LOCAL TOPOLOGY AND A SPECTRAL THEOREM

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1. Introduction.

The concepts of continuity and convergence pervade the study of functional analysis, and are based on the precise formulation of the (vague) concept of *closeness*. Traditionally, the concept of closeness has been defined in terms of topology. It is well known, however, that not every version of convergence can be expressed in the topological language.

In this note we present an approach somewhat more general than topological spaces, and illustrate this approach on three examples: the space of continuous functions, the space of measurable functions and the commutative von Neumann algebra. For the latter example we state a version of the Spectral Theorem.

The theory presented here was influenced by Takeuti's Boolean valued analysis [2] and is loosely related to our Boolean linear spaces [1]. I wish to thank Professor Louis de Branges for his invitation to submit my contribution to the volume honoring Ernst Hellinger.

2. An example.

Let E be the space of all (equivalence classes of) real-valued, Lebesgue measurable functions on $\mathbf{R}$, with the equivalence relation "$f(x) = g(x)$ a.e."

Let us consider the convergence

$$\lim_n f_n(x) = f(x) \text{ almost everywhere.}$$

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It is known that this notion of convergence is not determined by any topology on E, because there exists a sequence $\{f_n\}$ with the property that every subsequence has a subsequence converging to 0 yet the limit $\lim_n f_n$ does not exist.

We shall show how the concept of almost everywhere convergence can be defined from a partially ordered system of topologies, in fact from a system of seminorms. This will motivate a more general concept of *local topology.* Let $\mathcal{P}$ be the collection of all measurable subsets of $\mathbf{R}$, of positive measure. For each $P \in \mathcal{P}$ and every $f \in E$, let

$$\|f\|_P = \text{ess.}\sup_{x\in P} |f(x)|.$$

For each $P \in \mathcal{P}$, $\| \ \|_P$ is a seminorm (with values including ∞), and defines a topology of uniform convergence on E, namely $\lim_n f_n(x) = f(x)$ uniformly a.e. on P. The system $\{\| \ \|_P : P \in \mathcal{P}\}$ determines convergence a.e. as follows:

Proposition. *A sequence $\{f_n\}$ converges to f a.e. if and only if*

$$\forall P\, \forall \varepsilon > 0\, \exists Q \subseteq P\, \exists m\, \forall n \geq m \|f_n - f\|_Q < \varepsilon.$$

Proof. Assume first that the condition fails, and let P be a set of positive measure and $\varepsilon > 0$ be such that for all $Q \subseteq P$ of positive measure, $\|f_n - f\|_Q \geq \varepsilon$ for infinitely many n. It follows that for almost all $x \in P$, $|f_n(x) - f(x)| \geq \varepsilon/2$ for infinitely many n, and therefore $\lim_n f_n(x) = f(x)$ fails almost everywhere on P.

Conversely, assume that the condition holds, and let $k \geq 1$ be an interger. For every $P \in \mathcal{P}$ there is a $Q \subseteq P$ such that for eventually all n, $\|f_n - f\|_Q < \frac{1}{k}$. It follows that for almost all x, $|f_n(x) - f(x)| < \frac{1}{k}$ holds for eventually all n; let D_k be the set of all such x. Consequently, $\lim_n f_n(x) = f(x)$ holds for every $x \in \bigcap_k D_k$. □

In this example, we employ a class of topologies that forms a partially ordered set with respect to fineness. The general framework introduced in the next section involves a partially ordered system of "approximations" to a topology, which we call a *local topology.*

3. LOCAL TOPOLOGY

We consider a "space" E (a set of "points") and define a "local topology" on E. The "localization" is represented by a partially ordered set $(P, \leq)$. The idea is to define, for a point a, a set of points X and each $p \in P$, the concept "a is p-close to X", in a similar way as "a is close to X" is defined in terms of a topology on E, with the intention that if a is p-close to X and if $q \leq p$, then a is q-close to X. We do this by introducing a p-base of a local topology, for each $p \in P$. But first it is necessary to consider "local equality", namely relations *a is p-equal to b*, a system of equivalence relations on E.

Definition. For each $p \in P$, $=_p$ is an equivalence relation on E, and
(E1) if $a =_p b$ and $q \leq p$ then $a =_q b$
(E2) if $a \neq b$ then for some $p \in P$, $a \neq_p b$
(E3) if $a \neq_p b$ then there is some $q \leq p$ such that $a \neq_r b$ for all $r \leq q$

Example. $E = \mathbf{C}[0,1]$, *the space of all continuous functions on* $[0,1]$.

Let P be the set of all nonempty open intervals $p \subset [0,1]$ and let

$$p \le q \quad \text{if} \quad p \subseteq q.$$

Define

$$f =_p g \quad \text{if} \quad f(x) = g(x) \text{ for all } x \in p.$$

Conditions (E1)–(E3) are satisfied: For example, for (E3) assume that $f(x) \neq g(x)$ for some $x \in p$. Then there is a $q \subseteq p$ such that $f(x) \neq g(x)$ for all $x \in q$, and so $f \neq_r g$ for all $r \subseteq q$.

Example.

Let P be the set of all measurable subsets of $\mathbf{R}$, of positive measure, and again let

$$p \le q \quad \text{if} \quad p \subseteq q.$$

Define

$$f =_p g \quad \text{if} \quad f(x) = g(x) \text{ a.e. on } p.$$

Again, the equivalence relations $=_p$ satisfy the conditions (E1)–(E3).

We shall now introduce *local topology* on E. For a set $X \subseteq E$ and $p \in P$, let

$$X^p = \{z : z =_p x \text{ for some } x \in X\}.$$

Definition. For each $p \in P$, the *p-base for a local topology on E* is a set $\mathcal{B}_p$ of nonempty subsets of E, such that the following conditions are satisfied, for all p and q in P:

(LT1) If $U \in \mathcal{B}_p$, then $U^p = U$.

(LT2) If $U \in \mathcal{B}_p$ and if $q \le p$, then $U^q \in \mathcal{B}_q$.

(LT3) If $U \in \mathcal{B}_p, V \in \mathcal{B}_p$, and if $a \in U \cap V$, then there exists a $q \le p$ and some $W \in \mathcal{B}_q$ such that $a \in W$ and $W \subseteq U^q \cap V^q$.

(LT4) If $a \neq_p b$ then there exists a $q \le p$ and some U and V in $\mathcal{B}_q$ such that $a \in U$ and $b \in V$ and for every $r \le q$ and for all $x \in U$ and all $y \in V$, $x \neq_r y$.

Example.

Back to our example $E = \mathbf{C}[0,1]$: For every $f \in E, p \in P$ and $\varepsilon > 0$, let $U_p(f,\varepsilon)$ be as follows:

$$U_p(f,\varepsilon) = \{g : \sup_{x\in p} |f(x) - g(x)| < \varepsilon\}$$

and let

$$\mathcal{B}_p = \{U_p(f,\varepsilon) : f \in E, \varepsilon > 0\}.$$

We claim that $\{\mathcal{B}_p : p \in P\}$ is a local topology.

(LT1) is clearly satisfied as $(U_p(f,\varepsilon))^p = U_p(f,\varepsilon)$.

(LT2): We shall show that if $q \subseteq p$ then $(U_p(f,\varepsilon))^q = U_q(f,\varepsilon)$. If $g \in U_p(f,\varepsilon)$ and $h = g$ on q then $h \in U_q(f,\varepsilon)$. Conversely, if $h \in U_q(f,\varepsilon)$, then there exists a continuous function g such that $g = h$ on q and $\sup_{x\in q} |f(x) - g(x)| < \varepsilon$.

(LT3): Let $g \in U_p(f_1,\varepsilon_1) \cap U_p(f_2,\varepsilon_2)$. If we let, for $i = 1,2$, $\delta_i = \varepsilon_i - \sup_p |g(x) - f_i(x)|$, and $\delta = \min\{\delta_1,\delta_2\}$, then $U_p(g,\delta) \subseteq U_p(f_1,\varepsilon_1) \cap U_p(f_2,\varepsilon_2)$.

(LT4): Let p, f and g be such that $f(x) \neq g(x)$ for some $x \in p$. There exists $q \subseteq p$ and an $\varepsilon > 0$ such that $|f(x) - g(x)| \geq \varepsilon$ on q. If we let $U = U_p(f,\varepsilon/2)$ and $V = U_p(g,\varepsilon/2)$, then U and V satisfy the requirement, because for all $h \in U$ and $k \in V$, $h(x) \neq k(x)$ for all $x \in q$. □

Remarks.

1. When P is trivial, i.e. has a single point, then local topology is just a topology on E.

2. In the example $E = \mathbf{C}[0,1]$ above, each $\mathcal{B}_p$ is a topology base. In general, this is not necessarily the case. The systems $\mathcal{B}_p$, together with the partial ordering of P, approximate a topology; the "limit" of the system is a local topology.

4. CONTINUITY AND CONVERGENCE

Let $\{\mathcal{B}_p : p \in P\}$ be a local topology on E. For each $a \in E$ we let

$$\mathcal{B}_p(a) = \{U \in \mathcal{B}_p : a \in U\}$$

and call the elements of $\mathcal{B}_p(a)$ *p-neighborhoods* of a.

Definition. Let F be a function from E into E and let $a \in E$. We say that F is *continuous* at a if for every p and every $U \in \mathcal{B}_p(F(a))$ there exists a $q \leq p$ and a $V \in \mathcal{B}_q(a)$ such that $F(V) \subseteq U^q$.

Remarks.

1. If F is continuous and if $a_1 =_p a_2$ then $F(a_1) =_p F(a_2)$.

Proof. If $F(a_1) \neq_p F(a_2)$ then by (LT4) there are $q \leq p$ and $U \in \mathcal{B}_q(a_1)$ such that $a_2 \notin U^r$ for any $r \leq q$. By continuity at a_1, let $r \leq q$ and $V \in \mathcal{B}_r(a_1)$ be such that $F(V) \subseteq U^r$. It follows that $a_2 \neq_r a_1$ and so $a_2 \neq_p a_1$. □

2. We can similarly define continuous functions from E_1 into E_2, from $\mathbf{R}$ into E, and continuous functions of more variables.

3. An alternative definition of continuity uses the concept of "a is p-close to X":

Let $X \subseteq E$, $a \in E$ and $p \in P$. We say that a is *p-close to X* if for every $q \subseteq p$ and every $U \in \mathcal{B}_q(a)$ there is an $r \leq q$ such that $U^r \cap X \neq \emptyset$.

Then we can define:

F is *continuous* at $a \in E$ if for all p and all X whenever a is p-close to X then $F(a)$ is p-close to $F(X)$.

We now turn to convergence:

Definition. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in E, and let $a \in E$. We say that a_n *converges* to a,

$$\lim_{n\to\infty} a_n = a$$

if for every $p \in P$ and every p-neighborhood U of a there exists some $q \leq p$ such that eventually all a_n belong to U^q.

Remark.

Proof. Assume that a_n converges both to a and to b. If $a \neq b$ then $a \neq_p b$ for some p, and by (LT4) there exists a $q \leq p$, $U \in \mathcal{B}_q(a)$ and $V \in \mathcal{B}_q(b)$ such that U^r and V^r are disjoint for all $r \leq q$. An appeal to the definition yields a contradiction. ☐

We shall illustrate the concept of convergence on our example:

Example. *A sequence $\{f_n\}$ converges to f in $\mathbf{C}[0,1]$ if and only if the set $\{x \in [0,1] : \lim f_n(x) = f(x)\}$ is comeager.*

Proof. Assume that $\lim f_n = f$. For every open interval p and every $k \geq 1$ there exists $q \subseteq p$ such that $f_n \in (U_p(f, 1/k))^q$ for eventually all n. We have $(U_p(f,1/k))^q = U_q(f, 1/k)$, and it follows that the set

$$\{x : \text{ for eventually all } n \ |f_n(x) - f(x)| < 1/k\}$$

contains an open dense set, and so $\{x : \lim f_n(x) = f(x)\}$ is comeager.

Conversely, assume that $\lim f_n \neq f$. Then there is an open set p and an $\varepsilon > 0$ such that for all $q \subseteq p$ and for every n, there exists some $m \geq n$ such that $f_m \notin U_q(f, 2\varepsilon)$. Hence for every n, the set D_n of all $x \in p$ for which there exists some $m \geq n$ such that $|f_m(x) - f(x)| > \varepsilon$ is dense in p (and open). So the set $A = \bigcap_n D_n$ is nonmeager and for all $x \in A, \lim f_n(x) \neq f(x)$. ☐

5. Locally normed linear spaces

Many linear spaces such as the space of measurable functions described in Section 2 have a local topology given by a partially ordered system of seminorms. We call such space *locally normed.*

Definition. Let E be a (real) vector space, and let $(P, \leq)$ be a partially ordered set. A system of seminorms $\{\| \ \|_p : p \in P\}$ makes E a *locally normed linear space* if it satisfies the following:

(LN1) $0 \leq \|a\|_p \leq \infty$
$\|a+b\|_p \leq \|a\|_p + \|b\|_p$
$\|\lambda a\|_p = |\lambda| \cdot \|a\|_p$

(LN2) If $p \leq q$ then $\|a\|_p \leq \|a\|_q$.

(LN3) If $a \neq 0$ then for some p, $\|a\|_p > 0$

(LN4) If $\|a\|_p > 0$ then there exists some $q \leq p$ and $\varepsilon > 0$ such that $\|a\|_r \geq \varepsilon$ for all $r \leq q$.

(LN5) If $p \leq q$ and $\|a\|_p < \varepsilon$ then there exists some a' such that $\|a'\|_q < \varepsilon$ and $\|a' - a\|_p = 0$.

We note that the space of measurable functions from Section 2 is locally normed, by the seminorms $\|f\|_p = \operatorname{ess\,sup}_p |f(x)|$. Similarly, the space $\mathbf{C}[0,1]$ from Section 3 is locally normed, by $\|f\|_p = \sup_p |f(x)|$ (where p is an open set). In the next section we show that every Archimedean Riesz space is locally normed.

Local norms define local topology bases as follows:

For $a, b \,\varepsilon\, E$ and $p \in P$, let

$$a =_p b \quad \text{if} \quad \|a - b\|_p = 0.$$

Then $=_p$ is an equivalence relation on E and satisfies (E1), (E2) and (E3) (the proof of (E3) uses (LN4)).

Let

$$U_p(a, \varepsilon) = \{b : \|a - b\|_p < \varepsilon\}$$

and let

$$\mathcal{B}_p = \{U_p(a, \varepsilon) : a \in E, \varepsilon > 0\}.$$

We shall verify that the $\mathcal{B}_p$ satisfy (LT1)–(LT4).

(LT1): If $b \in U_p(a, \varepsilon)$ and $\|c = b\|_p = 0$ then $c \in U_p(a, \varepsilon)$, and so $(U_p(a, \varepsilon))^p = U_p(a, \varepsilon)$.

(LT2): We claim that if $p \leq q$ then $(U_q(a, \varepsilon))^p = U_p(a, \varepsilon)$. First, if $b \in (U_q(a, \varepsilon))^p$ then for some c, $\|b - c\|_p = 0$ and $\|c - a\|_q < \varepsilon$. Therefore $\|b - a\|_p \leq \|b - c\|_p + \|c - a\|_p < \varepsilon$ by (LN1) and (LN2).

Conversely, let $b \in U_p(a, \varepsilon)$, i.e. $\|b - a\|_p < \varepsilon$. By (LN5) there exists some c such that $\|c - a\|_q < \varepsilon$ and $\|c - b\|_p = 0$, and hence $b \in (U_q(a, \varepsilon))^p$.

(LT3): Let $a \in U \cap V$ where $U = U_p(b, \varepsilon_1)$ and $V = U_p(c, \varepsilon_2)$. If we let $\delta > 0$ be such that $\|a - b\|_p + \delta \leq \varepsilon_1$, and $\|a - c\|_p + \delta \leq \varepsilon_2$, then $U_p(a, \delta) \subseteq U \cap V$.

(LT4): Let a, b be such that $\|a-b\|_p > 0$. By (LN4) get a $q \leq p$ and $\varepsilon > 0$ such that $\|a-b\|_r \geq 2\varepsilon$ for all $r \leq q$, and let $U = U_q(a,\varepsilon)$ and $V = U_q(b,\varepsilon)$. Then for all $x \in U, y \in V$ and all $r \leq q$ we have $\|x-y\|_r > 0$. ☐

It is easy to see that for locally normed linear spaces, the definitions of continuity and convergence from Section 4 take the following form:

Proposition. *A function $F : E \to E$ is continuous at a iff*

$$\forall p \in P \;\; \forall \varepsilon > 0 \;\; \exists q \leq p \, \exists \delta > 0 \, \forall x (\|x-a\|_q < \delta \Rightarrow \|F(x) - F(a)\|_q < \varepsilon).$$

A sequence $\{a_n\}$ converges to a iff

$$\forall p \in P \;\; \forall \varepsilon > 0 \;\; \exists q \leq p \text{ (for eventually all } n, \|a_n - a\|_q < \varepsilon).$$

6. Archimedean Riesz spaces

We shall show that every Archimedean Riesz space (vector lattice) is locally normed. To simplify the argument, we assume that the space has a unit, that is an element 1 with the property that $1 \wedge a > 0$ for all $a > 0$. (A modification of the argument works in the general case.)

Thus let E be an Archimedean Riesz space with unit 1. For P take the positive cone of E, the set $\{p \in E : p > 0\}$, partially ordered by the lattice order. For $a \in E$ and $p \in P$ let

$$\|a\|_p = \inf\{\lambda > 0 : p \perp (|a| - \lambda \cdot 1)^+\}$$

(using the convention that $\inf \emptyset = \infty$). It is rather clear that $\| \;\; \|_p$ are seminorms satisfying (LN1) and (LN2). We shall verify (LN3)–(LN5).

(LN3): Here we use that E is Archimedean. If $a > 0$ we claim that $\|a\|_a > 0$. Since E is Archimedean, there exists some $\lambda > 0$ such that $a \not\leq \lambda \cdot 1$. Hence $a \wedge (a - \lambda \cdot 1)^+ > 0$, and so $\lambda \leq \|a\|_a$.

(LN4): Let $\|a\|_p > 0$, and assume, without loss of generality, that $a > 0$. Let $\varepsilon = \frac{1}{2}\|a\|_p$, and let $q = p \wedge (a - \varepsilon \cdot 1)^+$. Since $\varepsilon < \|a\|_p$, we have $q \neq 0$. Now if $r \leq q$ ($r \in P$), then $r \wedge (a - \varepsilon \cdot 1)^+ = r > 0$, and so $\varepsilon \leq \|a\|_r$.

(LN5): Let $\|a\|_p < \varepsilon$. First consider the case when $a > 0$. Let $\lambda = \|a\|_p$, and let $a' = a \wedge \lambda \cdot 1$. It is clear that $\|a'\|_q \leq \lambda < \varepsilon$ for all q, and since $p \wedge (a - a') = 0$, we have $\|a - a'\|_p = 0$.

In the general case, let again $\lambda = \|a\|_p$; then $a' = \lambda \cdot 1 \wedge a \vee -\lambda \cdot 1$ will do. ☐

7. Spectral theorem

We shall now apply local topology to commutative operator algebras. Let $\mathfrak{A}$ be either a commutative von Neumann algebra, or the algebra of all (possibly unbounded) normal operators associated with a commutative von Neumann algebra. Let P be the set of all nonzero projection operators in $\mathfrak{A}$. For a, b in $\mathfrak{A}$ and $p \in P$ we let

$$a =_p b \quad \text{if} \quad pa = pb.$$

For $p \in P$, $a \in \mathfrak{A}$ and $\varepsilon > 0$ we let

$$b \in U_p(a,\varepsilon) \quad \text{if} \quad \forall q \leq p \, |qa - qb| < \varepsilon q.$$

It is not difficult to verify that the local neighborhood systems $\{U_p(a,\varepsilon) : \varepsilon > 0\}$ form a local topology on $\mathfrak{A}$.

Theorem. *For every continuous function* $F\colon \mathbf{R} \to \mathbf{R}$ *there exists a unique continuous* $\hat{F}\colon \mathfrak{A} \to \mathfrak{A}$ *such that* $\hat{F}(\lambda I) = F(\lambda)I$ *for all* $\lambda \in \mathbf{R}$.

The existence of an $\hat{F}$ extending F is of course well known. Our claim is that $\hat{F}$ is continuous, and that such a continuous extension is unique.

Both the existence and the uniqueness of a continuous extension is proved, in a more general setting, in [1, Theorem 10.27]. As the general result uses a somewhat different definition of continuity, I shall sketch the proof that for the local topology on $\mathfrak{A}$, the present definition of continuity is equivalent to the definition used in [1], namely the definition using nets.

Lemma. *A function* $F\colon \mathfrak{A} \to \mathfrak{A}$ *is continuous if and only if for every net* $\{a_n\}_n$, *if* a_n *converges to* a *then* $F(a_n)$ *converges to* $F(a)$.

Proof. Assume that F is continuous, and let $\{a_n\}_n$ be a net converging to a. Let $U \in \mathcal{B}_p(F(a))$. First, by continuity, there is some $r \leq p$ and a $V \in \mathcal{B}_r(a)$ such that $F(V) \subseteq U^r$. Since a_n converges to a, there is a $q \leq r$ such that $a_n \in V^q$ for eventually all n. Since $x =_q y$ implies $F(x) =_q F(y)$, we have $F(V^q) \subseteq (U^r)^q = U^q$, proving that $F(a_n)$ converges to $F(a)$.

Conversely, assume that F is not continuous at a. There exists a p and some $U \in \mathcal{B}_p$ such that

$$\forall q \leq p \; \forall V \in \mathcal{B}_q(a) \; \exists a \in V \;\text{ such that }\; F(a) \notin U^q.$$

We claim that there is a directed set D and for each $d \in D$ some $a_d \in \mathfrak{A}$ such that $\{a_d\}_{d\in D}$ converges to a while $F(a_d)$ does not converge to $F(a)$. The set D consists of functions $d = \{(q, V_q) : q \in A_d\}$ where A_d is a maximal antichain in P, and for each q, $V_q \in \mathcal{B}_q(a)$. To get a bigger d in D, refine the partition and take $V_r \subseteq V_q^r$ for $r \leq q$. To get a_d, let a_q be, for each $q \in A_d$, an element of V_q such that $F(a_q) \notin U^q$, and let a_d be the element of $\mathfrak{A}$ defined by

$$qa_d = qa_q \qquad \text{for all } q \in A_d.$$

That the net $\{a_d\}_{d\in D}$ converges to a while $\{F(a_d)\}_d$ does not converge to $F(a)$ is a routine verification. $\square$

We conclude with some remarks about continuity of functions $F\colon \mathfrak{A} \to \mathfrak{A}$. First, it is not difficult to see that F is continuous at $a \in \mathfrak{A}$ if and only if

$$\forall \varepsilon > 0 \; \forall p \; \exists q \subseteq p \; \exists \delta > 0 \; \forall x \, (\text{if } |qx - qa| \leq \delta I \text{ then } |qF(x) - qF(a)| \leq \varepsilon I).$$

Somewhat less immediate is the following equivalence:

Definition. F is *uniform* if $\forall P \, \forall x \, \forall y (px = py$ implies $pF(x) = pF(y))$.
F is *norm-continuous* at a if $\forall \varepsilon \, \exists \delta \, \forall x (|x - a| \leq \delta$ implies $|F(x) - F(a))| \leq \varepsilon)$.

By Remark 1 in Section 3 every continuous function is uniform.

Theorem. *F is continuous if and only if there is a maximal antichain A in P and a collection $\{F_p : p \in A\}$ of uniform norm-continuous functions such that $F = \sum_{p\in A} F_p$.*

We omit the proof and refer the reader to [1, Section 7] for similar arguments (P is weakly distributive).

As a final comment we make the following observation. Let p and $q = I - p$ be nonzero projections. There exists a continuous function $F\colon \mathfrak{A} \to \mathfrak{A}$ such that $F(p) = q$ and $F(q) = p$, namely the function $F(x) = I - x$. On the other hand, let p_1, p_2 and p_3 be projections such that $p_i \cdot p_j = \delta_{ij} I$ and $p_1 + p_2 + p_3 = I$. We claim that there is no uniform function $F\colon \mathfrak{A} \to \mathfrak{A}$ such that $F(p_1) = p_2$, $F(p_2) = p_3$ and $F(p_3) = p_1$.

If such an F existed, we would have (using $p_i =_{p_j} \delta_{ij} I$)

$$p_2 = F(p_1) = p_1 F(p_1) + p_2 F(p_1) + p_3 F(p_1) = p_1 F(I) + p_2(F(0)) + p_3 F(0)$$

and so

$$F(0) =_{p_3} 0,$$

while

$$p_3 = F(p_2) = p_1 F(0) + p_2 F(I) + p_3 F(0)$$

and so

$$F(0) =_{p_3} I,$$

getting a contradiction.

References

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