)$ , then there exists a  $\lambda \in A$  such that  $\varphi(\lambda, X \cap \lambda)$ .

*Remark.* The property “ $A$  is 2-positive” is  $\Pi_3^1$ .

**Lemma 4.3.** *In  $V(Q|\kappa)$ ,  $E_1$  is positive.*

*Proof.* Let  $Q = Q|\kappa$ . Let  $\varphi$  be an absolute  $\Pi_2^1$  formula, and let  $\dot{X}$  be a  $Q$ -name for a subset of  $\kappa$ , and assume that in  $V(Q)$ ,  $R \Vdash \varphi(\kappa, \dot{X})$  for all order-1 iterations  $R$ . In particular, (taking  $R$  the empty iteration),  $V(Q) \models \varphi(\kappa, \dot{X})$ .

Using the  $\Pi_2^1$ -indescribability of  $\kappa$  in  $V$ , there exists a  $\lambda \in E_1$  such that  $V(Q|\lambda) \models \varphi(\lambda, \dot{X} \cap \lambda)$ . In order to prove that  $V(Q) \models \varphi(\lambda, \dot{X} \cap \lambda)$ , it is enough to show that  $V(Q|\lambda * Q_\lambda) \models \varphi(\lambda, \dot{X} \cap \lambda)$ . This however is true because  $\varphi$  is absolute for  $\lambda$ .  $\square$

**Lemma 4.4.** *The property “ $S$  is 1-positive” of a set  $S \subseteq E_0$  is an absolute  $\Pi_2^1$  property, and is preserved under forcing with iterations of order 1.*

*Proof.* The preservation of “1-positive” under iterations of order 1 was proved in Lemma 3.12. To show that the property is absolute for all  $\lambda \in E_1$ , first assume that  $S \in V(Q|\lambda * Q_\lambda|\alpha)$  is 1-positive. Since all longer initial segments of the iteration  $Q_\lambda$  are iterations of order 1, hence order 1 iterations over  $Q_\lambda|\alpha$  (by Lemma 3.13),  $S$  is 1-positive in each  $V(Q|\lambda * Q_\lambda|\beta)$ ,  $\beta > \alpha$ . However, the property “ $S$  is 1-positive” is  $\Pi_2^1$ , and so it also holds in  $V(Q|\lambda * Q_\lambda)$ , because every subset of  $\lambda$  in that model appears at some stage  $\beta$ . (We remark that this argument, using  $\Pi_2^1$ , does not work in higher cases).

Conversely, assume that  $S$  is not 1-positive in  $V(Q|\lambda * Q_\lambda|\alpha)$ . There exists a  $\Pi_1^1$  formula  $\varphi$  and some  $X \subseteq \lambda$  such that  $\varphi(\gamma, X \cap \gamma)$  fails for all  $\gamma \in S$ , while  $C_\lambda \Vdash \varphi(\lambda, X)$ . The rest of the argument is the same as the one in Lemma 3.11: Let  $P = Q_\lambda/(Q_\lambda|\alpha)$ ;  $C_\lambda$  commutes with  $P$  and forces that  $P$  is the iterated Cohen forcing. Hence by Lemma 3.5,  $P \Vdash (C_\lambda \Vdash \varphi)$ , i.e.  $V(Q|\lambda * Q_\lambda) \models (C_\lambda \Vdash \varphi)$ . Therefore  $S$  is not 1-positive in  $V(Q|\lambda * Q_\lambda)$ . (Again, this argument does not work in higher cases).  $\square$

**Lemma 4.5.** *The property “ $R$  is an iteration of order 1” is an absolute  $\Pi_2^1$  property, and is preserved under forcing with iterations of order 1. Moreover, in  $V(Q|\lambda * Q_\lambda)$ , if  $R$  is an iteration of order 1, then  $R$  is the Cohen forcing.*

*Proof.* The preservation of the property under iterations of order 1 was proved in Lemma 3.13. If  $R$  is an iteration of order 1 in  $V(Q|\lambda * Q_\lambda|\alpha)$ , shooting clubs through  $\dot{A}_0, \dot{A}_1, \dot{A}_2$ , etc., then  $R$  embeds in  $Q_\lambda$  above  $\alpha$  as a subiteration, i.e. there are  $\beta_0, \beta_1$ , etc. such that  $\dot{A}_0 = B_{\beta_0}, \dot{A}_1 = B_{\beta_1}$ , etc. Moreover, there is some  $\gamma > \alpha$  such that the  $A_0, A_1, A_2$ , etc. all contain a club. Hence  $R$  is the Cohen forcing in  $V(Q|\lambda * Q_\lambda|\gamma)$ . Therefore  $R$  is the Cohen forcing in  $V(Q|\lambda * Q_\lambda)$ , and consequently an iteration of order 1. As for the absoluteness

downward, we give the proof for iterations of length 2. Let  $M_\infty = V(Q|\lambda * Q_\lambda)$ , let  $R = (R_0, R_1)$  be an iteration given by  $A_0$  and  $\dot{A}_1 \in M_\infty(R_0)$ , such that in  $M_\infty$ ,  $A_0 \in F_1$  and  $R_0 \Vdash \dot{A}_1 \in F_1$ . Let  $R \in M_\alpha = V(Q|\lambda * Q_\lambda|\alpha)$ . We will show that in  $M_\alpha$ ,  $R$  is an iteration of order 1, and that in  $M_\infty$ ,  $R$  is the Cohen forcing.

First, since  $A_0 \in F_1$  is absolute, there is a  $\beta > \alpha$  such that  $M_\beta \Vdash A_0$  contains a club and such that  $A_0 = B_\beta$  ( $B_\beta$  is the set used at stage  $\beta$  of the iteration  $Q_\lambda$ ). Since  $M_\beta \Vdash (R_0 \text{ is Cohen})$ , we have  $M_\infty \Vdash R_0 \text{ is Cohen}$ .

Now, in  $M_\infty$  we have  $R_0 \Vdash \dot{A}_1 \in F_1$ . We claim that in  $M_\beta$ ,  $R_0 \Vdash \dot{A}_1 \in F_1$ . Then it follows that  $R$  is an iteration of order 1 in  $M_\beta$ .

It remains to prove the claim. Let  $\dot{X}$  denote  $\dot{A}_1$ , let  $\varphi(\dot{X})$  denote the absolute  $\Pi_2^1$  property  $\dot{A}_1 \in F_1$  and let  $C$  denote the Cohen forcing. We recall that  $M_{\beta+1} = M_\beta(C)$ .

**Sublemma 4.5.1.** *Let  $\dot{X}$  be a  $C$ -name in  $M_\beta$ , and assume that  $M_{\beta+1} = M_\beta(C)$ . If  $C \Vdash \varphi(\dot{X})$  in  $M_\infty$ , then  $C \Vdash \varphi(\dot{X})$  in  $M_\beta$ .*

*Proof.* Let  $P$  be the forcing such that  $M_\infty = M_{\beta+1}(P)$ , and assume, toward a contradiction, that  $C \Vdash \varphi(\dot{X})$  in  $M_\infty$  but  $C \Vdash \neg\varphi(\dot{X})$  in  $M_\beta$ . Let  $G_C \times G_P \times H$  be a generic on  $C * P * C$ , and let  $X = \dot{X}/H$ . Let  $C = C_1 \times C_2$  where both  $C_1$  and  $C_2$  are Cohen, and consider the generic  $H \times G_C \times G_P$  on  $C_1 \times C_2 \times P = C \times P$  (it is a generic because since  $H$  is generic over  $G_C \times G_P$ ,  $G_C \times G_P$  is generic over  $H$ ).

In  $M_\beta$ ,  $C_1$  forces  $\varphi$  false, hence  $\varphi(X)$  is false in  $M_\beta[H]$ . Since  $\neg\varphi$  is preserved by Cohen forcing (in fact by all order-1 iterations), so  $\varphi(X)$  is false in  $M_\beta[H \times G_C]$ . Now  $\varphi$  is absolute (between  $M_{\beta+1}$  and  $M_\infty$ ) and so  $\varphi(X)$  is false in  $M_\beta[H \times G_C \times G_P]$ . On the other hand, since  $C \Vdash \varphi(\dot{X})$  in  $M$ , we have  $M_\beta[G_C \times G_P \times H] \Vdash \varphi(\dot{X}/H)$ , so  $\varphi$  is true in  $M_\beta[G_C \times G_P \times H]$ , a contradiction.  $\square$

**Lemma 4.6.** *If  $S \subseteq E_0$  is 1-positive, then the set*

$$\{\lambda \in E_1 : S \cap \lambda \text{ is 1-positive}\}$$

*is in  $F_2$ . Therefore  $Tr(S) \cap E_1 \in F_2$ .*

*Proof.* The first sentence follows from the definition of  $F_2$  because “ $S$  is 1-positive” is absolute  $\Pi_2^1$  and if  $S$  is positive then it is positive after every order 1 iteration. The second sentence follows, since 1-positive subsets of  $\lambda$  are stationary.  $\square$

**Definition 4.7.** An *iteration of order 2* (for  $\kappa$ ) is an iteration of length  $< \kappa^+$  that at each stage  $\alpha$  shoots a club through some  $B_\alpha$  such that  $B_\alpha \supseteq Sing$ ,  $B_\alpha \cap E_0 \in F_1$ , and  $B_\alpha \cap E_1 \in F_2$ .

*Remarks.* 1. An iteration of order 2 is an iteration of order 1.

2. If we include the witnesses for  $B_\alpha$  to be in the filters, then the property “ $P$  is an iteration of order 2” is  $\Pi_3^1$ .

**Definition 4.8.**  $Q_\kappa$  is (in  $V(Q|\kappa)$ ) an iteration of length  $\kappa^+$ , such that for each  $\alpha < \kappa^+$ ,  $Q_\kappa|\alpha$  is an iteration of order 2, and such that each potential  $B$  is used as  $B_\beta$  at cofinally many stages  $\beta$ .

**Lemma 4.9.** *If  $B \in F_2$  and  $P$  is an iteration of order 2 then  $P \Vdash B \in F_2$  (and a witness  $(\varphi, X)$  remains a witness).*

*Proof.* Let  $B \supseteq B(\varphi, X)$  where  $\varphi$  is an absolute  $\Pi_2^1$ , and every iteration of order 1 forces  $\varphi$ ; let  $P$  be an iteration of order 2. Since  $P$  does not add subset of  $\kappa$ ,  $B(\varphi, X)$  remains the same and  $\varphi$  remains absolute. Thus it suffices to verify that for each  $P$ -name  $\dot{R}$  for an order 1 iteration,  $P \Vdash (\dot{R} \Vdash \varphi)$ . However,  $P$  is an iteration of order 1, so by Lemma 3.13,  $P * \dot{R}$  is an iteration of order 1, and by the assumption on  $\varphi$ ,  $P * \dot{R} \Vdash \varphi$ .  $\square$

**Lemma 4.10.** *If  $A \subseteq E_1$  is 2-positive and  $P$  is an iteration of order 2 then  $P \Vdash A$  is 2-positive.*

*Proof.* Let  $\varphi$  be an absolute  $\Pi_2^1$  property, let  $\dot{X}$  be a  $P$ -name for a subset of  $\kappa$  and let  $p \in P$  force that for all order-1-iterations  $R$ ,  $R \Vdash \varphi(\kappa, \dot{X})$ . We want a  $q \leq p$  and a  $\lambda \in A$  such that  $q \Vdash \varphi(\lambda, \dot{X} \cap \lambda)$ .

$P$  is an iteration of order 2 that at each stage  $\beta$  (less than the length of  $P$ ) shoots a club through a set  $\dot{B}_\beta$  such that  $P \upharpoonright \beta$  forces that

- (1)  $\dot{B}_\beta \supseteq \text{Sing}$ ,
- (2)  $\dot{B}_\beta \cap E_0 \supseteq \{\gamma \in E_0 : \varphi_\beta^1(\gamma, \dot{X}_\beta \cap \gamma)\}$ , and
- (3)  $\dot{B}_\beta \cap E_1 \supseteq \{\lambda \in E_1 : \varphi_\beta^2(\lambda, \dot{Y}_\beta \cap \lambda)\}$ ,

where  $\dot{X}_\beta$  and  $\dot{Y}_\beta$  are names for subsets of  $\kappa$ , the  $\varphi_\beta^1$  are  $\Pi_1^1$  formulas (with some extra property that make  $P$  an order-1 iteration) and the  $\varphi_\beta^2$  are absolute (in  $V(Q \upharpoonright \kappa)$ )  $\Pi_2^1$  properties, and  $P \upharpoonright \beta$  forces that  $\forall R$  (if  $R$  is an iteration of order 1 then  $R \Vdash \varphi_\beta^2(\kappa, \dot{Y}_\beta)$ ).

We shall reflect, to some  $\lambda \in E_1$ , the  $\Pi_2^1$  statement  $\psi$  that states (in addition to a first order statement in some parameter that produces the model  $M$  below):

- (a)  $P$  is an iteration of order 1 using the  $\varphi_\beta^1, \dot{X}_\beta, \varphi_\beta^2, \dot{Y}_\beta$ ,
- (b)  $p \Vdash \varphi(\kappa, \dot{X})$ ,
- (c) for every  $\beta < \text{length}(P)$ ,  $P \upharpoonright \beta \Vdash \varphi_\beta^2(\kappa, \dot{Y}_\beta)$ .

First we note that  $\psi$  is a  $\Pi_2^1$  property. Secondly, we claim that  $\psi$  is absolute for every  $\lambda \in E_1$ . Being an iteration of order 1 is absolute by Lemma 4.5. That (b) and (c) are absolute will follow once we show that if  $\varphi$  is an absolute  $\Pi_2^1$  property and  $R$  an iteration of order 1, then “ $R \Vdash \varphi$ ” is absolute:

**Sublemma 4.10.1.** *Let  $\varphi$  be absolute for  $\lambda$ , let  $\alpha < \lambda^+$ ,  $X, R \in V(Q \upharpoonright \lambda * Q \upharpoonright \alpha)$  be a subset of  $\lambda$  and an iteration of order 1. Then the property  $R \Vdash \varphi(\lambda, X)$  is absolute between  $M_\alpha = V(Q \upharpoonright \lambda * Q \upharpoonright \alpha)$  and  $M_\infty = V(Q \upharpoonright \lambda * Q \upharpoonright \lambda)$ .*

*Proof.* Let  $M_\alpha \models (R \Vdash \varphi(\lambda, X))$ . Then  $M_\alpha \models \varphi(\lambda, X)$  and by absoluteness,  $M_\infty \models \varphi(\lambda, X)$ . If in  $M_\infty$ ,  $R \Vdash \neg \varphi(\lambda, X)$ , then because  $R$  is in  $M_\infty$  the Cohen forcing, there is (by Sublemma 4.5.1) some  $\beta > \alpha$  such that  $R$  is the Cohen forcing in  $M_\beta$  and  $M_\beta \models (R \Vdash \neg \varphi)$ . By absoluteness again,  $M_\beta \models \neg \varphi$ , a contradiction.  $\square$

Thus  $\psi$  is an absolute  $\Pi_2^1$  property. Next we show that if  $R$  is an iteration of order 1 then  $R$  forces  $\psi(\kappa, \text{parameters})$ :

- (a)  $R \Vdash (P \text{ is an iteration of order 1})$ , by Lemma 3.13.

(b)  $R$  commutes with  $P$ , and by the assumption of the proof,  $p \Vdash (R \Vdash \varphi(\kappa, \dot{X}))$ . Hence  $R \Vdash (p \Vdash \varphi(\kappa, \dot{X}))$ .

(c) For every  $\beta$ ,  $R$  commutes with  $P \restriction \beta$ , and by the assumption on  $\varphi_\beta^2$ ,  $P \restriction \beta \Vdash (R \Vdash \varphi_\beta^2(\kappa, \dot{Y}_\beta))$ . Hence  $R \Vdash (P \restriction \beta \Vdash \varphi_\beta^2(\kappa, \dot{Y}_\beta))$ .

Now since  $A$  is 2-positive in the ground model, there exists a  $\lambda \in A$  such that  $\psi(\lambda, \text{parameters} \cap V_\lambda)$ . This gives us a model  $M$  of size  $\lambda$ , and its transitive collapse  $N = \pi(M)$ , with the following properties:

- (a)  $M \cap \kappa = \lambda$ ,
- (b)  $P, p, \dot{X} \in M$ ,
- (c)  $\pi(P)$  is an iteration of order 1 for  $\lambda$ ,
- (d)  $p \Vdash \varphi(\lambda, \pi(\dot{X}))$ ,
- (e)  $\forall \beta \in M \quad \pi(P \restriction \beta) \Vdash \varphi_\beta^2(\lambda, \pi(\dot{Y}_\beta))$ .

The rest of the proof is analogous to the proof of Lemma 3.12, as long as we prove the analog of Sublemma 3.12.1: after that, the proof is Case 1 generalizes with the obvious changes.

**Sublemma 4.10.2.** *There exists an  $N$ -generic filter  $G \ni p$  on  $\pi(P)$  such that if  $X = \pi(\dot{X})/G$  and  $Y_\beta = \pi(\dot{Y}_\beta)/G$  for each  $\beta \in M$ , then  $\varphi(\lambda, X)$  and  $\varphi_\beta^2(\lambda, Y_\beta)$  hold.*

*Proof.* We find  $G$  in  $V(Q \restriction \lambda * Q_\lambda)$ . Since  $\pi(P)$  is an iteration of order 1 and  $\varphi$  is absolute, there is an  $\alpha < \lambda^+$  such that  $V(Q \restriction \lambda * Q_\lambda \restriction \alpha)$  contains  $\pi(\dot{X}), \pi(\dot{Y}_\beta)$  ( $\beta \in M$ ) and the dense sets in  $N$ , thinks that  $\pi(P)$  is the Cohen forcing, such that the forcing  $Q_\lambda(\alpha)$  at stage  $\alpha$  is the Cohen forcing, and (by absoluteness and by Sublemma 4.10.1)  $Q_\lambda(\alpha)$  (or  $\pi(P)$ ) forces  $\varphi(\lambda, \pi(\dot{X}))$  and  $\varphi_\beta^2(\lambda, \pi(\dot{Y}_\beta))$ . The generic filter on  $Q_\lambda(\alpha)$  yields a generic  $G$  such that  $V(Q \restriction \lambda * Q_\lambda \restriction (\alpha + 1)) \Vdash \varphi(\lambda, X)$  and  $\varphi_\beta^2(\lambda, Y_\beta)$  where  $X = \pi(\dot{X})/G$ ,  $Y_\beta = \pi(\dot{Y}_\beta)/G$ . By absoluteness again,  $\varphi(\lambda, X)$  and  $\varphi_\beta^2(\lambda, Y_\beta)$  hold in  $V(Q \restriction \lambda * Q_\lambda)$ , and hence they hold.  $\square$

**Lemma 4.11.** (a) *If  $P$  and  $R$  are iterations, and  $P$  is of order 2, then  $P \Vdash (R \text{ is of order 2})$  if and only if  $R$  is order 2.*

(b) *If  $\dot{R}$  is a  $P$ -name then  $P * \dot{R}$  is an iteration of order 2 if and only if  $P$  is an iteration of order 2 and  $P \Vdash (\dot{R} \text{ is an iteration of order 2})$ .*

*Proof.* By Lemma 4.10 (just as Lemma 3.13 follows from Lemma 3.12).  $\square$

**Corollary 4.12.** *In  $V(Q \restriction \kappa * Q_\kappa)$ ,  $E_1$  is stationary, every stationary  $S \subseteq E_0$  reflects fully, and every stationary  $T \subseteq E_1$  reflects fully.*

*Proof.* The first part follows from Lemma 4.3 and 4.10. The second part is a consequence of Lemmas 3.8 and 4.6 and the construction that destroys non-1-positive as well as all non-2-positive sets.  $\square$

This concludes Case 2. We can now go on to Case 3 (and in an analogous way, to higher cases), with only one difficulty remaining. In analogy with definition 4.2 we can define a filter  $F_3$  and the associated with it 3-positive sets. All the proofs of Chapter 4 will generalize from Case 2 to Case 3, with the exception of Lemma 4.4 which proved that “1-positive” is an absolute  $\Pi_2^1$  property. The proof does not generalize, as it uses, in an essential way, the fact that the property is  $\Pi_2^1$ , while “2-positive” is a  $\Pi_3^1$  property.

However, we can replace the property “ $A \subseteq E_1 \cap \kappa$  is 2-positive” by another  $\Pi_3^1$  property that is absolute for  $\kappa$ , and that is equivalent to the definition 4.2 at all stages of the iteration  $Q_\kappa$  except possibly at the end of the iteration. The new property is as follows:

**(4.13).** *Either Full Reflection fails for some  $S \subseteq \text{Sing} \cap \kappa$  and  $A$  is 2-positive, or Full Reflection holds for all subsets of  $\text{Sing}$  and  $A$  is stationary.*

“Full Reflection” for  $S \subseteq \text{Sing}$  means that  $E_0 - \text{Tr}(S)$  is nonstationary. It is a  $\Sigma_1^1$  property of  $S$ , and so (4.13) is  $\Pi_3^1$ . We claim that Full Reflection fails at every intermediate stage of  $Q_\kappa$ . Hence (4.13) is equivalent to “2-positive” at the intermediate stages. At the end of  $Q_\kappa$ , every 2-positive set becomes stationary, and every non-2-positive set becomes nonstationary. Hence (4.13) is absolute.

Since for every  $\alpha < \kappa^+$ , the size of  $Q|_\kappa * Q_\kappa|_\alpha$  is  $\kappa$ , the following lemma verifies our claim:

**Lemma 4.14.** *Let  $\kappa$  be a Mahlo cardinal, and assume  $V = L[X]$  where  $X \subseteq \kappa$ . Then there exists a stationary set  $S \subseteq \text{Sing} \cap \kappa$  such that for every  $\gamma \in E_0$ ,  $S \cap \gamma$  is nonstationary.*

*Proof.* We define  $S \subseteq \text{Sing}$  by induction on  $\alpha < \kappa$ . Let  $\alpha \in \text{Sing}$  and assume  $S \cap \alpha$  has been defined. Let  $\eta(\alpha)$  be the least  $\eta < \alpha^+$  such that  $L_\eta[X \cap \alpha]$  is a model of  $\text{ZFC}^-$  and  $L_\eta[X \cap \alpha] \models \alpha$  is not Mahlo. Let

$$\alpha \in S \text{ iff } L_{\eta(\alpha)}[X \cap \alpha] \models S \cap \alpha \text{ is nonstationary.}$$

First we show that  $S$  is stationary.

Assume that  $S$  is nonstationary. Let  $\nu < \kappa^+$  be such that  $S \in L_\nu[X]$  and  $L_\nu[X] \models S$  is nonstationary. Also, since  $\kappa$  is Mahlo, we have  $L_\nu[X] \models \kappa$  is Mahlo. Using a continuous elementary chain of submodels of  $L_\nu[X]$ , we find a club  $C \subseteq \kappa$  and a function  $\nu(\xi)$  on  $C$  such that for every  $\xi \in C$ ,

$$L_{\nu(\xi)}[X \cap \xi] \models \xi \text{ is Mahlo and } S \cap \xi \text{ is not stationary.}$$

If  $\alpha \in \text{Sing} \cap C$ , then because  $\alpha$  is Mahlo in  $L_{\nu(\alpha)}[X \cap \alpha]$  but non Mahlo in  $L_{\eta(\alpha)}[X \cap \alpha]$ , we have  $\nu(\alpha) \leq \eta(\alpha)$ . Since  $S \cap \alpha$  is nonstationary in  $L_{\nu(\alpha)}[X \cap \alpha]$ , it is nonstationary in  $L_{\eta(\alpha)}[X \cap \alpha]$ . Therefore  $\alpha \in S$ , and so  $S \supseteq \text{Sing} \cap C$  contrary to the assumption that  $S$  is nonstationary.

Now let  $\gamma \in E_0$  be arbitrary and let us show that  $S \cap \gamma$  is nonstationary. Assume that  $S \cap \gamma$  is stationary. Let  $\delta < \gamma^+$  be such that  $S \cap \gamma \in L_\delta[X \cap \gamma]$ , that  $L_\delta[X \cap \gamma] \models S \cap \gamma$  is stationary and that  $L_\delta[X \cap \gamma] \models \gamma$  is not Mahlo. There is a club  $C \subseteq \gamma$  and a function  $\delta(\xi)$  on  $C$  such that for every  $\xi \in C$ ,

$$L_{\delta(\xi)}[X \cap \xi] \models \xi \text{ is not Mahlo and } S \cap \xi \text{ is stationary.}$$

Since  $S \cap \gamma$  is stationary, there is an  $\alpha \in S \cap C$ . Because  $\eta(\alpha)$  is the least  $\eta$  such that  $\alpha$  is not Mahlo in  $L_\eta[X \cap \alpha]$ , we have  $\eta(\alpha) \leq \delta(\alpha)$ . But  $S \cap \alpha$  is nonstationary in  $L_{\eta(\alpha)}[X \cap \alpha]$  and stationary in  $L_{\delta(\alpha)}[X \cap \alpha]$ , a contradiction.  $\square$

Now with the modification given by (4.13), the proofs of Chapter 4 go through in the higher cases, and the proof of Theorem B is complete.

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DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA  
16803, USA

INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY, JERUSALEM, ISRAEL