SINGULAR CARDINAL PROBLEM: SHELAH'S THEOREM ON $2^{\aleph_\omega}$

Thomas Jech

The Pennsylvania State University

Abstract. This is an expository paper giving a complete proof of a theorem of Saharon Shelah: If $2^{\aleph_n} < \aleph_\omega$ for all $n < \omega$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

1. Introduction.

The *singular cardinal problem* is to describe the possible size of $2^{\aleph_\alpha}$ for a singular cardinal $\aleph_\alpha$ under the assumption that $2^{\aleph_\xi} < \aleph_\alpha$ for all $\xi < \alpha$. The problem is still not completely solved, and various partial results involve forcing, large cardinals, inner models and combinatorial methods.

The present paper gives an exposition of Shelah's theory of *possible cofinalities* (pcf) and presents a complete proof of the following theorem:

Main Theorem. (Shelah 1989) *If* $2^{\aleph_n} < \aleph_\omega$ *for all* n *then* $2^{\aleph_\omega} < \aleph_{\omega_4}$.

The theorem is the most spectacular application of Shelah's theory of possible cofinalities of ultrapowers of sets of regular cardinals. If D is an ultrafilter on a set A then ordinal functions on A are linearly ordered by the relation

$$f \le_D g \qquad \text{if and only if} \qquad \{a \in A : f(a) \le g(a)\} \in D.$$

In particular, for every ultrafilter D on ω,

$$\prod_{n=0}^{\infty} \aleph_n / D$$

is a linearly ordered set, which has a cofinality $\operatorname{cof}\left(\prod_{n=0}^{\infty} \aleph_n / D\right)$. (This cofinality is a regular cardinal $\le \aleph_\omega^{\aleph_0}$.) It turns out that among cofinalities of the ultrapowers $\prod_n \aleph_n / D$ for all possible ultrafilters D there is a maximal one, which we denote

$$\max \operatorname{cof} \prod_{n=0}^{\infty} \aleph_n,$$

and its significance for cardinal arithmetic stems from the following:

1980 *Mathematics Subject Classification* (1985 *Revision*). 03E.

Key words and phrases. Singular cardinals, pcf.

Based on lectures given at Université Paris 7 and at Charles University in Prague in June and July 1990. Supported in part by an NSF grant DMS-8918299.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem A. (Shelah 1980) *If* $2^{\aleph_n} < \aleph_\omega$ *for all* n, *then*

$$2^{\aleph_\omega} = \max\operatorname{cof}\prod_{n=0}^{\infty}\aleph_n.$$

The Main Theorem thus follows from this:

Theorem B. (Shelah 1989) $\max\operatorname{cof}\prod_{n=0}^{\infty}\aleph_n < \aleph_{\omega_4}$.

We give a proof of Theorem B below under the assumption that $\aleph_\omega$ is a strong limit cardinal; it should be noted that the theorem is true even without this assumption.

The main reason for writing this paper was to present a self-contained proof of the Main Theorem. We develop just enough of Shelah's theory to be able to prove the theorem. The proof presented here is considerably shorter than either the original proof in [Sh400], or the proof given in [BM]. It turns out that the assumption that $\aleph_\omega$ is strong limit allows for major simplifications. The main ideas used in the proof are all due to S. Shelah.

The paper is based on lectures that I gave in June 1990 to the Paris Logic Group at UFR de Mathématiques, Université Paris 7, and in July 1990 to the set theory group in Prague, at Charles University and the Mathematical Institute of the Czechoslovak Academy of Sciences. I would like to thank these colleagues for their hospitality and attention.

The paper is organized as follows: Section 2 studies the partial ordering in reduced products of ordinals, and Section 3 deals with the nonstationary ideal. Section 4 gives a proof of Theorem A. Section 5 is an introduction to Shelah's pcf theory, and Section 6 applies the pcf theory to prove Theorem B.

2. Ordinal functions modulo an ideal.

Let A be an infinite set, and let I be an ideal on A. Let I^+ denote the set of all $X \subseteq A$ that are not in I. For ordinal functions f, g on A we define

$$\begin{array}{lll} f =_I g & \text{if} & \{a \in A : f(a) \neq g(a)\} \in I \\ f \leq_I g & \text{if} & \{a \in A : f(a) > g(a)\} \in I \\ f \lneqq_I g & \text{if} & f \leq_I g \text{ and } \{a \in A : f(a) < g(a)\} \in I^+ \\ f <_I g & \text{if} & \{a \in A : f(a) \geq g(a)\} \in I \end{array}$$

The relation $\leq_I$ is a partial ordering (of equivalence classes). If S is a set of ordinal functions on A then g is an *upper bound* of S in $\leq_I$ if $f \leq_I g$ for all $f \in S$, and g is a *least upper bound* of S if it is an upper bound and if $g \leq_I h$ for every upper bound h.

A transfinite sequence $\{f_\alpha : \alpha < \vartheta\}$ in $\leq_I$ is an *increasing* sequence if $f_\alpha \lneqq f_\beta$ whenever $\alpha < \beta$; it is *strictly increasing* if $f_\alpha < f_\beta$ for $\alpha < \beta$. Similarly for *decreasing*.

Lemma 2.1. *Let* λ *be a regular cardinal,* $\lambda > 2^{|A|}$. *Every increasing sequence* $\{f_\alpha : \alpha < \lambda\}$ *in* $\leq_I$ *has a least upper bound.*

Proof. Let $\{g_\xi : \xi < \vartheta\}$ be a maximal decreasing sequence of upper bounds of the set $\mathcal{F} = \{f_\alpha : \alpha < \lambda\}$. We will show that the sequence has a last element (which is the least upper bound).

First we show that the cardinality of ϑ is at most $2^{|A|}$. If not, consider the following partition F of $[(2^{|A|})^+]^2$ into $|A|$ classes:

$$F(\xi, \eta) = \text{ some } a \in A \text{ such that } g_\xi(a) > g_\eta(a) \qquad (\xi < \eta).$$

By the Erdős-Rado Theorem [ER], F has an infinite homogeneous set X; there is some $a \in A$ such that $F(\xi, \eta) = a$ for all $(\xi, \eta) \in [X]^2$. Let $\xi_0 < \xi_1 < \xi_2 < \dots$ be a sequence of ordinals in X. We have

$$g_{\xi_0}(a) > g_{\xi_1}(a) > g_{\xi_2}(a) > \cdots ,$$

a contradiction.

Thus $|\vartheta| \le 2^{|A|}$. We will show that for every limit ordinal $\eta \le \vartheta$, the set $\{g_\xi : \xi < \eta\}$ has a lower bound g that is an upper bound of $\mathcal{F}$. This will complete the proof.

Let $\eta \le \vartheta$ be a limit ordinal. For each $a \in A$, let $S_a = \{g_\xi(a) : \xi < \eta\}$, and let $H = \prod_{a \in A} S_a$. We have $|H| \le |\eta|^{|A|} = 2^{|A|} < \lambda$.

For each $h \in H$ let $\alpha_h < \lambda$ be such that $f_{\alpha_h} \not\le_I h$ if h is not an upper bound of $\mathcal{F}$, and let $\alpha = \sup\{\alpha_h : h \in H\}$. Define $g \in H$ as follows:

$$g(a) = \text{ least element of } S_a \text{ that is } \ge f_\alpha(a).$$

The function g is an upper bound of $\mathcal{F}$ because $g \ge_I f_\alpha \ge f_{\alpha_g}$, and if g were not an upper bound we would have $g \not\ge_I f_{\alpha_g}$.

Finally, g is a lower bound of $\{g_\xi : \xi < \eta\}$: If $\xi < \vartheta$ then for all a, $g_\xi(a) \in S_a$, and for almost all a, $g_\xi(a) \ge f_\alpha(a)$. Hence $g_\xi \ge_I g$. $\square$

Remark 2.2. The least upper bound g of the sequence $\{f_\alpha : \alpha < \lambda\}$ has the following property:

If $h <_I g$ then $h <_I f_\alpha$ for some $\alpha < \lambda$.

We say that the sequence $\{f_\alpha\}_\alpha$ is *cofinal* in g.

Proof. Let $h <_I g$. For each $\alpha < \lambda$ let $X_\alpha = \{a \in A : h(a) < f_\alpha(a)\}$. Since $\lambda > 2^{|A|}$, there exist an X and a set $K \subseteq \lambda$ of size λ such that $X_\alpha = X$ for all $\alpha \in K$. We claim that $A - X \in I$, which completes the proof.

If $A - X$ is not in I, let g' be the function that is equal to g on X and to h on $A - X$, and we have $g' \lneqq_I g$. But g' is an upper bound of $\{f_\alpha : \alpha \in K\}$, and therefore of $\{f_\alpha : \alpha < \lambda\}$. Thus g is not a least upper bound, a contradiction. $\square$

Let S be a set of ordinal functions on A and let g be an $\le_I$-upper bound of S. We say that S is *bounded below* g if there is an $h <_I g$ that is an upper bound of S.

If $X \subseteq A$ and $X \notin I$ then we relativize the various concepts discussed above to X: We let $I \restriction X$ be the ideal generated by $I \cup \{A - X\}$ and then $f \le_I g$ on X, $f <_I g$ on X, (least) upper bound on X, cofinal in g on X, bounded below g on X etc. all refer to the ideal $I \restriction X$ rather than I itself.

Corollary 2.3. (The splitting lemma.) *Let I be an ideal on A, let λ be a regular cardinal, $\lambda > 2^{|A|}$, and let $\{f_\alpha : \alpha < \lambda\}$ be an increasing sequence in $\leq_I$. Let g be an upper bound of $\{f_\alpha\}_\alpha$. Then either $\{f_\alpha\}_\alpha$ is bounded below g, or $\{f\}_\alpha$ is cofinal in g, or A splits into two sets $A = X \cup Y$ such that $\{f_\alpha\}_\alpha$ is bounded below g on X and is cofinal in g on Y.*

Proof. Let f be a least upper bound of $\{f_\alpha\}_\alpha$ and let $X = \{a \in A : f(a) < g(a)\}$. $\square$

Now let us consider reduced products

$$\prod_{a\in A} \lambda_a / I$$

where $\{\lambda_a : a \in A\}$ are limit ordinal numbers and I is an ideal on A. A set $S \subseteq \prod_{a\in A} \lambda_a$ is *bounded*, *cofinal* etc. if it is bounded, cofinal etc. below the function $\{\lambda_a : a \in A\}$.

Let λ be a regular cardinal. A λ-*scale* (for I) on a set $X \subseteq A$, $X \notin I$, is a sequence $\{f_\alpha : \alpha < \lambda\} \subseteq \prod_{a\in A} \lambda_a$ that is $<_I$-strictly increasing on X and cofinal on X. A λ-*scale* is a λ-scale on A. If there is a λ-scale we say that $\prod_{a\in A} \lambda_a/I$ has *true cofinality* λ.

We say that $\prod_a \lambda_a/I$ is λ-*directed* (or simply that I is λ-*directed*) if every subset S of size $< \lambda$ is bounded.

If $B \subset A$ is such that $A - B \in I^+$ then $I[B]$ denotes the ideal generated by $I \cup \{B\}$; $I[B] = I \restriction (A - B)$.

Lemma 2.4. *Let λ be a regular cardinal, $\lambda > 2^{|A|}$, and assume that $\prod_{a\in A} \lambda_a/I$ is λ-directed. Then either $\prod_{a\in A} \lambda_a/I$ is λ^+-directed, or there is a λ-scale, or there exists a set $B \in I^+$ with $A - B \in I^+$, such that I has a λ-scale on B and that $I[B]$ is λ^+-directed.*

Proof. Assume that $\prod_{a\in A} \lambda_a/I$ is λ not λ^+-directed, and let S be a set of size λ that is not bounded. We can easily construct a strictly increasing sequence $\{f_\alpha : \alpha < \lambda\}$ (using λ-directedness) such that for every $f \in S$ there is an $\alpha < \lambda$ such that $f \leq_I f_\alpha$. As $\{f_\alpha\}_\alpha$ is unbounded, there is some $Y \in I^+$ such that $\{f_\alpha\}_\alpha$ is a scale on Y.

Now consider the collection $\mathcal{S}$ of all sets $Y \notin I$ that have a λ-scale, and for every such Y let $\{f^Y_\alpha\}_{\alpha<\lambda}$ be a λ-scale on Y. Consider the set $S = \{f^Y_\alpha : \alpha < \lambda, Y \in \mathcal{S}\}$; since $2^{|A|} < \lambda$, we have $|S| = \lambda$. Again, we can construct a strictly increasing sequence $\{f_\alpha : \alpha < \lambda\}$ such that for every $f \in S$ there is an $\alpha < \lambda$ such that $f \leq_I f_\alpha$.

Either $\{f_\alpha\}_\alpha$ is a λ-scale, or A splits into $A = X \cup B$ such that $\{f_\alpha\}_\alpha$ is bounded on X and cofinal on B. The set B has a λ-scale, and we claim that every set of size λ is bounded on X. If not, we repeat the argument above and find a $Y \subseteq X$ that has a scale; this contradicts the fact that S is bounded on X. $\square$

3. Uncountable cofinality and the nonstationary ideal.

In this section we prove a theorem that is closely related to the theorems of Silver [S] and of Galvin and Hajnal [GH].

Theorem 3.1. *Let κ be a regular uncountable cardinal, let $\aleph_\eta$ be a singular cardinal of cofinality κ and assume that $2^\kappa < \aleph_\eta$. Let I be the ideal of nonstationary subsets of η. Then $\prod_{\xi<\eta} \aleph_{\xi+1}/I$ has true cofinality $\aleph_{\eta+1}$.*

Remark. The theorem holds even without the assumption $2^\kappa < \aleph_\eta$ (see [BM], Lemma 6.3 for a proof). On the other hand, if we assume that ${\aleph_\xi}^\kappa < \aleph_\eta$ for all $\xi < \eta$ then *every* strictly increasing sequence of length $\aleph_{\eta+1}$ in $\prod_{\xi<\eta} \aleph_{\xi+1}$ is a scale; this can be used to prove Silver's theorem.

Proof. Let us fix a continuous increasing sequence $\{\eta(\xi) : \xi < \kappa\}$ with limit η such that $2^\kappa < \aleph_\eta(0)$; it suffices to consider $\prod_{\xi<\kappa} \aleph_{\eta(\xi)+1}/I$ where I is the nonstationary ideal on κ. It is not difficult to see that the reduced product is $\aleph_{\eta+1}$-directed. If there is no $\aleph_{\eta+1}$-scale then by Lemma 2.4 there is a stationary set S such that $\prod_{\xi\in S} \aleph_{\eta(\xi)+1}/I$ is $\aleph_{\eta+2}$-directed.

Thus let us assume, toward a contradiction, that there exists a stationary set $S_0 \subseteq \kappa$ such that every set of $\aleph_{\eta+1}$ functions in $\prod_{\xi<\kappa} \aleph_{\eta(\xi)+1}/I$ is bounded on S_0 (and such that each $\xi \in S_0$ is a limit ordinal).

For each limit ordinal $\beta < \aleph_{\eta+1}$ choose a closed unbounded subset C_β of β of order type $\operatorname{cf}\beta$ (which is $< \aleph_\eta$) and for every $\alpha < \aleph_{\eta+1}$ let $\mathcal{E}_\alpha = \{C_\beta \cap \alpha : \beta < \aleph_{\eta+1}\}$. Each $\mathcal{E}_\alpha$ has size $\le \aleph_{\eta+1}$ and consists of sets of size $< \aleph_\eta$.

Now we construct, by induction on α, a strictly increasing sequence $\{f_\alpha : \alpha < \aleph_{\eta+1}\}$ on S_0. Assume that $\{f_\nu : \nu < \alpha\}$ has been constructed. For each $E \in \mathcal{E}_\alpha$, let g^α_E be the pointwise supremum of $\{f_\nu : \nu \in E\}$, i.e. for all sufficiently large $\xi \in S_0$ (namely if $\aleph_{\eta(\xi)+1} > |E|$), $g^\alpha_E(\xi) = \sup\{f_\nu(\xi) : \nu \in E\}$). Let $f_\alpha \in \prod_{\xi\in S_0} \aleph_{\eta(\xi)+1}$ be an upper bound of the set (of size $\le \aleph_{\eta+1}$) $\{g^\alpha_E : E \in \mathcal{E}_\alpha\} \cup \{f_\nu : \nu < \alpha\}$.

Now let $h \in \prod_{\xi\in S_0} \aleph_{\eta(\xi)+1}$ be the least upper bound of $\{f_\alpha : \alpha < \aleph_{\eta+1}\}$. Since $h(\xi) < \aleph_{\eta(\xi)+1}$ for all $\xi \in S_0$, and $\aleph_{\eta(\xi)}$ is singular for every limit ξ, we have $\operatorname{cf} h(\xi) < \aleph_{\eta(\xi)}$ for every $\xi \in S_0$. Therefore there exist a stationary set $S \subseteq S_0$ and some $\gamma < \eta$ such that $\operatorname{cf} h(\xi) \le \aleph_\gamma$ for all $\xi \in S$ (and such that $\gamma < \eta(\xi)$ for all $\xi \in S$). For each $\xi \in S$ we choose a cofinal set $D_\xi \subseteq h(\xi)$ of cardinality $\le \aleph_\gamma$.

Now we construct, by induction on $\nu < \aleph_{\gamma+1}$, a strictly increasing (on S) sequence of functions $h_\nu \in \prod_{\xi\in S} D_\xi/I$ and an increasing continuous sequence $\{\alpha(\nu)\}_\nu$ such that $f_{\alpha(\nu)} <_I h_\nu <_I f_{\alpha(\nu+1)}$ on S, for all $\nu < \aleph_{\gamma+1}$. Let $\beta = \lim_{\nu\to\aleph_{\gamma+1}} \alpha(\nu)$.

Consider the (previously chosen) club $C = C_\beta \subseteq \beta$ (of size $\aleph_{\gamma+1}$). For every $\nu < \aleph_{\gamma+1}$, let ν' be the least $\nu' > \nu$ such that $\alpha(\nu') \in C$. Let $\xi_\nu \in S$ be such that

$$g^{\alpha(\nu)}_{C\cap\alpha(\nu)} \le f_{\alpha(\nu)}(\xi_\nu) < h_\nu(\xi_\nu) < f_{\alpha(\nu')}(\xi_\nu).$$

Since $|S| < \aleph_{\gamma+1}$, there exist a set $Z \subseteq \aleph_{\gamma+1}$ of cardinality $\aleph_{\gamma+1}$ and some $\xi \in S$ such that $\xi_\nu = \xi$ for every $\nu \in Z$. Moreover, we may require that if $\nu_1 < \nu_2$ are in Z then $\nu_1' < \nu_2$.

Now if $\nu_1 < \nu_2$ are in Z then $\alpha(\nu_1') \in C \cap \alpha(\nu_2)$, and so

$$f_{\alpha(\nu_1')}(\xi) \le g^{\alpha(\nu_2)}_{C\cap\alpha(\nu_2)}(\xi).$$

Therefore

$$h_{\nu_1}(\xi) < f_{\alpha(\nu_1')}(\xi) \le g^{\alpha(\nu_2)}_{C\cap\alpha(\nu_2)}(\xi) \le f_{\alpha(\nu_2)}(\xi) < h_{\nu_2}(\xi).$$

Consequently, $\{h_\nu(\xi) : \nu \in Z\}$ is a subset of D_ξ of size $\aleph_{\gamma+1}$, a contradiction. □

Remark. The collection $\{\mathcal{E}_\alpha : \alpha \in \aleph_{\eta+1}\}$ was dubbed "silly square" by Shelah.

4. Possible cofinalities and cardinal arithmetic.

In this section we prove Theorem A. We show that if $\aleph_\omega$ is a strong limit cardinal then there exists an ultrafilter D on ω such that cof $\prod_{n=0}^{\infty} \aleph_n/D = 2^{\aleph_\omega}$.

Let A be a set of regular cardinals. If D is an ultrafilter on A, we denote

$$\text{cof } D = \text{ the cofinality of } \prod_{a\in A} \{a : a \in A\}/D$$

and define

$$\text{pcf } A = \{\text{cof } D : D \text{ an ultrafilter on } A\}.$$

The set pcf A (the set of all *possible cofinalities* of $\prod A$) is a set of regular cardinals, includes A (via principal ultrafilters), has cardinality at most $2^{2^{|A|}}$ and satisifies

$$\text{pcf } (A_1 \cup A_2) = \text{pcf } A_1 \cup \text{pcf } A_2.$$

If pcf A has a largest element, we call it the *maximal cofinality* of $\prod A$:

$$\max\text{cof } A = \max(\text{pcf } A)$$

We say that A is an *interval* if it contains every regular λ such that $\min A \le \lambda < \sup A$.

Lemma 4.1. *Assume that A is an interval and that* $\min A = (2^{|A|})^+$. *Then* pcf A *is an interval.*

Proof. Let D be an ultrafilter on A and let λ be a regular cardinal such that $\min A \le \lambda < \text{cof } D$. We shall find an ultrafilter E on A with cof $E = \lambda$.

Let $\{f_\alpha : \alpha < \text{cof } D\}$ be a D-increasing sequence in $\prod A$. Since $\lambda > 2^{|A|}$, the sequence $\{f_\alpha : \alpha < \lambda\}$ has a least upper bound g in $\le_D$. For each $a \in A$ let $h(a) = \text{ cf } g(a)$ and let S_a be a cofinal subset of $g(a)$, of order type $h(a)$. It is easy to see that $\prod_{a\in A} S_a/D$ has an increasing λ-sequence cofinal in g, and therefore $\prod_{a\in A} h(a)/D$ has a cofinal sequence $\{h_\alpha : \alpha < \lambda\}$.

For D-almost all a, $h(a)$ is greater than $2^{|A|}$: this is because the number of functions from A into $2^{|A|}$ is less than λ. Thus, without loss of generality, $h(a) \in A$ for all $a \in A$. Let E be the ultrafilter on A defined as follows:

$$X \in E \quad \text{if} \quad h^{-1}(X) \in D.$$

We can now construct, inductively, functions g_α, $\alpha < \lambda$, such that the sequence $\{g_\alpha \circ h : \alpha < \lambda\}$ is D-increasing and cofinal in h. Then $\{g_\alpha : \alpha < \lambda\}$ is E-increasing and cofinal in $\prod A$. □

Corollary 4.2. *If $\aleph_\omega$ is a strong limit cardinal then* pcf $\{\aleph_n\}_{n=0}^{\infty}$ *is an interval and* sup pcf $\{\aleph_n\}_{n=0}^{\infty} < \aleph_{\aleph_\omega}$.

Proof. The first statement is a consequence of Lemma 4.1 (applied to the interval $A = [(2^{\aleph_0})^+,\ \aleph_\omega)$). The second follows from $|\text{pcf } \{\aleph_n\}_n| < \aleph_\omega$. □

For the proof of Theorem A we need the following fact:

Lemma 4.3. *Let $k < \omega$ and let λ be a cardinal such that $2^{\aleph_k} \le \lambda < \aleph_{\aleph_k}$. There exists a family $\mathcal{F}_\lambda$ of λ subsets of λ, each $X \in \mathcal{F}_\lambda$ of size $\aleph_k$ such that for every subset Z of λ of size $\aleph_k$ there exists an $X \in \mathcal{F}_\lambda$ such that $X \subseteq Z$.*

Proof. We prove, by induction on λ, that for every *ordinal* λ such that $2^{\aleph_k} \le \lambda < \aleph_{\aleph_k}$ there exists a family $\mathcal{F}_\lambda$ which has the stated property. First, such $\mathcal{F}_\lambda$ exists for $\lambda = 2^{\aleph_k}$. Also, if λ is not a cardinal then $\mathcal{F}_\lambda$ can easily be constructed by a one-to-one transformation from $\mathcal{F}_{|\lambda|}$. If λ is a cardinal and $2^{\aleph_k} \le \lambda < \aleph_{\aleph_k}$, we observe that $\operatorname{cf} \lambda \ne \aleph_k$, and it follows that $\mathcal{F}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{F}_\alpha$ has the required property. $\square$

We now proceed to prove Theorem A. We assume that $\aleph_\omega$ is a strong limit cardinal. We shall show that $2^{\aleph_\omega} = \sup \operatorname{pcf} \{\aleph_n\}_{n=0}^\infty$. Since $\operatorname{cf} 2^{\aleph_\omega} > \aleph_\omega$ (by König's theorem) and $\sup \operatorname{pcf} \{\aleph_n\}_n < \aleph_{\aleph_\omega}$, it then follows that $2^{\aleph_\omega}$ is a successor cardinal, and therefore $2^{\aleph_\omega} = \max \operatorname{cof} \{\aleph_n\}_n$.

Let $\lambda = \sup \operatorname{pcf} \{\aleph_n\}_n$.

Lemma 4.4. *There exists a family $\mathcal{F}$ of functions in $\prod_{n=0}^\infty \aleph_n$, $|\mathcal{F}| = \lambda$, such that for every $g \in \prod_{n=0}^\infty \aleph_n$ there is some $f \in \mathcal{F}$ such that $g(n) \le f(n)$ for all n.*

Proof. For every ultrafilter D on ω choose a sequence $\{f_\alpha^D : \alpha < \operatorname{cof} D\}$ that is cofinal in $\prod_{n=0}^\infty \aleph_n / D$, and let $\mathcal{F}$ be the set of all $f = \max\{f_{\alpha_1}^{D_1}, f_{\alpha_2}^{D_2}, \dots, f_{\alpha_m}^{D_m}\}$ where $\{D_1, \dots, D_m\}$ is a finite set of ultrafilters and $\{\alpha_1, \dots, \alpha_m\}$ a finite set of ordinals. Since $\lambda > \aleph_\omega > 2^{2^{\aleph_0}}$, we have $|\mathcal{F}| = \lambda$.

To see that each g is majorized by some $f \in \mathcal{F}$, assume the contrary, and let g be a counterexample. Considering the sets $X_\alpha^D = \{n : g(n) > f_\alpha^D(n)\}$, we note that the family $\{X_\alpha^D\}_{\alpha,D}$ has the finite intersection property, and so extends to an ultrafilter U. Then $g \le_U f_\alpha^U$ for some α, a contradiction. $\square$

Now let $k < \omega$ be large enough, so that $\aleph_k \ge 2^{\aleph_0}$ and that $\aleph_{\aleph_k} > \lambda$.

For every countable subset a of $\aleph_\omega$ we shall construct an elementary chain of models M_α^a, of length ω_k. (By a *model* we mean an elementary submodel of a Skolem expansion of V_ϑ for some large ϑ.) Each M_α^a will have size $\aleph_k$ and will be such that $M_\alpha^a \supseteq a \cup \omega_k$.

We choose M_0^a so that $M_0^a \supseteq a \cup \omega_k$. If $\alpha < \omega_k$ is a limit ordinal, we let $M_\alpha^a = \bigcup_{\beta<\alpha} M_\beta^\alpha$. Given M_α^a, we find $M_{\alpha+1}^a$ as follows: Let

$$\chi_\alpha^a(n) = \sup(M_\alpha^a \cap \omega_n) \qquad \text{(all } n > k\text{)}.$$

There exists a function $f_\alpha^a \in \mathcal{F}$ such that $f_\alpha^a(n) \ge \chi_\alpha^a(n)$ for all $n > k$; let $M_{\alpha+1}^a$ be so that $f_\alpha^a \in M_{\alpha+1}^a$.

For each countable $a \subset \aleph_\omega$, let $M^a = \bigcup_{\alpha<\omega_k} M_\alpha^a$, and let

$$\chi^a(n) = \sup(M^a \cap \omega_n) \qquad \text{(all } n > k\text{)}.$$

Lemma 4.5. *If a and b are countable subsets of $\aleph_\omega$ and if $\chi^a = \chi^b$ then $M^a \cap \aleph_\omega = M^b \cap \aleph_\omega$.*

Proof. By induction on n we show that $M^a \cap \aleph_n = M^b \cap \aleph_n$, for all $n \ge k$. This is true for $n = k$; thus assume that it is true for n and prove it for $n+1$. Both $M^a \cap \aleph_{n+1}$ and $M^b \cap \aleph_{n+1}$ contain a closed unbounded subset of the ordinal

$\chi^a(n+1) = \chi^b(n+1)$ (of cofinality $\aleph_k$), and so there is a cofinal subset C of this ordinal with the property that $C \subseteq M^a$ and $C \subseteq M^b$. Note that for every $\gamma \geq \omega_n$ in C there is one-to-one function $\pi_\gamma \in M^a \cap M^b$ that maps ω_n onto γ, and therefore $\gamma \cap M^a = \gamma \cap M^b$. Consequently, $\omega_{n+1} \cap M^a = \omega_{n+1} \cap M^b$, and the Lemma follows. □

Each M^a has ${\aleph_k}^{\aleph_0} = \aleph_k$ countable subsets, and $\aleph_\omega$ has $2^{\aleph_\omega}$ countable subsets. We complete the proof of Theorem A by showing that the number of functions χ^a is at most λ:

Lemma 4.6. *The set* $\{\chi^a : a \subset \aleph_\omega$ *countable*$\}$ *has size at most* λ.

Proof. For each a let $Z_a = \{f^a_\alpha : \alpha < \omega_k\}$. If S is any subset of ω_k of size $\aleph_k$ then for all $n > k$,

$$\chi^a(n) = \sup_{\alpha \in S} \chi^a_\alpha(n) = \sup_{\alpha \in S} f^a_\alpha(n)$$

and so the set $X = \{f^a_\alpha : \alpha \in S\}$ determines χ^a. Applying Lemma 4.3 to the set $\mathcal{F}$ (of size λ) we get a family $\mathcal{F}_\lambda$ of λ sets $X \subset \mathcal{F}$ such that for each a there is some $X \in \mathcal{F}_\lambda$ with $X \subseteq Z_a$. It follows that the number of the χ^a's is at most λ. □

5. The structure of pcf.

In this section we assume that A is a set of regular cardinals and that $2^{|A|} < \min A$.

Theorem 5.1. *Assume that* $2^{|A|} < \min A$. *There exist sets* $B_\lambda \subseteq A$, $\lambda \in$ *pcf* A, *such that for every* $\lambda \in$ *pcf* A

(5.1) $\max \operatorname{cof} B_\lambda = \lambda$.
(5.2) *for every ultrafilter* D *on* A, *if* $\operatorname{cof} D = \lambda$ *then* $B_\lambda \in D$.

Remark. While we prove the theorem under the assumption $2^{|A|} < \min A$, it is true under the weaker assumption $|A| < \min A$. The sets B_λ are called *generators* of pcf A.

First we prove several consequences of Theorem 5.1.

Corollary 5.2. *If* $2^{|A|} < \min A$ *then* $|\text{pcf } A| \leq 2^{|A|}$.

Proof. By (5.1), the generators corresponding to distinct λ's are distinct.

Corollary 5.3. *If* $\aleph_\omega$ *is a strong limit cardinal then* $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.

This upper bound (proved in [Sh]) follows from Theorem A.

Corollary 5.4. *For every ultrafilter* D *on* A,

$$\operatorname{cof} D = \textit{least } \lambda \textit{ such that } B_\lambda \in D.$$

Proof. Let $\lambda = \operatorname{cof} D$. We have $B_\lambda \in D$ by (5.2), and if $\mu < \lambda$ then $B_\mu \notin D$ by (5.1).

Corollary 5.5. *If* $2^{|A|} < \min A$ *then* $\max \operatorname{cof} A$ *exists.*

Proof. Assume that pcf A does not have a largest element. Then the set $\{A - B_\lambda : \lambda \in \text{pcf } A\}$ has the finite intersection property, and so extends to an ultrafilter D. By (5.2), $B_{\operatorname{cof} D} \in D$, a contradiction. □

Definition. For every cardinal $\kappa \leq \max\operatorname{pcf} A$, let

$$J_\kappa = \text{ the ideal generated by } \{B_\lambda : \lambda < \kappa \text{ and } \lambda \in \operatorname{pcf} A\}.$$

To see that J_κ is an ideal, we observe that if $X \in J_\kappa$ then $X \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n}$ and so $\max\operatorname{cof} X < \kappa$, and therefore $X \neq A$.

Corollary 5.6. *For every $X \subseteq A$,*

$$X \in J_\kappa \quad \textit{if and only if} \quad \operatorname{cof} D < \kappa \textit{ for every ultrafilter } D \textit{ on } X.$$

Proof. As we just observed, if $X \in J_\kappa$ than max cof $X < \kappa$. On the other hand, if $X \notin J_\kappa$ then the set $\{X - B_\lambda : \lambda < \kappa\}$ has the finite intersection property, and so there exists an ultrafilter D on X such that $B_\lambda \notin D$ for all $\lambda < \kappa$. By (5.2), cof $D \geq \kappa$. □

Proof of Theorem 5.1.

We construct the B_λ's by induction, so that for each cardinal $\kappa \leq \sup\operatorname{pcf} A$, the following conditions hold:

(5.3) the ideal J_κ generated by $\{B_\lambda : \lambda < \kappa$ and $\lambda \in \operatorname{pcf} A\}$ is κ-directed (i.e. the partial ordering $\leq_{J_\kappa}$ of $\prod A$ is κ-directed)

(5.4) if $\kappa \notin \operatorname{pcf} A$ then J_κ is κ^+-directed

(5.5) if $\kappa \in \operatorname{pcf} A$ then there exists a $B_\kappa \in J_\kappa^+$ such that J_κ has a κ-scale on B_κ, and

(5.6) if $\kappa = \max\operatorname{pcf} A$ then $B_\kappa = A$, and if not then $J_\kappa[B_\kappa]$ is a κ^+-directed ideal.

First note that if (5.3)–(5.6) are satisfied then the generators satisfy (5.1) and (5.2). Let $\lambda \in \operatorname{pcf} A$. To show that $\lambda \in \operatorname{pcf} B_\lambda$, let D be any ultrafilter on B_λ that extends the filter dual to J_λ (i.e. $D \cap J_\lambda = \emptyset$). Any λ-scale on B_λ for $\leq_{J_\lambda}$ is a scale for $\leq_D$, and so we have cof $D = \lambda$. Also, if D is any ultrafilter on B_λ, then either $D \cap J_\lambda = \emptyset$ in which case cof $D = \lambda$, or let ν be the least ν such that $B_\nu \in D$; then D is an ultrafilter on B_ν such that $J_\nu \cap D = \emptyset$, and since B_ν has ν-scale for J_ν, we have cof $D = \nu$. This proves (5.1).

If D is an ultrafilter on A such that $B_\lambda \notin D$, then either $D \ni B_\nu$ for some $\nu < \lambda$ in which case cof $D < \lambda$, or else $D \cap J_\lambda[B_\lambda] = \emptyset$, and since $J_\lambda[B_\lambda]$ is λ^+-directed, so D is λ^+-directed, and we have cof $D > \lambda$. This proves (5.2).

We shall now prove (5.3)–(5.6), by induction on κ. We use Lemma 2.4 and the assumption that $2^{|A|} < \min A$.

(5.3): If $\kappa \leq \min A$ then $J_\kappa = \{\emptyset\}$ is κ-directed. If κ is a limit cardinal then $J_\kappa = \bigcup_{\lambda<\kappa} J_\lambda$ and the claim follows easily. If $\kappa = \lambda^+$ then the statement follows either from (5.4) or from (5.6).

If κ is a singular cardinal, then κ-directed implies κ^+-directed and so J_κ is κ^+-directed. If κ is regular then by Lemma 2.4 either J_κ is κ^+-directed or J_κ has a κ-scale on some $B \in J_\kappa^+$. To prove (5.4), we show that if there is a κ-scale on some B then $\kappa \in \operatorname{pcf} A$: Let D be any ultrafilter on B such that $D \cap J_\kappa = \emptyset$. Then the κ-scale is a κ-scale for $\leq_D$, and so cof $D = \kappa$.

For (5.5) we prove that if J_κ is κ^+-directed then $\kappa \notin \text{pcf } A$. So let D be any ultrafilter on A. Either $D \ni B_\lambda$ for some $\lambda < \kappa$, in which case $\text{cof } D < \kappa$, or else $D \cap J_\kappa = \emptyset$ in which case $\leq_D$ is κ^+-directed and $\text{cof } D > \kappa$. Hence $\kappa \notin \text{pcf } A$.

Finally, (5.6) follows from Lemma 2.4, provided we show that when $\kappa \in \text{pcf } A$ then $\kappa = \max \text{pcf } A$ if and only if J_κ has a κ-scale on A. If there is a κ-scale, then for every D on A either $D \ni B_\lambda$ for some $\lambda < \kappa$ in which case $\text{cof } D < \kappa$ or else $D \cap J_\kappa = \emptyset$ and D has a κ-scale, and so $\kappa = \max \text{cof } A$. If there is no κ-scale then $J_\kappa[B_\kappa]$ is a κ^+-directed ideal, and if D is any ultrafilter such that $D \cap J_\kappa[B_\kappa] = \emptyset$ then $\leq_D$ is κ^+-directed and so $\kappa < \text{cof } D$. □

Lemma 5.7. (Compactness.) *For every $X \subseteq A$ there exists a finite set $\lambda_1, \ldots, \lambda_n \in \text{pcf } X$ such that $X \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n}$.*

Proof. Assume the contrary. Then the set $\{X - B_\nu : \nu \in \text{pcf X}\}$ has the finite intersection property and so there is an ultrafilter D on X such that $B_\nu \notin D$ for all $\nu \in \text{pcf } X$. But $B_{\text{cof } D} \in D$ by (5.2), a contradiction. □

By Corollary 5.6, the ideals J_κ are independent of the choice of generators. Moreover, each generator B_λ is uniquely determined up to the equivalence mod J_λ: if $B \Delta B_\lambda \in J_\lambda$ then B also satisfies (5.1) and (5.2). To see this, note that by Corollary 5.6, if $X \Delta Y \in J_\lambda$ then $\text{pcf } X - \lambda = \text{pcf } Y - \lambda$, and that (5.2) can be written as $\lambda \notin \text{pcf } (A - B_\lambda)$.

We shall now produce better generators for pcf A.

Theorem 5.8. *Assume that $2^{|A|} < \min A$. There exist generators B_λ for* pcf A *such that for every $\lambda \in \text{pcf } A$,*

$$\max \text{cof } (\bigcup\{B_\mu : \mu \in B_\lambda\}) \leq \lambda. \tag{5.7}$$

We remark that even better generators can be found, namely such that $B_\mu \subseteq B_\lambda$ whenever $\mu \in B_\lambda$, and that the assumption can be weakened to $|A| < \min A$.

Proof. Let B_λ, $\lambda \in \text{pcf } A$, be generators for pcf A. We shall replace each B_λ by an equivalent generator, so that (5.7) is satisfied.

For each λ, there exists (by (5.5)) an increasing (mod J_λ) sequence $\{f_\alpha^\lambda : \alpha < \lambda\}$ of functions on A that is cofinal on B_λ. Moreover, by Lemma 2.1 we may assume that for each λ and each α of cofinality $> 2^{|A|}$, f_α^λ is a least upper bound of $\{f_\beta^\lambda : \beta < \alpha\}$.

Let $\kappa = (2^{|A|})^+$, and assume, with no harm, that $\kappa < \min A$. We consider an elementary chain $\langle M_\xi : \xi \leq \kappa \rangle$ of models of size κ with the property that M_0 contains (as elements) A, pcf A, the generators B_λ, the scales $\{f_\alpha^\lambda : \alpha < \lambda\}$, and every function $\varphi : A \to A$. Moreover, we assume $\langle M_\xi : \xi \leq \eta \rangle \in M_{\eta+1}$ for every η. Let $M = M_\kappa$.

For each $\xi \leq \kappa$, we define $\chi_\xi \in \prod A$ as follows:

$$\chi_\xi(\nu) = \sup(M_\xi \cap \nu) \qquad (\nu \in A).$$

Let $\chi = \chi_\kappa$; we also define $\chi(\lambda) = \sup(M \cap \lambda)$ for all $\lambda \in \text{pcf } A$.

Each χ_ξ belongs to M_ξ and therefore to M, and if $\xi < \eta$ then $\chi_\xi(\nu) < \chi_\eta(\nu)$ for all $\nu \in A$. The function χ is the pointwise least upper bound of $\{\chi_\xi : \xi < \kappa\}$.

Let $\lambda \in \operatorname{pcf} A$, and consider the function $f^\lambda_{\chi(\lambda)}$. We have $\operatorname{cf} \chi(\lambda) = \kappa$, and therefore $f^\lambda_{\chi(\lambda)}$ is a least upper bound (in $\leq_{J_\lambda}$) of $\{f^\lambda_\alpha : \alpha \in M \cap \lambda\}$. Now if $\alpha \in M \cap \lambda$ then $f^\lambda_\alpha \in M$ and so $f^\lambda_\alpha(\nu) < \chi(\nu)$ for all $\nu \in A$. On the other hand, if $\xi < \kappa$ than there exists an α such that $\chi_\xi \leq_{J_\lambda} f^\lambda_\alpha$ on B_λ, and since M is an elementary submodel, there exists such an α in M. It follows that the function χ is a least upper bound (in $\leq_{J_\lambda}$) of $\{f^\lambda_\alpha : \alpha \in M \cap \lambda\}$ on B_λ, and consequently $f^\lambda_{\chi(\lambda)} = \chi$ almost everywhere (mod J_λ) on B_λ.

Thus we replace each generator B_λ by the generator

$$B^*_\lambda = \{\nu \in B_\lambda : f^\lambda_{\chi(\lambda)}(\nu) = \chi(\nu)\} \tag{5.8}$$

and we proceed to show that the generators B^*_λ satisfy (5.7).

Let $E = \bigcup\{B^*_\mu : \mu \in B^*_\lambda\}$; we will show that $E \in J_{\lambda^+}$. For each $\nu \in E$ let $\mu = \varphi(\nu)$ be such that $\nu \in B^*_\mu$ and $\mu \in B^*_\lambda$ (and $\varphi(\nu) \in A$ arbitary if $\nu \notin E$). By our assumption on M, φ is in M. For each $\alpha < \lambda$, let $g_\alpha \in \prod A$ be the function defined as follows:

$$g_\alpha(\nu) = f^\mu_\beta(\nu), \text{ where } \mu = \varphi(\nu) \text{ and } \beta = f^\lambda_\alpha(\mu) \qquad (\nu \in A).$$

The set $\{g_\alpha : \alpha < \lambda\}$ is in M and since J_{λ^+} is λ^+-directed, there exists a $g \in M$ such that $g_\alpha < g$ mod J_{λ^+} for all $\alpha < \lambda$. Since $g \in M$, we have $g(v) < \chi(\nu)$ for all ν.

Now let $\alpha = \chi(\lambda)$. Since $g_\alpha < \chi$ mod J_{λ^+}, we shall complete the proof by showing that $g_\alpha = \chi$ on E. Thus let $\nu \in E$ be arbitrary, and let $\mu = \varphi(\nu)$ and $\beta = f^\lambda_\alpha(\mu)$. Since $\mu \in B^*_\lambda$, we have $\beta = f^\lambda_\alpha(\mu) = f^\lambda_{\chi(\lambda)}(\mu) = \chi(\mu)$ and since $\nu \in B^*_\mu$, it follows that $g_\alpha(\nu) = f^\mu_\beta(\nu) = f_{\chi(\nu)}(\nu) = \chi(\nu)$. □

We conclude this Section by proving two consequences of the structure of pcf, which will be used in the proof of the Main Theorem.

Theorem 5.9. *Let κ be a regular uncountable cardinal, and let $\aleph_\eta$ be a singular cardinal of cofinality κ such that $2^\kappa < \aleph_\eta$. Then there is a closed unbounded set $C \subset \eta$ such that* $\max\operatorname{cof}\{\aleph_{\alpha+1} : \alpha \in C\} = \aleph_{\eta+1}$.

Proof. Let C_0 be any club subset of η of order type κ, and consider the structure of pcf A where $A = \{\aleph_{\alpha+1} : \alpha \in C_0\}$. Let $\lambda = \aleph_{\eta+1}$, let B_λ be a generator for λ, and let $X = \{\alpha \in C_0 : \aleph_{\alpha+1} \in B_\lambda\}$. If D is any ultrafilter on C_0 that extends the closed unbounded filter then by Theorem 3.1, $\operatorname{cof} \prod_{\alpha \in C_0} \aleph_{\alpha+1}/D = \lambda$, and by (5.2), $X \in D$. Thus X contains a club subset C. By (5.1), $\max\operatorname{cof}\{\aleph_{\alpha+1} : \alpha \in C\} \leq \lambda$ and therefore $= \lambda$. □

Theorem 5.10. *Let C be a subset of* pcf *A such that $|A| < |C|$, and assume that $2^{|C|} < \min A$. Than there exists a set $B \subset C$ such that $|B| \leq |A|$ and such that* $\max\operatorname{cof} B \geq \sup C$.

Proof. Let B_λ, $\lambda \in \operatorname{pcf}(A \cup C)$ be generators for $\operatorname{pcf}(A \cup C)$ that satisfy (5.7). For each $\lambda \in C$, let $B^A_\lambda = A \cap B_\lambda$. As $\lambda \in \operatorname{pcf} A$, there is an ultrafilter D on A of cofinality λ. By (5.1), $B_\lambda \in D$, and so $B^A_\lambda \in D$, therefore $\lambda \in \operatorname{pcf} B^A_\lambda$.

Let $E = \bigcup\{B^A_\lambda : \lambda \in C\}$; we have $C \subset \text{pcf } E$, and so max cof $E \geq \sup C$. Let $B \subset C$ be any set of cardinality at most $|A|$ such that $E = \bigcup\{B^A_\lambda : \lambda \in B\}$. We will finish by showing that max cof $B \geq$ max cof E.

By the compactness lemma 5.7 there are $\lambda_1, \dots, \lambda_n \in \text{pcf } B$ such that $B \subseteq B_{\lambda_1} \cup \dots \cup B_{\lambda_n}$, and so

$$E \subseteq \bigcup\{B_\mu : \mu \in B_{\lambda_1}\} \cup \dots \cup \bigcup\{B_\mu : \mu \in B_{\lambda_n}\}. \tag{5.9}$$

Applying (5.7) to the set in (5.9), we get

$$\max\text{cof } E \leq \max\{\lambda_1, \dots, \lambda_n\} \leq \max\text{cof } B. \quad \square$$

6. Proof of Theorem B.

We shall now use the results of Section 5 to prove, under the assumption that $\aleph_\omega$ is a strong limit cardinal, that max cof $\{\aleph_n\}_{n<\omega}$ is less than $\aleph_{\omega_4}$. Since $2^{\aleph_0} < \aleph_\omega$ we know by Lemma 4.1 and by Corollaries 5.2 and 5.5 that pcf $\{\aleph_n\}_{n<\omega}$ is an interval and has a largest element $\aleph_{\Theta+1}$ where $\Theta < (2^{\aleph_0})^+ < \aleph_\omega$.

Theorem 6.1. *Assume that $\aleph_\omega$ is a strong limit cardinal, and let Θ be the ordinal such that $2^{\aleph_\omega} = \aleph_{\Theta+1}$. There exists an ordinal function F on $P(\Theta)$ with the following properties*

(6.1) *if $X \subseteq Y$ then $F(X) \leq F(Y)$,*
(6.2) *if $\vartheta < \Theta$ is a limit ordinal of uncountable cofinality then there exists a closed unbounded set $C \subseteq \vartheta$ such that $F(C) = \vartheta$,*
(6.3) *if $X \subseteq \Theta$ is a set of order type ω_1 then there exists some $\gamma \in X$ such that $F(X \cap \gamma) \geq \sup X$.*

Proof. Let X be a subset of Θ, and consider the set $B = \{\aleph_{\xi+1} : \xi \in X\}$. As $2^{|B|} = \aleph_k$ for some finite k, max cof B exists and is equal to some $\aleph_{\eta+1}$. We define $F(X)$ to be this η.

It is clear that $X \subseteq Y$ implies $F(X) \leq F(Y)$. Property (6.2) follows from Theorem 5.9: If $\kappa = \text{cf } \eta$ then $\kappa < \aleph_\omega$ and so $2^\kappa < \aleph_\omega < \aleph_\eta$ and Theorem 5.9 applies. Finally, property (6.3) is a consequence of Theorem 5.10: If $X \subseteq \Theta$ then $\{\aleph_{\xi+1} : \xi \in X\} \subseteq \text{pcf } \{\aleph_n\}_{n<\omega}$, and since $2^{|X|} < \aleph_\omega$, Theorem 5.10 applies and so X has a countable subset Y such that $F(Y) \geq \sup X$. $\square$

We shall complete the proof by showing that properties (6.1)–(6.3) imply that $\Theta < \omega_4$.

We need one more lemma. Let

$$\mathbf{E}^3_1 = \{\alpha < \omega_3 : \text{ cf } \alpha = \omega_1\}$$

Lemma 6.2. *There exists a family $\{C_\alpha : \alpha \in \mathbf{E}^3_1\}$ such that each C_α is a club subset of α, and such that for every club $C \subseteq \omega_3$, the set $\{\alpha \in \mathbf{E}^3_1 : C_\alpha \subset C\}$ is stationary.*

Proof. It suffices to find a family $\{C_\alpha : \alpha \in \mathbf{E}^3_1\}$ such that each C_α is a subset of α and so that for each club $C \subset \omega_3$, the set $\{\alpha \in \mathbf{E}^3_1 : C_\alpha$ is a club in α and $C_\alpha \subset C\}$ is stationary.

Assume that no such $\{C_\alpha\}_\alpha$ exists. Let $\{C^0_\alpha : \alpha \in \mathbf{E}^3_1\}$ be any collection of club subsets of the α's, such that $|C^0_\alpha| = \aleph_1$. By induction on $\nu < \omega_2$, we construct clubs $E_\nu \subseteq \omega_3$ and collections $\{C^\nu_\alpha : \alpha \in \mathbf{E}^3_1\}$ as follows: $C^\nu_\alpha = C^0_\alpha \cap \bigcap_{\xi\in\nu} E_\xi$, and E_ν is such that the set $\{\alpha \in \mathbf{E}^3_1 : C^\nu_\alpha \text{ is a club in } \alpha \text{ and } C^\nu_\alpha \subset E_\nu\}$ is nonstationary.

Let E be the club $E = \bigcap_{\nu<\omega_2} E_\nu$, and for each α let $C_\alpha = C^0_\alpha \cap E$. The set $S = \{\alpha \in \mathbf{E}^3_1 : E \cap \alpha \text{ is club in } \alpha\}$ is stationary, and for each $\alpha \in S$ there exists a $\nu(\alpha) < \omega_2$ such that $C_\alpha = C^{\nu(\alpha)}_\alpha$ (because $C^0_\alpha \supseteq C^1_\alpha \supseteq \ldots$ of length ω_2).

There is a $\nu < \omega_2$ and a stationary set $T \subseteq S$ such that $C_\alpha = C^\nu_\alpha$ for all $\alpha \in T$. If $\alpha \in T$ then $C^\nu_\alpha = C^{\nu+1}_\alpha = C^\nu_\alpha \cap E_\nu$, and so $C^\nu_\alpha \subset E_\nu$, contrary to the choice of E_ν. □

The following lemma completes the proof:

Lemma 6.3. *Let F be an ordinal function on $P(\Theta)$ with properties (6.1)–(6.3). Then $\Theta < \omega_4$.*

Proof. Assume that $\Theta \geq \omega_4$. Let $\{C_\alpha : \alpha \in \mathbf{E}^3_1\}$ be some family that satisfies Lemma 6.2. Let M_α, $\alpha < \omega_3$, be an elementary chain of models of size $\aleph_3$ that contain the family $\{C_\alpha\}_\alpha$, are closed under F, such that $\langle M_\xi : \xi \leq \alpha\rangle \in M_{\alpha+1}$ for each α, and that for each α, $\eta_\alpha = M_\alpha \cap \omega_4$ is an ordinal. Let $\eta : \omega_3 \to \omega_4$ be the continuous function $\eta(\alpha) = \eta_\alpha$.

By (6.2) there is a club $C \subseteq \omega_3$ such that $F(\eta[C]) = \sup_\alpha \eta_\alpha$. Let $\alpha \in \mathbf{E}^3_1$ be such that $C_\alpha \subset C$. By (6.3) there exists a $\beta < \alpha$ such that $F(\eta[C_\alpha \cap \beta]) \geq \eta(\alpha)$. Let $X = \eta[C_\alpha \cap \beta]$.

Since $C_\alpha \in M_\alpha$ and $\eta \restriction \beta \in M_\alpha$, we have $X \in M_\alpha$. Since $X \subseteq \eta[C]$ we have, by (6.1), $F(X) \leq F(\eta[C]) < \omega_4$. As M_α is closed under F we have $F(X) \in M_\alpha$, and since $\omega_4 \cap M_\alpha = \eta(\alpha)$, it follows that $F(X) < \eta(\alpha)$, a contradiction. □

REFERENCES

[BM] M. Burke and M. Magidor, *Shelah's pcf theory and its applications*, Annals of Pure and Applied Logic.

[ER] P. Erdős and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489.

[GH] F. Galvin and A. Hajnal, *Inequalities for cardinal powers*, Annals of Math. **101** (1975), 491–498.

[S] J. Silver, *On the singular cardinals problem*, Proc. Int. Congress of Math. (Vancouver 1974), vol. 1, 265–268.

[Sh] S. Shelah, *Proper forcing*, Lecture Notes in Mathematics vol. 940, Springer-Verlag, New York, 1982.

[Sh400] S. Shelah, *Cardinal arithmetic*, preprint.

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802

E-mail: jech@math.psu.edu