

POSSIBLE PCF ALGEBRAS

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ABSTRACT. There exists a family $\{B_\alpha\}_{\alpha < \omega_1}$ of sets of countable ordinals such that

- (1) $\max B_\alpha = \alpha$,
- (2) if $\alpha \in B_\beta$ then $B_\alpha \subseteq B_\beta$,
- (3) if $\lambda \leq \alpha$ and λ is a limit ordinal then $B_\alpha \cap \lambda$ is not in the ideal generated by the B_β , $\beta < \alpha$, and by the bounded subsets of λ ,
- (4) there is a partition $\{A_n\}_{n=0}^\infty$ of ω_1 such that for every α and every n , $B_\alpha \cap A_n$ is finite.

1. Introduction.

In [3], [4], [5] and [6] the second author developed the theory of possible cofinalities (pcf), and proved, among others, that if \aleph_ω is a strong limit cardinal then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$ as well as $2^{\aleph_\omega} < \aleph_{\omega_4}$. The latter inequality is established via an analysis of the structure of pcf; in particular, it is shown that if $\aleph_4 \leq |\text{pcf}\{\aleph_n\}_{n=0}^\infty|$ then a certain structure exists on ω_4 , and then it is proved that such a structure is impossible. (Cf. [5], [1] and [2] for details.) One might hope that by investigating this structure one could possibly derive a contradiction for \aleph_3 , \aleph_2 or even \aleph_1 .

A major open problem in the theory of singular cardinals (or in the pcf theory) is whether it is consistent that \aleph_ω is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$; or whether the set $\text{pcf}\{\aleph_n\}_{n=1}^\infty$ can be uncountable. If we make this assumption, we obtain a certain structure on ω_1 . The structure is described in Theorem 2.1. Unlike in the ω_4 -case, the structure so obtained is not impossible: in Theorem 3.1 we show that

The first author was supported in part by an NSF grant DMS-8918299, and by the U.S.-Israel Binational Science Foundation.

The second author was partially supported by the U.S.-Israel Binational Science Foundation. Publication No. 476.

there exists a structure on ω_1 which has the properties listed in the abstract, and consequently has the properties listed in Theorem 2.1.

In Section 2, all facts on Shelah's pcf theory not proved explicitly can be found in the expository articles [1] and [2]. In Section 3 we assume rudimentary knowledge of forcing.

2. A consequence of “pcf $\{\aleph_n\}_{n=0}^\infty$ is uncountable”.

Theorem 2.1. *If pcf $\{\aleph_n\}_{n=0}^\infty$ is uncountable, then there exist sets B_α , $\alpha < \omega_1$, of countable ordinals with the following properties:*

- (a) *For every $\alpha < \omega_1$, $\max B_\alpha = \alpha$.*
- (b) *For all $\alpha, \beta < \omega_1$, if $\alpha \in B_\beta$ then $B_\alpha \subseteq B_\beta$.*
- (c) *For every limit ordinal $\lambda < \omega_1$, $B_\lambda \cap \lambda$ is unbounded in λ .*
- (d) *There is a closed unbounded set C of countable limit ordinals such that for all $\lambda \in C$ and for all $\alpha \geq \lambda$, the set $B_\alpha \cap \lambda$ is not in the ideal generated by the sets B_β , $\beta < \alpha$, and by bounded subsets of λ . (I.e. $B_\alpha \cap \lambda \not\subseteq \gamma \cup B_{\beta_1} \cup \dots \cup B_{\beta_k}$, for any $\gamma < \lambda$, and any $\beta_1, \dots, \beta_k < \alpha$.)*
- (e) *Every unbounded set $X \subseteq \omega_1$ has an initial segment $X \cap \gamma$ that is not in the ideal generated by the sets B_α , $\alpha < \omega_1$.*
- (f) *Moreover, (e) remains true in every extension M of the ground model that preserves cardinals and cofinalities, and has the property that every countable set of ordinals in M is covered by a countable set in the ground model.*

Proof. Let $a = \text{pcf}\{\aleph_n\}_{n=0}^\infty$ and assume that a is uncountable. Applying the pcf theory, one obtains (cf. [6], Main Theorem) sets b_λ , $\lambda \in a$, (*generators*) together with sequences of functions f_i^λ ($i < \lambda$) in $\prod a$. As a contains all regular cardinals $\lambda < \aleph_{\omega_1}$, we let, for each $\alpha < \omega_1$

$$B_\alpha = \{\xi : \aleph_{\xi+1} \in b_{\aleph_{\alpha+1}}\}.$$

Property (a) is immediate. Property (b) is the *transitivity* of generators; such generators can be found (cf. [1], Lemma 6.9).

Property (c) is a consequence of the fact that for every countable limit ordinal λ , there exists an increasing sequence α_n , $n < \omega$, with limit λ , and an ultrafilter D on ω such that $\text{cof}\left(\prod_{n=0}^{\infty} \aleph_{\alpha_n+1}/D\right) = \aleph_{\lambda+1}$ (cf. [1], Theorem 2.1).

Property (d): Let γ_i , $i < \omega_1$, be a continuous increasing sequence of countable ordinals constructed as follows: Given γ_i , we first note that $\aleph_{\omega_1+1} \in \text{pcf}[\aleph_{\gamma_i+1}, \aleph_{\omega_1})$ (by [1], Theorem 2.1), and by the Localization Theorem [6], there is a $\gamma_{i+1} < \omega_1$ such that $\aleph_{\omega_1+1} \in \text{pcf}[\aleph_{\gamma_{i+1}}, \aleph_{\gamma_i+1})$. Let C be the set of all limit points of the sequence $\{\gamma_i\}_{i < \omega_1}$.

Now let $\lambda \in C$, $\alpha \geq \lambda$, $\gamma < \lambda$, and $\beta_1, \dots, \beta_k < \alpha$. We find γ_i such that $\gamma < \gamma_i < \lambda$. By [1], Theorem 2.1, we have $\aleph_{\alpha+1} \in \text{pcf}[\aleph_{\gamma_i+1}, \aleph_{\gamma_i+1})$ and so there is an ultrafilter D on $[\gamma_i + 1, \gamma_i)$ such that $\text{cof}(\prod \aleph_{\xi+1}/D) = \aleph_{\alpha+1}$. By the definition of generators, we have $B_\alpha \in D$ while $B_{\beta_i} \notin D$ ($i = 1, \dots, k$), and (d) follows.

Property (e): If $X \subseteq \omega_1$ is unbounded, then $\max \text{pcf}\{\aleph_{\alpha+1} : \alpha \in X\} \geq \aleph_{\omega_1}$, and by the Localization Theorem, there is a countable γ such that $\max \text{pcf}\{\aleph_{\alpha+1} : \alpha \in X \cap \gamma\} \geq \aleph_{\omega_1}$. Now if $\alpha_1, \dots, \alpha_k$ are countable ordinals, we cannot have $X \cap \gamma \subseteq B_{\alpha_1} \cup \dots \cup B_{\alpha_k}$, because $\max \text{pcf}(b_{\aleph_{\alpha_1+1}} \cup \dots \cup b_{\aleph_{\alpha_k+1}}) = \max\{\aleph_{\alpha_i+1} : i = 1, \dots, k\} < \aleph_{\omega_1}$.

Property (f): Let M be an extension of the ground model V that preserves cardinals and cofinalities, and assume further that every countable set of ordinals in M is covered by a countable set in V .

To show that (e) is true in M , it suffices to show that the generators b_λ are generators of the pcf structure in M . For that, it is enough to verify that the sequences f_i^λ ($i < \lambda$) are increasing cofinal sequences in $\prod a$ (modulo the appropriate ideals $J_{<\lambda}$). Since M has the same cardinals and cofinalities, the claim follows upon the observation that for every regular $\lambda < \aleph_{\omega_1}$, every function $f \in \prod b_\lambda$ in M is majorized by some function $g \in \prod b_\lambda$ in V .

3. Existence of the family $\{B_\alpha\}_{\alpha < \omega_1}$.

Theorem 3.1. *There exist a partition $\{A_n\}_{n=0}^\infty$ of ω_1 , and a family $\{B_\alpha\}_{\alpha < \omega_1}$ of countable sets of countable ordinals such that*

- (a) *For every $\alpha < \omega_1$, $\max B_\alpha = \alpha$.*
- (b) *For all $\alpha, \beta < \omega_1$, if $\alpha \in B_\beta$ then $B_\alpha \subseteq B_\beta$.*
- (c) *For every limit ordinal $\lambda < \omega_1$ and for all $\alpha \geq \lambda$, $B_\alpha \cap \lambda \not\subseteq \gamma \cup B_{\beta_1} \cup \dots \cup B_{\beta_k}$ for any $\gamma < \lambda$ and any $\beta_1, \dots, \beta_k < \alpha$.*
- (d) *For all $\alpha < \omega_1$ and all n , $B_\alpha \cap A_n$ is finite.*

Corollary 3.2. *If M is any \aleph_1 -preserving extension of V , then every unbounded set $X \subseteq \omega_1$ in M has an initial segment $X \cap \gamma$ that is not in the ideal generated by the sets B_α , $\alpha < \omega_1$.*

Proof. By (d), any set in the ideal has a finite intersection with each A_n . If $X \subseteq \omega_1$ is unbounded then some $X \cap A_n$ is uncountable, and so some $(X \cap \gamma) \cap A_n$ is infinite. Hence $X \cap \gamma$ is not in the ideal.

To construct the structure described in Theorem 3.1 we shall first define a forcing notion and prove that it forces such a structure to exist in the generic extension. The forcing notion that we use satisfies the countable chain condition and consists of finite conditions consisting of countable ordinals and relations between countable ordinals. Using a general method due to the second author [7] we then conclude that such a structure exists in V .

Definition 3.3.

A forcing condition is a quadruple $p = (S_p, \pi_p, b_p, u_p)$ such that

- (i) S_p is a finite subset of ω_1 ,
- (ii) b_p is a function from $S_p \times S_p$ into $\{0, 1\}$ such that

$$b_p(\alpha, \alpha) = 1 \quad (\alpha \in S_p)$$

$$b_p(\alpha, \beta) = 0 \quad (\alpha, \beta \in S_p, \alpha < \beta)$$
 if $b_p(\alpha, \beta) = 1$ and $b_p(\beta, \gamma) = 1$ then $b_p(\alpha, \gamma) = 1 \quad (\alpha, \beta, \gamma \in S_p)$

- (iii) u_p is a natural number,
- (iv) π_p is a function from S_p into $\{0, \dots, u_p - 1\}$ such that for all α and β in S_p , if $b_p(\alpha, \beta) = 1$ and $\beta < \alpha$ then $\pi_p(\beta) \neq \pi_p(\alpha)$,

[Motivation: S is the support of the condition, $\pi(\alpha) = n$ forces $\alpha \in A_n$, $b(\alpha, \beta) = 1$ forces $\beta \in B_\alpha$ and $b(\alpha, \beta) = 0$ forces $\beta \notin B_\alpha$.]

A condition $r = (S_r, \pi_r, b_r, u_r)$ is stronger than $p = (S_p, \pi_p, b_p, u_p)$ if

- (i) $S_r \supseteq S_p$,
- (ii) b_r extends b_p ,
- (iii) π_r extends π_p
- (iv) $u_r \geq u_p$,
- (v) for all $\alpha \in S_p$ and all $\beta \in S_r - S_p$, if $b_r(\alpha, \beta) = 1$ then $\pi_r(\beta) \geq u_p$.

It is easy to verify that “stronger than” is a transitive relation.

Definition 3.4.

If $p = (S_p, \pi_p, b_p, u_p)$ is a condition and η is a countable ordinal, we let

$$p \upharpoonright \eta = (S_p \cap \eta, \pi_p \upharpoonright \eta, b_p \upharpoonright (\eta \times \eta), u_p).$$

Clearly, $p \upharpoonright \eta$ is a condition and p is stronger than $p \upharpoonright \eta$.

Lemma 3.5 (Amalgamation). *If p and q are conditions and η a countable ordinal such that q is stronger than $p \upharpoonright \eta$ and $S_q \subseteq \eta$ then there exists a condition r such that r is stronger than both p and q (and such that $S_r = S_p \cup S_q$).*

Proof. Note that $u_q \geq u_p$. We let $S_r = S_p \cup S_q$, $\pi_r = \pi_p \cup \pi_q$ and $u_r = u_q$. We define b_r as follows: if α and β are both in S_p (both in S_q) then we let $b_r(\alpha, \beta) = b_p(\alpha, \beta)$ (we let $b_r(\alpha, \beta) = b_q(\alpha, \beta)$.) If $\alpha \geq \eta$ is in S_p and if $\beta < \eta$ is in $S_q - S_p$ then we let $b_r(\alpha, \beta) = 1$ if and only if there exists a $\gamma < \eta$ in S_p such that $b_p(\alpha, \gamma) = 1$ and $b_q(\gamma, \beta) = 1$. Otherwise we let $b_r(\alpha, \beta) = 0$.

Next we verify that r is a condition. It is easy to see that requirement (ii) from the definition is satisfied. To verify (iv), the only case we need to worry about is

when $b_r(\alpha, \beta) = 1$ where $\alpha \geq \eta$ is in S_p and $\beta < \eta$ is in $S_q - S_p$. In this case, $\pi_q(\beta) \geq u_p$ (because q is stronger than $p \upharpoonright \eta$ and $b_q(\gamma, \beta) = 1$ for some $\gamma \in S_p \cap \eta$) while $\pi_p(\alpha) < u_p$, and so $\pi_r(\beta) \neq \pi_r(\alpha)$.

Since $r \upharpoonright \eta = q$, r is stronger than q . In order to show that r is stronger than p we only need to verify condition (v), and only for the case when $\alpha \geq \eta$ is in S_p and $\beta < \eta$ is in $S_q - S_p$. This is however exactly the argument in the preceding paragraph.

Lemma 3.6. *The forcing satisfies the countable chain condition.*

Proof. Given \aleph_1 conditions, we first find \aleph_1 of them whose supports form a Δ -system, with a root A , i.e. $S_{p_\xi} \cap S_{p_\eta} = A$ whenever $\xi < \eta$, and such that $\beta < \alpha$ whenever $\beta \in S_{p_\xi}$ and $\alpha \in S_{p_\eta} - A$. Then \aleph_1 of them have the same restrictions of π and b to the root A , and the same u .

Now it follows from Lemma 3.5 that any two such conditions are compatible.

Let G be a generic set of conditions. In $V[G]$, we let, for each $\alpha < \omega_1$ and each $n < \omega$,

$$(3.7) \quad B_\alpha = \{\beta : b(\alpha, \beta) = 1 \text{ for some condition } (S, \pi, b, u) \in G\},$$

$$(3.8) \quad A_n = \{\alpha : \pi(\alpha) = n \text{ for some condition } (S, \pi, b, u) \in G\}.$$

Clearly, $\max B_\alpha = \alpha$, and if $\alpha \in B_\beta$ then $B_\alpha \subseteq B_\beta$. The sets A_n are mutually disjoint subsets of ω_1 .

Lemma 3.9. *For every $\alpha < \omega_1$ the set of all conditions p with $\alpha \in S_p$ is dense. For every n the set of all conditions p with $u_p \geq n$ is dense.*

Proof. If q is a condition and $\alpha \notin S_q$ then let $S_p = S_q \cup \{\alpha\}$, let $b_p(\alpha, \alpha) = 1$, $u_p = u_q + 1$ and $\pi_p(\alpha) = u_q$. Then p is a condition stronger than q . The proof of the second statement is similar.

Corollary 3.10. $\{A_n\}_{n=0}^\infty$ is a partition of ω_1 .

Lemma 3.11. For all $\alpha < \omega_1$ and all n , $B_\alpha \cap A_n$ is finite.

Proof. Let α and n be given, and let $p = (S_p, \pi_p, b_p, u_p)$ be a condition. We shall find a stronger condition q that forces that $B_\alpha \cap A_n$ is finite.

There is a condition $q = (S_q, \pi_q, b_q, u_q)$ stronger than p such that $\alpha \in S_q$ and that $u_q > n$. We claim that q forces that $B_\alpha \cap A_n \subseteq S_q$.

If β is an ordinal not in S_q and if $r = (S_r, \pi_r, b_r, u_r)$ is a stronger condition that forces $\beta \in B_\alpha$ then because $b_r(\alpha, \beta) = 1$, we have $\pi_r(\beta) \geq u_q > n$, and so r forces $\beta \notin A_n$. Thus q forces $B_\alpha \cap A_n \subseteq S_q$.

Lemma 3.12. Let $\lambda < \omega_1$ be a limit ordinal, let $\alpha \geq \lambda$, and let $\gamma < \lambda$ and $\alpha_1, \dots, \alpha_k < \alpha$. There exists a $\beta \geq \gamma, \beta < \lambda$, such that $\beta \in B_\alpha$ and $\beta \notin B_{\alpha_1}, \dots, \beta \notin B_{\alpha_k}$.

Proof. Let $p = (S_p, \pi_p, b_p, u_p)$ be a condition. We may assume that $\alpha, \alpha_1, \dots, \alpha_k \in S_p$. Let $\beta < \lambda$ be such that $\beta \geq \gamma$ and $\beta \notin S_p$.

Let $\eta = \alpha + 1$ and $S = S_p \cap \eta$. We let $S_q = S \cup \{\beta\}$, $u_q = u_p + 1$, $\pi_q \upharpoonright S = \pi_p \upharpoonright S$, $\pi_q(\beta) = u_p$, $b_q \upharpoonright (S \times S) = b_p \upharpoonright (S \times S)$, $b_q(\alpha, \beta) = b_q(\beta, \beta) = 1$, and $b_q(\beta, \xi) = b_q(\xi, \beta) = 0$ otherwise. The condition $q = (S_q, \pi_q, b_q, u_q)$ is stronger than $p \upharpoonright \eta$, has $S_q \subseteq \eta$ and forces $\beta \in B_\alpha$, $\beta \notin B_{\alpha_1}, \dots, \beta \notin B_{\alpha_k}$. By Lemma 3.5 there is a condition r that is stronger than both p and q .

This concludes the proof that the forcing from Definition 3.3 adjoins a structure described in Theorem 3.1. That such a structure exists in V is a consequence of the general theorem (Theorem 1.9) in [7]. Our forcing is ω_1 -uniform in the sense of Definition 1.1 in [7] and the dense sets needed to produce the B_α and the A_n in Theorem 3.1 conform to Definition 1.4 in [7] and hence the method of [7] applies.

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