FULL REFLECTION AT A MEASURABLE CARDINAL

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Abstract. A stationary subset S of a regular uncountable cardinal κ *reflects fully* at regular cardinals if for every stationary set $T \subseteq \kappa$ of higher order consisting of regular cardinals there exists an $\alpha \in T$ such that $S \cap \alpha$ is a stationary subset of α. *Full Reflection* states that every stationary set reflects fully at regular cardinals. We will prove that under a slightly weaker assumption than κ having Mitchell order κ^{++} it is consistent that Full Reflection holds at every $\lambda \leq \kappa$ and κ is measurable.

1. Definitions and results.

It has been proved in [M82] that reflection of stationary sets is a large cardinal property. Reflection of stationary subsets of ω_n $(n \geq 2)$ and $\omega_{\omega+1}$ has been investigated in [M82] and [JS90] and consistency strength of Full Reflection at regular cardinals at a Mahlo cardinal has been characterized in [JS92]. In this paper we address the question of Full Reflection at a measurable cardinal.

If S is a stationary subset of a regular uncountable cardinal κ then *the trace of* S is the set

$$Tr(S) = \{\alpha < \kappa;\ \ S \cap \alpha \text{ is stationary in } \alpha\}$$

(and we say that S *reflects at* α). If S and T are both stationary, we define

$$S < T \text{ if for almost all } \alpha \in T,\ \ \alpha \in Tr(S)$$

and say that S *reflects fully* in T. (Throughout the paper, "for almost all" means "except for a nonstationary set of points").

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Lemma 1.1. ([J84]) *The relation $<$ is well founded.*

Proof. By contradiction suppose there is a sequence of stationary sets such that

$$A_1 > A_2 > A_3 > \cdots$$

It means that there are clubs C_n such that

$$A_n \cap C_n \subseteq Tr(A_{n+1}) \text{ for } n = 1, 2, \ldots$$

If $C \subseteq \kappa$ is a club let us denote $C' = \{\alpha < \kappa;\ C \cap \alpha \text{ is unbounded in } \alpha\}$, (C' is again a club) and put

$$\tilde{A}_n = A_n \cap C_n \cap C'_{n+1} \cap C''_{n+2} \cap \cdots \quad \text{for } n = 1, 2, \ldots$$

Then all $\tilde{A}_n$ are stationary. Observe that $\alpha \in Tr(S)$ implies $cf(\alpha) > \omega$ and $Tr(S \cap C) = Tr(S) \cap C'$ where C is any club. Now it is easy to verify that

$$\tilde{A}_n \subseteq Tr(\tilde{A}_{n+1}) \quad \text{for } n = 1, 2, \ldots$$

Let $\alpha_n = \min(\tilde{A}_n)$. Since $\tilde{A}_{n+1} \cap \alpha_n$ is stationary in α_n, the ordinal α_{n+1} must be less then α_n. We obtain a sequence of ordinals

$$\alpha_1 > \alpha_2 > \alpha_3 > \cdots$$

- a contradiction. □

The order $o(S)$ of a stationary set of regular cardinals is defined as the rank of S in relation $<$:

$$o(S) = \sup\{o(T) + 1;\ T \subseteq Reg(\kappa) \text{ stationary and } T < S\}.$$

For a stationary set T such that $T \cap Sing(\kappa)$ is stationary we define $o(T) = -1$. *The order of* κ is then defined as

$$o(\kappa) = \sup\{o(S) + 1\ ;\ S \subseteq \kappa \text{ is stationary}\}.$$

Note that if $Tr(S)$, where $S \subseteq Reg(\kappa)$, is stationary then $o(S) < o(Tr(S))$ because $S < Tr(S)$. It follows from [J84] that the order $o(\kappa)$ provides a natural generalization of the Mahlo hierarchy: κ is exactly $o(\kappa)$-Mahlo if $o(\kappa) < \kappa^+$ and greatly Mahlo if $o(\kappa) \geq \kappa^+$.

We say that a stationary set S *reflects fully* at regular cardinals if for any stationary set T of regular cardinals $o(S) < o(T)$ implies $S < T$.

Axiom of Full Reflection at κ. *Every stationary subset of κ reflects fully at regular cardinals.*

Following [J84] we say that a stationary set E is *canonical of order ν* if E is hereditarily of order ν (i.e. $o(X) = \nu$ for every stationary $X \subseteq E$) and E meets every stationary set of order ν.

The existence of canonical stationary sets of order less then κ^+ (if a set of such order exists) is proved in [BTW76] and [J84]. In the model constructed in Section 3 we get a sequence of stationary sets with the following properties:

Lemma 1.2. *Let $\langle E_\delta;\ -1 \le \delta < \theta\rangle$ be a maximal antichain of stationary subsets of λ such that*

(i) $E_{-1} = Sing(\lambda)$, $E_\delta \subseteq Reg(\lambda)$ for $\delta \ge 0$,

(ii) for any $\delta \ge 0$ the set $Tr(E_\delta) \cap E_\delta$ is nonstationary,

(iii) if $S \subseteq E_\delta$ is stationary and $-1 \le \delta < \delta'$ then $S < E_{\delta'}$.

Then each E_δ is a canonical stationary set of order δ, $o(\lambda) = \theta$ and Full Reflection holds at λ.

Proof. We will prove the lemma in several steps. Obviously $E_\delta < E_{\delta'}$ if $\delta < \delta'$.

Claim 1. *Let $T \subseteq Reg(\lambda)$ be a stationary set such that $T \cap E_{\delta'}$ is nonstationary for $\delta' \le \delta$ and $S \subseteq E_\delta$ stationary. Then $S < T$.*

Proof. We need to prove that $T \setminus Tr(S)$ is nonstationary. But $(T \setminus Tr(S)) \cap E_{\delta'}$ is nonstationary for $\delta' \le \delta$ because $T \cap E_{\delta'}$ is nonstationary, and for $\delta' > \delta$ the set $(T \setminus Tr(S)) \cap E_{\delta'}$ is nonstationary because $E_{\delta'} \setminus Tr(S)$ is nonstationary. Consequently $T \setminus Tr(S)$ is nonstationary.

Claim 2. *If $S \subseteq E_\delta$ is stationary then $o(S) = o(E_\delta)$.*

Proof. Suppose the claim holds for $\delta' < \delta$ and that for some $S \subseteq E_\delta$ stationary $o(S) > o(E_\delta)$. Then there is $T < S$ such that $o(T) = o(E_\delta)$. By the induction hypothesis $T \cap E_{\delta'}$ must be nonstationary for $\delta' < \delta$ ($T_1 \subseteq T_2$ stationary implies $o(T_1) \ge o(T_2)$). Moreover $E_\delta \cap Tr(T \cap E_\delta)$ is nonstationary. Thus $S \setminus Tr(T \setminus E_\delta)$ must be nonstationary because $S \setminus Tr(T) = (S \setminus Tr(T \setminus E_\delta)) \cap (\lambda \setminus (E_\delta \cap Tr(T \cap E_\delta))$ is nonstationary. It means that $(T \setminus E_\delta) < S$ but by claim 1 $S < (T \setminus E_\delta)$ which is a contradiction with well-foundedness of $<$.

Claim 3. *$o(E_\delta) = \delta$ for $\delta < \theta_\lambda$.*

proof. Suppose by induction that $o(E_{\delta'}) = \delta'$ for $\delta' < \delta$. Certainly $o(E_\delta) \ge \delta$, suppose by contradiction that there is a set $T < E_\delta$ such that $o(T) = \delta$. As in the proof of claim 2 we can suppose that $T \cap E_{\delta'}$ is nonstationary for $\delta' \le \delta$. But it implies by claim 1 that $E_\delta < T$ - a contradiction.

It follows from these claims that each E_δ is a canonical stationary set of order δ. Any $S \subseteq Reg(\lambda)$ stationary must have a nonstationary intersection with some E_δ which means $o(S) \le \delta$ and so $o(\lambda) = \theta_\lambda$, actually

$$o(S) = \min\{\delta < \theta_\lambda;\ E_\delta \cap S \text{ is stationary}\}.$$

Finally let $S \subseteq \lambda$, $T \subseteq Reg(\lambda)$ be stationary and $\delta = o(S) < o(T)$ then by claim 1 $S \cap E_\delta < T$ which implies $S < T$. □

To state our result we need to review the definition of Mitchell order and of a coherent sequence.

If $\mathcal{U}$, $\mathcal{V}$ are two measures on κ then $\mathcal{U} \lhd \mathcal{V}$ is defined as $\mathcal{U} \in V^\kappa/\mathcal{V}$. The transitive relation $\lhd$ is known to be well-founded (see [Mi74]). *The Mitchell order of* κ is then defined as the rank of this relation on measures over κ.

A coherent sequence of measures is a function $\overrightarrow{\mathcal{U}}$ with domain of the form $\{(\alpha,\beta);\ \alpha < l(\mathcal{U})$ and $\beta < o^{\mathcal{U}}(\alpha)\}$ for some ordinal $l(\mathcal{U})$ and a function $o^{\mathcal{U}}(\cdot)$ such that

(i) For all $(\alpha,\beta)\in \operatorname{dom}\overrightarrow{\mathcal{U}}\quad \mathcal{U}^\alpha_\beta = \overrightarrow{\mathcal{U}}(\alpha,\beta)$ is a measure on α,

(ii) if j is the canonical embedding $j : V \to V^\alpha/\mathcal{U}^\alpha_\beta$ then

$j(\overrightarrow{\mathcal{U}}) \restriction (\alpha+1) = \overrightarrow{\mathcal{U}} \restriction (\alpha,\beta)$ where

$\overrightarrow{\mathcal{U}} \restriction (\alpha,\beta) = \overrightarrow{\mathcal{U}} \restriction \{(\alpha',\beta');\ \alpha' < \alpha \text{ or } \alpha' = \alpha \text{ and } \beta' < \beta\}$ and

$\overrightarrow{\mathcal{U}} \restriction (\alpha+1) = \overrightarrow{\mathcal{U}} \restriction (\alpha+1, 0)$.

Observe that in particular $\mathcal{U}^\alpha_0 \lhd \mathcal{U}^\alpha_1 \lhd \cdots \lhd \mathcal{U}^\alpha_\beta \lhd \cdots \quad (\beta < o^{\mathcal{U}}(\alpha))$. The following is proved in [Mi83].

Proposition 1.3. *There is a class sequence $\overrightarrow{\mathcal{U}}$ such that $L[\overrightarrow{\mathcal{U}}] \models$ " For every α $\overrightarrow{\mathcal{U}} \restriction (\alpha+1)$ is a coherent sequence, every measure on α is equal to some $\mathcal{U}^\alpha_\beta$ and $o^{\mathcal{U}}(\alpha) = \min\{(\text{Mitchell order})^V(\alpha),\ \alpha^{++}\}$" . Moreover, $L[\overrightarrow{\mathcal{U}}]$ satisfies GCH.*

We say that κ has *a repeat point* (see [Mi82]) if there is a coherent sequence $\overrightarrow{\mathcal{U}}$ up to κ and an ordinal $\theta < o^{\mathcal{U}}(\kappa)$ such that

$$\forall X \in \mathcal{U}^\kappa_\theta\ \exists \alpha < \theta\ :\ X \in \mathcal{U}^\kappa_\alpha\ .$$

It can be proved that such θ must be greater than κ^+. Suppose we have a coherent sequence $\overrightarrow{\mathcal{U}}$ such that $o^{\mathcal{U}}(\kappa) = \kappa^{++}$; then using a simple counting argument we can prove the existence of a repeat point for κ. Consequently, if Mitchell order of (κ) is κ^{++} then there is an inner model satisfying GCH where κ has a repeat point.

Our result is the following:

Theorem. *If κ has a repeat point in the ground model V satisfying GCH then there is a generic extension of V preserving cardinalities, cofinalities and GCH in which Full Reflection holds at all $\lambda \le \kappa$ and κ is measurable.*

Actually if we start with $V = L[\overrightarrow{\mathcal{U}}]$ from proposition 1.3 then our construction provides a class generic extension $V[G]$ preserving cardinalities, cofinalities and GCH such that for any cardinal λ $V[G]$ satisfies Full Reflection at λ, $(\text{Mitchell order})^V(\lambda) = o^{V[G]}(\lambda)$ and if λ has a repeat in V then λ is measurable in $V[G]$.

2. The Forcing $P_{\kappa+1}$.

From now on we work in a ground model V satisfying GCH with a coherent sequence $\overrightarrow{\mathcal{U}}$ up to κ and a repeat point at κ. For $\lambda \le \kappa$ let θ_λ be $o^{\mathcal{U}}(\lambda)$ if λ does not have a repeat point, or otherwise the least θ such that $\mathcal{U}^\lambda_\theta$ is a repeat point.

As usual, if P is a forcing notion then $V(P)$ denotes either the Boolean valued model or a generic extension by a P-generic filter over V.

$P_{\kappa+1}$ will be an Easton support iteration of $\langle Q_\lambda;\ \lambda \le \kappa\rangle$, Q_λ will be nontrivial only for λ Mahlo. Q_λ (for λ Mahlo) is defined in $V(P_\lambda)$, where P_λ denotes the iteration below λ, as an iteration of length λ^+ with $<\lambda$-support of forcing notions shooting clubs through certain sets $X \subseteq \lambda$ (we will denote this standard forcing notion $CU(X)$), always with the property that $X \supseteq Sing(\lambda)$. This condition will guarantee Q_λ to be essentially $<\lambda$-closed (i.e. for any $\gamma < \lambda$ there is a dense γ-closed subset of Q_λ). Q_λ will also satisfy the λ^+-chain condition. Consequently P_λ will satisfy λ-c.c. and will have size λ. Cardinalities, cofinalities and GCH will be preserved, stationary subsets of λ can be made nonstationary only by the forcing at λ, not below λ, and not after the stage λ - after stage λ no subsets of λ are added.

We use the λ^+-chain condition of Q_λ to get a canonical enumeration of length λ^+ of all the λ^+ Q_λ-names for subsets of λ so that the βth name appears in $V(P_\lambda * Q_\lambda|\beta)$. Moreover for $\delta < \theta_\lambda$ we will define filters $F_\delta = F_\delta^\lambda$ in $V(P_\lambda * Q_\lambda|\beta)$. Their definition will not be absolute, however the filters will extend the V-measures $\mathcal{U}_\delta^\lambda$ and will increase coherently during the iteration.

Definition. *An iteration Q of $\langle CU(B_\alpha);\ \alpha < \alpha_0\rangle$ with $<\lambda$-support and length $\alpha_0 < \lambda^+$ is called an iteration of order δ_0 if for all $\alpha < \alpha_0$,*

$$V(P_\lambda * Q|\alpha) \models\ B_\alpha \in F_\delta \text{ for any } \delta < \delta_0 \text{ and } Sing(\lambda) \subseteq B_\alpha \quad .$$

(Note that an iteration of order δ_0 is also an iteration of order δ, for all $\delta < \delta_0$.)

Q_λ is then defined as an iteration of $\langle CU(B_\alpha);\ \alpha < \lambda^+\rangle$ with $<\lambda$-support and length λ^+ so that every $Q_\lambda|\alpha$ is an iteration of order θ_λ and all potential names $\dot{X} \subseteq \lambda$ are used cofinally many times in the iteration as some B_α.

Observe that Q_λ can be represented in $V(P_\lambda)$ as a set of sequences of closed bounded subsets of λ in $V(P_\lambda)$ rather than in $V(P_\lambda * Q_\lambda|\alpha)$. Moreover if $\dot{q}$ is a P_λ-name such that $1\Vdash_{P_\lambda} \dot{q} \in Q_\lambda$ then using the λ-chain condition of P_λ there is a set $A \subseteq \lambda^+$ (in V) of cardinality $<\lambda$ and $\gamma_0 < \lambda$ so that

$$1\Vdash_{P_\lambda} \text{ supp } \dot{q} \subseteq A \text{ and } \forall\alpha \in A:\ \dot{q}(\alpha) \subseteq \gamma_0 \quad .$$

Consequently, Q_λ can be represented as a set of functions $g : A \times \gamma_0 \to [P_\lambda]^{<\lambda}$ where $A \subseteq \lambda^+$, $|A| < \lambda$ and $\gamma_0 < \lambda$. In this sense Q_λ has cardinality λ^+ and any $Q_\lambda|\alpha$ has cardinality at most λ.

We will need to lift various elementary embeddings to generic extensions. For a review of basic methods see [WoC92]. We will often use the following simple fact: Let N be a submodel of M such that $M \cap {}^\kappa N \subseteq N$ and let G be a filter P-generic/M where P satisfies κ^+-c.c. Then

$$M[G] \cap {}^\kappa N[G] \subseteq N[G] \quad .$$

Moreover if Q is κ-closed and H is Q-generic/$M[G]$ then

$$M[G * H] \cap {}^\kappa N[G * H] \subseteq N[G * H] \quad .$$

Definition of filters F_δ.

The filters F_δ in $V(P_\lambda * Q)$, where Q is any iteration of order $\delta+1$, are defined by induction so that the following is satisfied:

Proposition 2.1. *Let Q, $Q' = Q * R$ be two iterations of order $\delta'+1$ then*

$$F_{\delta'}^{V(P_\lambda * Q)} = F_{\delta'}^{V(P_\lambda * Q')} \cap V(P_\lambda * Q) .$$

Moreover $F_{\delta'}^{V(P_\lambda)} \cap V = \mathcal{U}_{\delta'}^\lambda$.

Proposition 2.2. *Let $j = j_{\delta'}$ be the canonical embedding from V into $V^\lambda/\mathcal{U}_{\delta'}^\lambda = M$ and Q an iteration of order $\delta'+1$. Then j can be lifted to an elementary embedding from the generic extension $V(P_\lambda * Q)$ of V to a generic extension $M(jP_\lambda * jQ)$ of M.*

Lemma 2.3. *Let $N = V^\lambda/\mathcal{U}_\beta^\lambda$ for some $\beta > \delta'$ and Q be an iteration of order $\delta'+1$. Then*

$$F_{\delta'}^{V(P_\lambda * Q)} = F_{\delta'}^{N(P_\lambda * Q)} .$$

Note that it also means that the definition of $F_{\delta'}$ relativized to $N(P_\lambda * Q)$ makes sense.

Lemma 2.4. *Let $j = j_{\delta'} : V \to M$. Then any iteration Q of order δ' is an subiteration of $(jP_\lambda)^\lambda$, where $(jP_\lambda)^\lambda$ is the factor of $jP_\lambda = P_\lambda * (jP_\lambda)^\lambda * (jP_\lambda)^{>\lambda}$. Consequently for any G^* jP_λ-generic$/V$ and any $q \in Q$ there is an $H \in M[G^*]$ Q-generic$/V[G]$ containing q given by an embedding of Q as a subiteration of $(jP_\lambda)^\lambda$, where $G = G^* \restriction P_\lambda$.*

Definition. *Let j, Q, G^*, G be as in the lemma. Then $Gen_j(Q, G^*)$ is the set of all filters $H \in M[G^*]$ Q-generic$/V[G]$ given by an embedding of Q as a subiteration of $(jP_\lambda)^\lambda$.*

Lemma 2.5. *Let j be as above, Q an iteration of order $\delta'+1$, G^* jP_λ-generic$/V$, $H \in Gen_j(Q, G^*)$. For every $\beta < l(Q)$ let $C_\beta \subset \lambda$ be the club $\bigcup\{r_\beta; r \in H\}$, and let $[H]^j$ denote the $j(l(Q))$-sequence given by*

$$[H]^j_\gamma = \begin{cases} C_\lambda \cup \{\lambda\}, & \text{if } \gamma = j(\beta) \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $[H]^j \in jQ_{/G^*}$.

Propositions 2.1 and 2.2 will be essential to prove Full Reflection in the generic extension. We will later prove that if lemma 2.3 holds for $\delta' < \delta$ then lemma 2.4 holds for all $\delta' \le \delta$.

Now suppose that the filters $F_{\delta'}^{\lambda'}$ were defined for all $\lambda' < \lambda$ and $\delta' < \theta_{\lambda'}$ and for $\lambda' = \lambda$ and $\delta' < \delta$ so that 2.1-2.5 holds. Moreover let $\alpha < \lambda^+$ and $F_\delta = F_\delta^\lambda$ be defined for all iterations of order $\delta+1$ of length $< \alpha$ so that lemma 2.5 holds for δ and iterations of length $\le \alpha$. Let $j = j_\delta : V \to M = V^\lambda/\mathcal{U}_\delta^\lambda$. Then we can define F_δ for iterations of order $\delta+1$ and length α.

Definition. *Let Q be an iteration of order $\delta+1$ and length α, $j = j_\delta : V \to M$. For a $P_\lambda * Q$-name $\dot X$ of a subset of λ and $(p,q) \in P_\lambda * Q$ we define*

$$(p,q) \Vdash_{P_\lambda * Q} \dot X \in F_\delta$$

if the following holds in V:

$$j(p) \Vdash_{jP_\lambda} \text{“For any } H \in Gen_j(Q, G^*) \text{ containing } q :$$
$$[H]^j \Vdash_{jQ} \check\lambda \in j\dot X \quad \text{”.}$$

The definition says that $(p,q) \Vdash \dot X \in F_\delta$ if $\lambda \in j^* X$ whenever $j^* : V[G*H] \to M[G^* * H^*]$ is a lifting of j of certain kind and $(p,q) \in G * H$. To verify soundness of the definition let us first prove lemma 2.4 for δ.

Proof of lemma 2.4. Let $j = j_\delta : V \to M = V^\lambda/\mathcal{U}^\delta_\lambda$, Q be an iteration of order δ. We assume that lemma 2.3 holds for $\delta' < \delta$. Observe that $jP_\lambda = P_\lambda * (jP_\lambda)^\lambda * (jP_\lambda)^{>\lambda}$ is an Easton support iteration (in M) below $j\lambda$ and $(jP_\lambda)^\lambda$ is an iteration of length λ^+ with $< \lambda$-support such that for any $\alpha < \lambda^+$ $(jP_\lambda)^\lambda | \alpha$ is an iteration of order $\delta = \theta^M_\lambda$ and all potential names $\dot X \subseteq \lambda$ are used cofinally many times in the iteration. That is true in $M(P_\lambda)$ as in $V(P_\lambda)$.

Let us now define what it means for Q to be a subiteration of $(jP_\lambda)^\lambda$. Suppose that P, Q are iterations of legths $l(P) \le l(Q)$ of $\langle \dot R_\gamma;\ \gamma < l(P)\rangle$ and $\langle \dot S_\alpha;\ \alpha < l(Q)\rangle$ with $< \lambda$-support essentially $< \lambda$-closed. Then we say that P is *a subiteration of Q* if there is an increasing sequence $\langle \alpha_\gamma;\ \gamma < l(P)\rangle$ of ordinals below $l(Q)$ such that

1) $Q_{\alpha_0} \Vdash \check R_0 = \dot S_{\alpha_0}$
   consequently for $\beta > \alpha_0$ $\ Q_\beta \simeq P_1 * Q'_\beta$,
2) and by induction $Q_{\alpha_\gamma} \Vdash \check{\dot R}_\gamma = \dot S_{\alpha_\gamma}$,
   using the inductive assumption that $Q_{\alpha_\gamma} \simeq P_\gamma * Q'_{\alpha_\gamma}$,
   consequently again for $\beta > \alpha_\gamma$ we have $Q_\beta \simeq P_{\gamma+1} * Q'_\beta$.

It is now obvious that in this sense Q is a subiteration of $(jP_\lambda)^\lambda$. Moreover for any $\alpha < \lambda^+$ there is a sequence $\langle \alpha_\gamma;\ \gamma < l(Q)\rangle$ determining an embedding of Q into $(jP_\lambda)^\lambda$ such that $\alpha_0 > \alpha$.

Finally let G^* be jP_λ-generic/V and $q \in Q$. Then $(G^*)^\lambda$ is $(jP_\lambda)^\lambda$-generic/$V[G]$. Note that the set

$$D = \{ r \in (jP_\lambda)^\lambda;\ \text{there is a sequence } \langle \alpha_\gamma; \gamma < l(Q)\rangle \text{ determining an embedding of } Q \text{ into } (jP_\lambda)^\lambda \text{ such that } q \text{ corresponds to } r \restriction \langle \alpha_\gamma; \gamma < l(Q)\rangle \ \}$$

is dense in $(jP_\lambda)^\lambda$. Thus let $r \in D \cap (G^*)^\lambda$ and $\langle \alpha_\gamma; \gamma < l(Q)\rangle$ be the sequence. Then $(G^*)^\lambda \restriction \langle \alpha_\gamma; \gamma < l(Q)\rangle$ gives the Q-generic/$V[G]$ filter $H \ni q$. $\square$

We have defined F_δ for iterations of length $\le \alpha$. Let us now prove lemma 2.5 for iterations of length $\alpha+1$.

Proof of lemma 2.5. Let $j = j_\delta : V \to M$, Q be an iteration of order $\delta+1$ of $\langle CU(B_\beta); \beta < \alpha+1\rangle$, G^* jP_λ-generic$/V$, $H \in Gen_j(Q, G^*)$, $[H]^j$ as in the lemma. Note that $[H]^j$ is a $j(\alpha+1)$-sequence of closed bounded subsets of $j\lambda$, supp $[H]^j = j"(\alpha+1)$. Since $|j"(\alpha+1)| = \lambda < j\lambda$ in M, $[H]^j$ has a small support and $[H]^j \in M[G^*]$. It follows from the induction hypothesis that $[H]^j \restriction j(\alpha) \in j(Q \restriction \alpha)$. We only have to verify that in $M[G^*]$

$$[H]^j \restriction j(\alpha) \Vdash_{j(Q)\restriction j(\alpha)} [H]^j_{j(\alpha)} \in j(\langle CU(B_\beta); \beta < \alpha+1\rangle)_{j(\alpha)}$$

which means just that

$$[H]^j \restriction j(\alpha) \Vdash [H]^j_{j(\alpha)} \subseteq j(B_\alpha)$$

where $[H]^j_{j(\alpha)} = \cup\{r_\alpha;\ r \in H\} \cup \{\lambda\}$. If $\gamma \in [H]^j_{j(\alpha)}$, $\gamma < \lambda$, then $\gamma \in r_\alpha$ for some $r \in H$ and

$$V[G] \models r \restriction \alpha \Vdash_{Q|\alpha} r_\alpha \subseteq B_\alpha$$

so

$$M[G^*] \models j(r \restriction \alpha) \Vdash_{j(Q|\alpha)} r_\alpha = jr_\alpha \subseteq jB_\alpha \ .$$

Since $[H]^j \restriction j(\alpha) \le j(r \restriction \alpha)$ we get $[H]^j \restriction j(\alpha) \Vdash \gamma \in jB_\alpha$. So we only have to prove that in $M[G^*]$

$$(*) \qquad [H]^j \restriction j(\alpha) \Vdash \lambda \in jB_\alpha$$

Here we use the fact that Q is an iteration of order $\delta+1$. It implies that $Q \restriction \alpha \Vdash B_\alpha \in F_\delta$ and that exactly gives (*) in $V[G^*]$ and so in $M[G^*]$ by the definition of F_δ in $V(P_\lambda * Q|\alpha)$. ▫

F_δ is now well defined in $V(P_\lambda * Q)$ for any iteration Q of order $\delta+1$. We have to verify Proposition 2.1, 2.2 and Lemma 2.3 for F_δ.

Proof of Lemma 2.3.

Let $\beta > \delta$, $N = V^\lambda/\mathcal{U}^\lambda_\beta$, $j = j_\delta : V \to M$, Q an iteration of order $\delta+1$. We want to prove that

$$F_\delta^{V(P_\lambda * Q)} = F_\delta^{N(P_\lambda * Q)} \ .$$

Let $j' : N \to N^\lambda/\mathcal{U}^\lambda_\delta = N'$, observe that $j' = j \restriction N$. We need to prove that the following two conditions are equivalent:

$$(1) \qquad V \models jp \Vdash_{jP_\lambda} \forall H \in Gen_j(Q, G^*),\ H \ni q$$
$$[H]^j \Vdash_{jQ} \check{\lambda} \in j\dot{X} \ ,$$
$$(2) \qquad N \models jp \Vdash_{jP_\lambda} \forall H \in Gen_j(Q, G^*),\ H \ni q$$
$$[H]^j \Vdash_{jQ} \check{\lambda} \in j\dot{X} \ .$$

Observe that $V[G^*] \cap {}^\lambda N[G^*] \subseteq N[G^*]$, and so

$$Gen_j(Q, G^*)^{V[G^*]} = Gen_j(Q, G^*)^{N[G^*]}.$$

From that the equivalence of (1) and (2) easily follows. □

Proof of Proposition 2.2. Let $j = j_\delta : V \to M$, Q be an iteration of order $\delta + 1$. Let G^* be jP_λ-generic/V, $G = G^* \upharpoonright P_\lambda$. Then j is lifted to $j^* : V[G] \to M[G^*]$. Using Lemma 2.4 find $H \in M[G^*]$ Q-generic/$V[G]$. By Lemma 2.5 $[H]^j \in j^*Q$ and $r \in H$ implies $j^*r \geq [H]^j$. Thus let H^* be jQ-generic/$V[G^*]$ containing $[H]^j$. Then $r \in H$ implies $j^*r \in H^*$ and j^* is lifted to $j^{**} : V[G * H] \to M[G^* * H^*]$. □

Proof of Proposition 2.1. Let Q, $Q' = Q * R$ be iterations of order $\delta + 1$, we want to prove

$$F_\delta^{V(P_\lambda * Q)} = F_\delta^{V(P_\lambda * Q')} \cap V(P_\lambda * Q) \ .$$

Firstly let us prove the easy direction: $\dot{X} \subseteq \lambda$ a $P_\lambda * Q$-name, $(p, q) \in P_\lambda * Q$, $(p, q) \Vdash_{P_\lambda * Q} \dot{X} \in F_\delta$. Then it is straightforward that $(p, q^\frown 1) \Vdash_{P_\lambda * Q'} \dot{X} \in F_\delta$.

Now let $\dot{X} \subseteq \lambda$ be a $P_\lambda * Q$-name, $(p, q') \in P_\lambda * Q'$ and $(p, q') \Vdash_{P_\lambda * Q'} \dot{X} \in F_\delta$. We prove that $(p, q) \Vdash_{P_\lambda * Q} \dot{X} \in F_\delta$ where $q = q' \upharpoonright l(Q)$. Let $G^* \ni p$ be jP_λ-generic/V, $j = j_\delta : V \to M$, $H \in Gen_j(Q, G^*)$ and $q \in H$. We want to prove that $[H]^j \Vdash_{j(Q)} \check{\lambda} \in j\dot{X}$. Suppose not, then there is $\tilde{q} \leq [H]^j$ such that $\tilde{q} \Vdash_{j(Q)} \check{\lambda} \notin j\dot{X}$. Obviously there exists $\tilde{H} \in Gen_j(Q', G^*)$ such that $\tilde{H} \upharpoonright Q = H$ and $\tilde{H} \ni q'$. Note that $\tilde{q}^\frown 1$ and $[\tilde{H}]^j$ are compatible, $\tilde{q}^\frown 1 \cup [\tilde{H}]^j \in jQ'$. But $[\tilde{H}]^j \Vdash_{jQ'} \check{\lambda} \in j\dot{X}$ and $\tilde{q}^\frown 1 \Vdash_{jQ'} \check{\lambda} \notin j\dot{X}$ - a contradiction.

Finally let us prove that $\mathcal{U}_\delta^\lambda = F_\delta^{V(P_\lambda)} \cap V$. The inclusion $\mathcal{U}_\delta^\lambda \subseteq F_\delta$ is obvious. Now let $X \in V$, $X \subseteq \lambda$ and $p \Vdash_{P_\lambda} \check{X} \in F_\delta$. Then it means that $j(p) \Vdash_{jP_\lambda} \check{\lambda} \in j(\check{X})$ which can be true only if $\lambda \in j(X)$. So $X \in \mathcal{U}_\delta^\lambda$. □

3. Full Reflection in $V(P_{\kappa+1})$.

To prove that Full Reflection holds in $V(P_{\kappa+1})$ at some $\lambda \leq \kappa$ it is enough to prove that in $V(P_{\lambda+1})$. Fix $\lambda \leq \kappa$.

Firstly let us prove the existence of sets E_δ ($\delta < \theta_\lambda$) separating the measures $\mathcal{U}_\delta^\lambda$ in the sense that $E_\delta \in \mathcal{U}_{\delta'}^\lambda$ iff $\delta = \delta'$.

Proposition 3.1. *Suppose that $\mathcal{U}_\delta^\lambda$ is not a repeat point. Then there is a set $E \in \mathcal{U}_\delta^\lambda$ such that $E \notin \mathcal{U}_{\delta'}^\lambda$ for any $\delta' \neq \delta$, $\delta' < o^{\mathcal{U}}(\lambda)$.*

Proof. Since $\mathcal{U}_\delta^\lambda$ is not a repeat point there is a set $X \in \mathcal{U}_\delta^\lambda$ such that $X \notin \mathcal{U}_{\delta'}^\lambda$ for all $\delta' < \delta$. Define for $\xi < \lambda$

$$f_\delta(\xi) = \sup\{\eta;\ \eta \leq o^{\mathcal{U}}(\xi) \quad \& \quad \forall \eta' < \eta : X \cap \xi \notin \mathcal{U}_{\eta'}^\xi\}$$

and

$$Y = \{\xi < \lambda;\ f_\delta(\xi) = o^{\mathcal{U}}(\xi)\}.$$

Claim. *$Y \in \mathcal{U}_\delta^\lambda$ but $Y \notin \mathcal{U}_{\delta'}^\lambda$ for any δ' such that $\delta < \delta' < o^{\mathcal{U}}(\lambda)$.*

Proof. 1) Let $j : V \to N = V^\lambda / \mathcal{U}_\delta^\lambda$. Then

$$Y \in \mathcal{U}_\delta^\lambda \text{ iff } \lambda \in jY = \{\xi < j\lambda;\ N \models o^{j\mathcal{U}}(\xi) = j(f_\delta)(\xi)\}$$

$$\text{iff } N \models o^{j\mathcal{U}}(\lambda) = j(f_\delta)(\lambda)$$

By the definition of a coherent sequence $o^{j\mathcal{U}}(\lambda) = \delta$. Moreover

$$N \models j(f_\delta)(\lambda) = sup\{\eta;\ \eta \le \delta\ \ \&\ \forall \eta' < \eta :\ jX \cap \lambda \notin j\mathcal{U}^\lambda_{\eta'}\}.$$

By coherence $j\mathcal{U}^\lambda_{\eta'} = \mathcal{U}^\lambda_{\eta'}$ for $\eta' < \delta$, and also $jX \cap \lambda = X$. Consequently $j(f_\delta)(\lambda) = \delta$ and $Y \in \mathcal{U}^\lambda_\delta$.

2) Let $\delta < \delta' < o^{\mathcal{U}}(\lambda)$ and $j : V \to N = V^\lambda/\mathcal{U}^\lambda_{\delta'}$. Again $Y \in \mathcal{U}^\lambda_{\delta'}$ iff $N \models o^{j\mathcal{U}}(\lambda) = j(f_\delta)(\lambda)$. As above $o^{j\mathcal{U}}(\lambda) = \delta'$ but $j(f_\delta)(\lambda) = \delta$ since $X \notin \mathcal{U}^\lambda_\delta$. Thus $Y \notin \mathcal{U}^\lambda_{\delta'}$ and the claim is proved.

Finally put $E = Y \cap X$. ◻

So we have a separating sequence. We can suppose that the E_α are sets of regular cardinals because $Reg(\lambda) \in \mathcal{U}^\lambda_\delta$ for any δ. We are going to prove the following:

Proposition 3.2. *In $V(P_{\lambda+1})$ the sets $\langle E_\alpha;\ \alpha < \theta_\lambda\rangle$ form a maximal antichain of stationary subsets of $Reg(\lambda)$. Moreover if $S \subseteq E_{-1} = Sing(\lambda)$ or $S \subseteq E_\alpha$ is stationary then S reflects in any E_β for $\beta > \alpha$ and $Tr(E_\alpha) \cap E_\alpha$ is nonstationary if $\alpha > -1$. Consequently each E_α is a canonical stationary set of order α, $o(\lambda) = \theta_\lambda$ and Full Reflection at λ holds.*

To prove the proposition we need the following lemmas:

Lemma 3.3. *$V(P_\lambda * Q_\lambda|\alpha) \models Club(\lambda) \subseteq F_\delta$ for any $\delta < \theta_\lambda$.*

Proof. Let $\delta < \theta_\lambda$ and $(p,q) \Vdash_{P_\lambda * Q_\lambda|\alpha}$ " $\dot X \subseteq \lambda$ is a club". If (p,q) does not force $\dot X \in F_\delta$ then by the definition of F_δ there is a G^* jP_λ-generic$/V$, $G^* \ni jp$, where $j : V \to V^\lambda/\mathcal{U}^\lambda_\delta = M$, and $H \in Gen_j(Q_\lambda|\alpha, G^*)$, $H \ni q$, such that in $V[G^*]\ [H]^j \not\Vdash_{j(Q_\lambda|\alpha)} \check\lambda \in j\dot X$. So there is $H^* \ni [H]^j$ $j(Q_\lambda|\alpha)$-generic $/V[G^*]$ so that $\lambda \notin j\dot X_{/G^* * H^*}$. The embeding j is by the proof of Propostion 2.2 lifted to the elementary embeding $j^{**} : V[G * H] \to M[G^* * H^*]$, $X = \dot X_{/G*H}$ is a club in $V[G*H]$, thus $j^{**}X = j\dot X_{/G^* * H^*}$ is a club in $M[G^* * H^*]$. Since $j^{**}X \cap \lambda = X$ necessarily $\lambda \in j^{**}X$ - a contradiction. ◻

Lemma 3.4. *If S in $V(P_\lambda * Q_\lambda|\alpha)$ is F_δ-positive then $Tr(S) \in F_{\delta'}$ for any $\delta' > \delta$. If $S \subseteq Sing(\lambda)$ is stationary then $Tr(S) \in F_\delta$ for any δ. Moreover $V(P_\lambda) \models E_\delta \setminus Tr(E_\delta)\ \in\ F_\delta$.*

Proof. Suppose $(p,q) \Vdash_{P_\lambda * Q_\lambda|\alpha}$ "$\dot S$ is F_δ-positive" but $(p,q) \not\Vdash_{P_\lambda * Q_\lambda|\alpha}$ "$Tr(\dot S) \in F_{\delta'}$" for some $\delta' > \delta$. As usual denote $j : V \to M = V^\lambda/\mathcal{U}^\lambda_{\delta'}$. Then there is a filter G^* jP_λ-generic$/V$, $G^* \ni jp$, and $H \in Gen_j(Q_\lambda|\alpha, G^*)$, $H \ni q$, and a filter H^* $j(Q_\lambda|\alpha)$-generic$/V[G^*]$, $H^* \ni [H]^j$, so that $\lambda \notin jTr(\dot S)_{/G^* * H^*}$. As above j is lifted to

$$j^{**} : V[G*H] \to M[G^* * H^*]\ .$$

$S = \dot S_{/G*H}$ is F_δ-positive in $V[G*H]$ and

$$jTr(\dot S)_{/G^* * H^*}\ =\ j^{**}(Tr(S))\ =\ Tr^{M[G^* * H^*]}(j^{**}S)\ .$$

Thus $\lambda \notin Tr^{M[G^* * H^*]}(j^{**}S)$ which means that

$$M[G^* * H^*] \models \text{`` } S \text{ is not stationary in } \lambda\text{''}$$

because $S = j^{**}S \cap \lambda$. Consequently also

$$V[G^* * H^*] \models \text{``}S \text{ is not stationary in } \lambda\text{''}.$$

Observe that $j(Q_\lambda|\alpha)$ has a dense subset λ-closed in $M[G^*]$ and thus also in $V[G^*]$. Moreover $jP_\lambda = P_\lambda * (jP_\lambda)^\lambda * R$ where R is essentially λ-closed in $V[G^*|\lambda+1]$. It implies that already $V[G^*|\lambda+1] \models$ "S is not stationary in λ". Let us now consider the isomorphism $(jP_\lambda)^\lambda \simeq (Q_\lambda|\alpha) * \tilde{Q}$ from the proof of Lemma 2.4 giving the filter $H = G^* \restriction (Q_\lambda|\alpha)$, let $\tilde{H} = G^* \restriction \tilde{Q}$. Since every subset of λ in $V[G * H * \tilde{H}]$ is already in some $V[G * H * \tilde{H}|\beta]$ there is a $\beta < \lambda^+$ so that

$$V[G * H * \tilde{H}|\beta] \models \text{``}S \text{ is not stationary in } \lambda\text{''}.$$

But since $(Q_\lambda|\alpha) * (\tilde{Q}|\beta)$ is an iteration of order $\delta' \geq \delta + 1$ it follows from Proposition 2.1 that

$$V[G * H * \tilde{H}|\beta] \models \text{``}S \text{ is } F_\delta\text{-positive''}$$

which contradicts Lemma 3.3.

The proof for $S \subseteq Sing(\lambda)$ is the same using the following fact instead of Proposition 2.1.

Claim. *Stationary subsets of $Sing(\lambda)$ are preserved by iterations of order 0.*

Proof. For simplicity assume that $R = CU(X)$ where $X \supseteq Sing(\lambda)$; the generalization for an iteration of order 0 is straightforward. We closely follow the proof of 7.38 in [J86].

Let $S \subseteq Sing(\lambda)$ be stationary, $\dot{C}$ an R-name and $p \Vdash_R \dot{C} \subseteq \lambda$ is a club. We need a $\tilde{q} \leq p$ and $\beta \in S$ so that $\tilde{q} \Vdash \beta \in \dot{C}$. Put $A_0 = \{p\}$, $\gamma_0 = \max(p)$, and inductively for $q \in A_\alpha$ find $r(q) \leq q$ and $\beta(q) > \gamma_\alpha$ so that $\max(r(q)) > \gamma_\alpha$ and $r(q) \Vdash \beta(q) \in \dot{C}$. Put

$$A_{\alpha+1} = A_\alpha \cup \{r(q);\ q \in A_\alpha\} \text{ and}$$

$$\gamma_{\alpha+1} = \sup(\{\max(q);\ q \in A_{\alpha+1}\} \cup \{\beta(q);\ q \in A_\alpha\}).$$

For β limit put

$$A_\beta = \bigcup_{\alpha<\beta} A_\alpha \ \cup\ \{\text{unions of all decreasing sequences}$$

$$\subseteq \bigcup_{\alpha<\beta} A_\alpha \text{ that are in } R\} \text{ and}$$

$$\gamma_\beta = \sup\{\gamma_\alpha;\ \alpha < \beta\}.$$

Find a $\beta \in S$ such that $\gamma_\beta = \beta$. Observe that $\mathrm{cf}(\beta) < \beta$ and all unions of increasing sequences $\subseteq \bigcup_{\mathrm{cf}(\beta)<\alpha<\beta} A_\alpha$ of length $\leq cf(\beta)$ are in R. Now it is easy to find an increasing sequences $\beta_\alpha \nearrow \beta$ and decreasing $q_\alpha \searrow \tilde{q} \in R$ $(\alpha < cf(\beta))$ so that $q_\alpha \Vdash \beta_\alpha \in \dot{C}$. Consequently $\tilde{q} \Vdash \beta \in \dot{C}$.

Let us now prove that $V(P_\lambda) \models E_\delta \setminus Tr(E_\delta) \in F_\delta$. Let $j = j_\delta : V \to M$ then $(jP_\lambda)^\lambda$ is an iteration of length λ^+ such that $(jP_\lambda)^\lambda|\alpha$ is always an iteration of order δ and every potential name is used cofinally many times. Thus a club is shot through $\lambda \setminus E_\delta$ in the iteration. It implies that

$$V[G^*] \models E_\delta \subseteq \lambda \text{ is nonstationary}$$

and consequently

$$V[G^*] \models \lambda \in j(E_\delta \setminus Tr(E_\delta)).$$

□

Proof of Proposition 3.2. That each E_δ is stationary in $V(P_{\lambda+1})$ folows easily from Proposition 2.1 and Lemma 3.3. Let $\delta \neq \delta' < \theta_\lambda$, then $\lambda \setminus (E_\delta \cap E_{\delta'}) \in \mathcal{U}^\lambda_\eta$ for any $\eta < \theta_\lambda$ and $\lambda \setminus (E_\delta \cap E_{\delta'}) \supseteq Sing(\lambda)$, so $\lambda \setminus (E_\delta \cap E_{\delta'})$ contains a club in $V(P_{\lambda+1})$, and $E_\delta \cap E_{\delta'}$ is nonstationary. Let now $A \subseteq Reg(\lambda)$, $A \in V(P_{\lambda+1})$ be such that $A \cap E_\delta$ is nonstationary in $V(P_{\lambda+1})$ for any $\delta < \theta_\lambda$. We know that $A \in V(P_\lambda * Q_\lambda|\beta)$ for some $\beta < \lambda^+$.

Claim. *$V(P_\lambda * Q_\lambda|\beta) \models \lambda \setminus A \in F_\delta$ for any $\delta < \theta_\lambda$.*

Proof. If A was F_δ-positive then $E_\delta \cap A$ would be F_δ-positive in $V(P_\lambda * Q_\lambda|\alpha)$ for $\alpha \geq \beta$. Therefore $E_\delta \cap A$ would be stationary in $V(P_{\lambda+1})$.

Since also $Sing \subseteq \lambda \setminus A$ there is a club $C \subseteq \lambda \setminus A$ in $V(P_{\lambda+1})$, and so A is nonstationary.

We have proved that $\langle E_\delta;\ \delta < \theta_\lambda \rangle$ forms a maximal antichain of stationary subsets of $Reg(\lambda)$ in $V(P_{\lambda+1})$.

Now let $S \subseteq E_\delta$ be stationary, $\delta' > \delta$, $S \in V(P_\lambda * Q_\lambda|\alpha)$. S is F_δ-positive (or just stationary if $\delta = -1$) in $V(P_\lambda * Q_\lambda|\alpha)$ and so by Lemma 3.4 $Tr(S) \in F_{\delta'}$. Consequently $E_{\delta'} \setminus Tr(S)$ is nonstationary in $V(P_{\lambda+1})$ - $(\lambda \setminus E_{\delta'}) \cup (E_{\delta'} \cap Tr(S))$ contains a club - which exactly means that $S < E_{\delta'}$. Since by Lemma 3.4 $E_\delta \setminus Tr(E_\delta) \in F_\delta$ the set $Tr(E_\delta) \cap E_\delta$ is nonstationary in $V(P_{\lambda+1})$ - $(\lambda \setminus E_\delta) \cup (E_\delta \setminus Tr(E_\delta))$ contains a club. □

The following easy observation tells us more about the properties of the algebra $\mathcal{P}(\kappa)/NS$ in the resulting model.

Proposition 3.5. *Let $-1 \leq \alpha < \beta < \theta_\lambda$, then the sum of the sets $\{E_\delta; \alpha < \delta \leq \beta\}$ in the algebra $\mathcal{P}(\kappa)/NS$ exists. Moreover for any normal measure over κ this sum has measure zero.*

Proof. It follows immediately from proposition 3.2 that $Tr(E_\alpha)$ is the sum of $\{E_\delta; \alpha < \delta < \theta_\lambda\}$. Hence the desired sum is just $Tr(E_\alpha) \setminus Tr(E_\beta)$. For any normal measure over κ the measure of $Tr(E_\beta)$ is one, consequently the measure of $Tr(E_\alpha) \setminus Tr(E_\beta)$ must be zero. □

4. Measurability of κ in $V(P_{\kappa+1})$.

Let $\mathcal{U}^\kappa_\theta$ be the first repeat point of κ and $j = j_\theta : V \to M = V^\kappa/\mathcal{U}^\kappa_\theta$. Then $(j\overrightarrow{\mathcal{U}}) \restriction \kappa+1 = \overrightarrow{\mathcal{U}} \restriction (\kappa,\theta)$ and it follows from Lemma 2.3 that $(jP_\kappa)^\kappa$ is an iteration of length κ^+ with $< \kappa$-support such that any initial segment is an iteration of order θ and any potential name is used cofinally many times in $M(P_\kappa)$ as well as in $V(P_\kappa)$. Consequently we can suppose that $Q_\kappa = (jP_\kappa)^\kappa$.

Using methods for extending elementary embeddings (see [WoC92] and [JWo85]) we will prove that κ is actually measurable in $V(P_{\kappa+1})$. Let G be a P_κ-generic filter$/V$, G_κ a Q_κ-generic$/V[G]$. We know that $jP_\kappa = P_\kappa * Q_\kappa * R$, where the factor R is κ-closed in $M[G][G_\kappa]$ and consequently in $V[G][G_\kappa]$.

Lemma 4.1. *There is a filter $H \in V[G][G_\kappa]$ R-generic$/M[G][G_\kappa]$.*

Proof. Since $V \models |P_\kappa| = \kappa$ we have $M \models |jP_\kappa| = j\kappa$ and therefore the factor R has cardinality $j\kappa$ in $M[G][G_\kappa]$. Thus $M[G][G_\kappa] \models |\mathcal{P}(R)| = (j\kappa)^+$ because of GCH. Put

$$\mathcal{D} = \{D \in M[G][G_\kappa];\ D \text{ is a dense subset of } R\}$$

then the cardinality of $\mathcal{D}$ in $V[G][G_\kappa]$ is same as the cardinality of $(j\kappa)^{+M}$ which is κ^+. Now use the fact that R is κ-closed to get a generic filter $H \in V[G][G_\kappa]$. $\square$

Consequently j can be lifted to

$$j^* : V[G] \to M[G][G_\kappa][H]$$

where j^* is defined in $V[G][G_\kappa]$. Next we need to prove the following important lemma:

Lemma 4.2. *For any $\alpha < \kappa^+$*

$$[G_\kappa \restriction \alpha]^j \ \in\ j^*(Q_\kappa|\alpha)$$

where $[G_\kappa \restriction \alpha]^j$ is defined as in Lemma 2.5.

Proof. Let j_δ denote the elementary embedding $j_\delta : V \to M_\delta = V^\kappa/\mathcal{U}^\kappa_\delta$ for $\delta < \theta$. It follows from Lemma 2.5 and the proof of Lemma 2.3 that for any $\delta < \theta$

$$(*)\qquad M_\delta \models 1 \Vdash_{j_\delta P_\kappa} \forall H \in M_\delta[G^*]\ \ Q_\kappa|\alpha\text{-generic}/M_\delta[G] \\ [H]^j \in j_\delta(Q_\kappa|\alpha)_{/G^*}$$

Denote this formula $\varphi(j_\delta P_\kappa, Q_\kappa|\alpha, j_\delta(Q_\kappa|\alpha))$.

Now we need to introduce the notion of a canonical name. We say that $f \in V^\kappa$ is a canonical name for $x \in V$ iff for any measure $\mathcal{U}$ over κ the set x belongs to the transitive collapse $V^\kappa/\mathcal{U}$ and is equal to $[f]_\mathcal{U}$. Let

$$C = \{x \in V;\ x \text{ has a canonical name}\}.$$

Obviously $V_\kappa \subseteq C$ and $C^{\leq\kappa} \subseteq C$. Since $P_\alpha \in V_\kappa$ for $\alpha < \kappa$ we get that $P_\kappa \in C$ and $Q_\kappa|\alpha \in C$. Let f be the canonical name for $Q_\kappa|\alpha$. Then by the Loś Theorem (*) is equivalent to

$$\{\beta < \kappa;\ V \models \varphi(P_\kappa, f(\beta), Q_\kappa|\alpha)\} \in \mathcal{U}^\kappa_\delta.$$

Since this is true for any $\delta < \theta$ and $\mathcal{U}^\kappa_\theta$ is a repeat point it follows

$$\{\beta < \kappa;\ V \models \varphi(P_\kappa, f(\beta), Q_\kappa|\alpha)\} \in \mathcal{U}^\kappa_\theta$$

which (again by the Loś Theorem) means that

$$M \models \varphi(jP_\kappa, Q_\kappa|\alpha, j(Q_\kappa|\alpha))$$

In particular for $G^* = G * G_\kappa * H$ and $G_\kappa \restriction \alpha \in M[G^*]$ it says that

$$[G_\kappa \restriction \alpha]^j \in j(Q_\kappa|\alpha)_{/G^*} = j^*(Q_\kappa|\alpha).$$

□

Lemma 4.3. *There is a j^*Q_κ-generic/$M[G*G_\kappa*H]$ filter H^* such that for every $\alpha < \kappa^+$ the condition $[G_\kappa \restriction \alpha]^j$ is in H^*.*

Proof. Put

$$\tilde{Q} = \{q \in j^*Q_\kappa;\ \forall\beta < j(\kappa^+) : q_\beta = \emptyset \text{ or } \max(q_\beta) \geq \kappa\}$$

and

$$\mathcal{D} = \{a \in M[G * G_\kappa * H];\ a \subseteq \tilde{Q} \text{ is a maximal antichain}\}.$$

It follows from the κ^+-c.c. of Q_κ that

$$V[G] \models \forall a \subseteq Q_\kappa :\ \text{if } a \text{ is an antichain then } \exists\alpha < \kappa^+ :\ a \subseteq Q_\kappa|\alpha$$

so

$$M[G * G_\kappa * H] \models \forall a \subseteq j^*Q_\kappa :\ \text{if } a \text{ is an antichain then } \exists\alpha < j(\kappa^+) :\ a \subseteq j^*Q_\kappa|\alpha\ .$$

Moreover the cardinality of the power set of $j^*Q_\kappa|\alpha$ in $M[G * G_\kappa * H]$ is at most $j(\kappa^+)$. Thus the cardinality of $\mathcal{D}$ in $M[G * G_\kappa * H]$ is $j(\kappa^+)$ and in $V[G * G_\kappa]$ the cardinality is κ^+. Let $\langle a_\alpha;\ \alpha < \kappa^+\rangle$ be an enumeration of $\mathcal{D}$ in which each maximal antichain occurs cofinally many times. Observe that $\tilde{Q}$ is κ-closed in $V[G*G_\kappa]$. Now it is easy to construct in $V[G * G_\kappa]$ a descending sequence of conditions $\langle q_\alpha;\ \alpha < \kappa^+\rangle \subseteq \tilde{Q}$ with the following properties:

$$(i)\ q_\alpha \in j^*(Q_\kappa|\alpha),$$
$$(ii)\ q_\alpha \leq [G_\kappa \restriction \alpha]^j,$$
$$(iii)\ \text{if } a_\alpha \subseteq j^*(Q_\kappa|\alpha) \text{ then } q_\alpha \text{ strenghtens a condition in } a_\alpha.$$

The sequence $\langle q_\alpha;\ \alpha < \kappa^+\rangle$ generates a j^*Q_κ-generic/$M[G * G_\kappa * H]$ filter H^* such that each $[G_\kappa \restriction \alpha]^j$ is in H^*. □

It means that $p \in G_\kappa$ implies $j^*(p) \in H^*$ and consequently j^* is lifted to

$$j^{**} : V[G * G_\kappa] \to M[G * G_\kappa * H * H^*]$$

in $V[G * G_\kappa]$. We have proved that κ is measurable in $V[G * G_\kappa]$.

5. Generalizations and questions.

We say that $S = \langle S_\lambda; \lambda \leq \kappa \rangle$ is a generalized coherent sequence of measures if for any $\lambda \leq \kappa$ the set S_λ is a set of measures over λ and for any $U \in S_\lambda$

$$j_U(S)(\lambda) = S_\lambda \restriction U = \{V \in S_\lambda; V \lhd U\}.$$

For example if each S_λ is the set of all measures over λ then the sequence is coherent. Suppose now that GCH holds, S is a generalized coherent sequence and moreover there are separating sets of regular cardinals $\langle X_U; U \in S_\kappa \rangle$, i.e. $X_U \in V$ iff $U = V$ for $U, V \in S_\kappa$. By a straightforward modification of our construction we get a generic extension $V(P_{\kappa+1})$ preserving cardinalities, cofinalities and GCH with the following properties:

1. $\langle X_U; U \in S_\kappa \rangle$ forms a maximal antichain of stationary subsets of Reg in $V(P_{\kappa+1})$.
2. If $U, V \in S_\kappa$ and $S \subseteq X_U, T \subseteq X_V$ are stationary then $S < T$ (in $V(P_{\kappa+1})$) iff $U \lhd V$. If $U \in S_\kappa$ and $S \subseteq Sing, T \subseteq X_U$ are stationary then $S < T$. Consequently $(\text{Mitchell order})^V(\kappa) = o^{V(P_{\kappa+1})}(\kappa)$.

Moreover if κ has a repeat point (in a generalized sense) then κ is measurable in the generic extension. It is shown in [Ba85] that any prewellordering P with $|P| < \kappa$ can be represented as the set of all measures over κ. That does not give us anything new - in that case Full Reflection again holds in the resulting model. However recent papers of Cummings ([Cu92a], [Cu92b]) provide models with a rather complex structure of the Mitchell order. Using the model of [Cu92a] that satisfies GCH we can for example construct a generalized coherent sequence S such that S_κ is isomorphic to the four element poset of the type

$$\begin{array}{ccc} \circ & & \circ \\ \downarrow & \searrow & \downarrow \\ \circ & & \circ \end{array}$$

Thus in the resulting model κ is 2-Mahlo and we get two disjoint sets of inaccessible non-Mahlo cardinals $X_1, X_2 \subset E_0$ and two disjoint sets of 1-Mahlo cardinals $Y_1, Y_2 \subset E_1$ so that for any stationary $S_1 \subseteq X_1$, $S_2 \subseteq X_2$ the following holds:

$$S_1 < Y_1 \text{ but } S_1 \not< Y_2, Tr(S_1) = Y_1 \text{ (mod NS)}$$

$$S_2 < Y_1 \text{ and } S_2 < Y_2, Tr(S_2) = E_1 \text{ (mod NS)}.$$

The following question immediately comes to mind:

Question 1. *Does the consistency of Full Reflection at a measurable cardinal imply the consistency of a cardinal with a repeat point?*

Let $V(P_{\kappa+1})$ be our generic extension, let U denote the measure on κ and $C[E^\lambda_\delta]$ the filters of subsets of λ generated in $V(P_{\kappa+1})$ by closed unbounded sets and the canonical stationary set E^λ_δ. Let $\mathcal{F}$ code all these filters, then in $L[\mathcal{F}, U]$ we get back the original measures and U becomes a repeat point of κ. Hence we can ask a more specific question and conjecture that the answer is yes.

Question 2. *Suppose that Full Reflection holds at all $\lambda \leq \kappa$, κ is measurable and canonical stationary sets E^λ_δ of all orders exist. Let $C[E^\lambda_\delta], \mathcal{F}, U$ be as above. Is it then true that all $C[E^\lambda_\delta] \cap L[\mathcal{F}, U]$ are measures in $L[\mathcal{F}, U]$ and $U \cap L[\mathcal{F}, U]$ is a repeat point of κ?*

Another way to state an equiconsistency result would be to improve our construction so that the filters $C[E^\lambda_\delta]$ are λ^+-saturated. If we add this property of the filters to the assumptions of question 2 then using a method of [J84] or [JWo85] we can prove that the answer is yes. Unfortunately if we analyze our construction we find out that already the filters F^λ_δ are not λ^+-saturated. We can try to use the ideas of [JWo85] and instead of extensions of $j = j_\delta : V \to M$ into $j^* : V(P_\kappa * Q) \to V(j(P_\kappa * Q))$ constructed in $V(j(P_\kappa * Q))$ work only with extensions constructed in $V(P_\kappa * (jP_\kappa)^\kappa)$. We can get the construction to work but the filters still will not be saturated. Hence we conjecture that the answer of the following question is no.

Question 3. *Is it consistent that Full Reflection holds at κ measurable, all canonical stationary sets E^κ_δ exist and the filters $C[E^\kappa_\delta]$ are κ^+-saturated?*

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