ON COUNTABLY CLOSED COMPLETE BOOLEAN ALGEBRAS

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Abstract. It is unprovable that every complete subalgebra of a countably closed complete Boolean algebra is countably closed.

Introduction. A partially ordered set $(P, <)$ is *σ-closed* if every countable chain in P has a lower bound. A complete Boolean algebra B is *countably closed* if $(B^+, <)$ has a dense subset that is σ-closed. In [2] the first author introduced a weaker condition for Boolean algebras, *game-closed*: the second player has a winning strategy in the infinite game where the two players play an infinite descending chain of nonzero elements, and the second player wins if the chain has a lower bound. In [1], Foreman proved that when B has a dense subset of size $\aleph_1$ and is game-closed then B is countably closed. (By Vojtáš [5] and Veličković [4] this holds for every B that has a dense subset of size $2^{\aleph_0}$.) We show that, in general, it is unprovable that game-closed implies countably closed. We construct a model in which a B exists that is game-closed but not countably closed. It remains open whether a counterexample exists in ZFC.

Being game-closed is a hereditary property: If A is a complete subalgebra of a game-closed complete Boolean algebra B then A is game-closed. It is observed in [3] that every game-closed algebra is embedded in a countably closed algebra; in fact, for a forcing notion $(P, <)$, being game-closed is equivalent to the existence of a σ-closed forcing Q such that $P \times Q$ has a dense σ-closed subset. Hence the statement "every game-closed complete Boolean algebra is countably closed" is equivalent to the statement "every complete subalgebra of a countably closed complete Boolean algebra is countably closed".

Below we construct (by forcing) a model of ZFC+GCH and in it a partial ordering P of size $\aleph_2$ such that $B(P)$, the completion of P, is not countably closed, but $B(P \times Col)$ is, where Col is the Lévy collapse of $\aleph_2$ to $\aleph_1$ (with countable conditions).

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Theorem. *It is consistent that there exists a partial ordering $(P,<)$ such that $B(P)$ is not countably closed but $B(P \times Col)$ is countably closed.*

Forcing Conditions.

We assume that the ground model satisfies GCH.

We want to construct, by forcing, a partially ordered set $(P,<_P)$ of size $\aleph_2$ that has the desired properties. We shall use as forcing conditions countable approximations of P. One part of a forcing condition will thus be a countable partial ordering $(A,<_A)$ with the intention that A be a subset of P and that the relation $<_A$ on A be the restriction of $<_P$. As P will have size $\aleph_2$, we let $P=\omega_2$, and so A is a countable subset of ω_2.

The second part of a forcing condition will be a countable set $B \subset A \times Col$, a countable approximation of a dense set in the product ordering $P \times Col$. The third part of a forcing condition will be a countable set C of countable descending chains in A that have no lower bound. Finally, a forcing condition includes a function that guarantees that the limit of the B's is σ-closed (and so $P \times Col$ has a σ-closed dense subset).

Whenever we use $<$ without a subscript, we mean the natural ordering of ordinal numbers.

Definition. For any set X, $Col(X)$ is the set of all countable functions q such that $dom(q) \in \omega_1$ and range $(q) \subset X$; $Col = Col(\omega_2)$.

Definition. The set R of forcing conditions r consists of quadruples $r=((A_r,<_r), B_r, C_r, F_r)$ such that

(1) A_r is a countable subset of ω_2,
(2) $(A_r,<_r)$ is a partially ordered set,
(3) if $b <_r a$ then $a < b$,
(4) B_r is a countable subset of $A_r \times Col(A_r)$, and for every $(p,q) \in B_r$, $p \in \text{range}(q)$,
(5) C_r is a countable set of countable sequences $\{a_n\}_{n=0}^{\infty}$ in A_r with the property that $a_0 >_r a_1 >_r \cdots >_r a_n >_r \cdots$ and that $\{a_n\}_n$ has no lower bound in A_r,
(6) F_r is a function of two variables, $\{a_n\}_n \in C_r$ and $(p,q) \in B_r$ such that $p \geq a_0$, and range$(F_r) \subset \omega$. If $m = F_r(\{a_n\}_n,(p,q))$ then for every $(p',q') \in B_r$ stronger than (p,q),

(*) $$\text{if } p' <_r a_m \text{ then } p' \perp_r \{a_n\}_n \text{ (i.e. } p' \perp_r a_k \text{ for some } k).$$

If $r,s \in R$ then $r <_R s$ (r is stronger than s) if

(7) $A_r \supseteq A_s$,
(8) $<_r$ and $<_s$ agree on A_s, and $\perp_r$ and $\perp_s$ agree on A_s; i.e. if $a,b \in A_s$ then $a <_r b$ iff $a <_s b$ and $a \perp_r b$ iff $a \perp_s b$ for all $a,b \in A_s$,
(9) $B_r \supseteq B_s$,
(10) $C_r \supseteq C_s$,
(11) $F_r \supseteq F_s$.

The relation $<_R$ on R is a partial ordering. We shall prove that the forcing extension by R contains a desired example $(P,<_P)$. Assuming the GCH in the ground model, the forcing R preserves cardinals and V^R is a model of $ZFC+GCH$; this follows from the next two lemmas:

Lemma 1. *R is σ-closed.*

Proof. Let $\{r_n\}_n$ be a sequence of conditions such that $r_0 >_R r_1 >_R \cdots >_R r_n >_R \cdots$. We show that $\{r_n\}_n$ has a lower bound.

Assuming that for each n, $r_n = ((A_n, <_n), B_n, C_n, F_n)$, we let $A_r = \bigcup_{n=0}^{\infty} A_n$, $B_r = \bigcup_{n=0}^{\infty} B_n$, $C_r = \bigcup_{n=0}^{\infty} C_n$, $F_r = \bigcup_{n=0}^{\infty} F_n$ and $<_r = \bigcup_{n=0}^{\infty} <_n$; we claim that $r = ((A_r, <_r), B_r, C_r, F_r)$ is a condition, and is stronger than each r_n.

The quadruple r clearly has properties (1)–(4). It is also easy to see that for every n, $<_r$ agrees with $<_n$ and $\perp_r$ agrees with $\perp_n$ on A_n. To verify (5), let $\{a_n\}_n \in C_r$. There is an m such that $\{a_n\}_n \in C_k$ for all $k \geq m$, and therefore $\{a_n\}_n$ has no lower bound in any A_k. Thus $\{a_n\}_n$ has no lower bound in A_r. Finally, to verify (6), let $F_r(\vec{a}, (p, q)) = m$ and let (p', q') be stronger than (p, q). Since (*) holds in r_n where n is large enough so that $\vec{a} \in C_n$ and $(p, q), (p', q') \in B_n$, (*) holds in r as well.

Therefore r is a condition and for every n, r is stronger than r_n.

Lemma 2. *R has the $\aleph_2$-chain condition.*

Proof. If W is a set of conditions of size $\aleph_2$, then a Δ-system argument (using CH) yields two conditions $r, s \in W$ such that if $r = ((A_r, <_r), B_r, C_r, F_r)$ and $s = ((A_s, <_s), B_s, C_s, F_s)$, then there is a D (the root of the Δ-system) such that $D = A_r \cap A_s$, $\sup D < \min(A_r - D)$, $\sup A_r < \min(A_s - D)$, $<_r$ and $<_s$ agree on D, $\perp_r$ and $\perp_s$ agree on D, $B_r \cap (D \times Col(D)) = B_s \cap (D \times Col(D))$, $C_r \cap D^\omega = C_s \cap D^\omega$, and $F_r(\vec{a}, (p, q)) = F_s(\vec{a}, (p, q))$ whenever $\vec{a} \in C_r \cap D^\omega$ and $(p, q) \in B_r \cap (D \times Col(D))$.

Moreover, there exists a mapping π of A_s onto A_r that is an isomorphism between s and r and is the identity on D.

Let $t = ((A_t, <_t), B_t, C_t, F_t)$ where $A_t = A_r \cup A_s$, $B_t = B_r \cup B_s$, $C_t = C_r \cup C_s$, $<_t = <_r \cup <_s$, and F_t will be defined below such that $F_t \supseteq F_r \cup F_s$. We claim that t is a condition, and is stronger than both r and s; thus r and s are compatible. Properties (1)–(4) are easy to verify. It is also easy to see that $<_t$ agrees with $<_r$ on A_r and with $<_s$ on A_s, and $\perp_t$ agrees with $\perp_r$ on A_r and with $\perp_s$ on A_s.

Note that if $a \in A_r - D$ and $b \in A_s - D$ then $a \perp_t b$. Thus if $\{a_n\}_n$ is in C_r but not in C_s (or vice versa) then $\{a_n\}_n$ has no lower bound in $A_r \cup A_s$, and so (5) holds.

In order to deal with (6), we first verify it for the values of F_t inherited from either r or s. Thus let $\vec{a} \in C_r$, $(p, q) \in B_r$, $m = F_r(\vec{a}, (p, q))$ and let $(p', q') \in B_t$ be stronger than (p, q). (The argument for s in place of r is completely analogous.) If $(p', q') \in B_r$ then (*) holds in r and therefore in t. Thus assume that $(p', q') \in B_s$.

Since $p' \in A_s$ and $p' <_t p$, it follows that $p \in D$, and since $\text{range}(q) \subseteq \text{range}(q') \subseteq A_s$, we have $(p, q) \in B_s$. Now if $\vec{a} \in C_s$ then $F_s(\vec{a}, (p, q)) = F_r(\vec{a}, (p, q))$ and so p' satisfies (*) in s and hence in t.

If $\vec{a} \notin C_s$ and $p' \notin A_r$ then $p' \perp_t \vec{a}$ and again p' satisfies (*).

The remaining case is when $p' \in D$ and $(p, q) \in B_r \cap B_s$. Since $(p', \pi q') = (\pi p', \pi q')$ is stronger than $(p, q) = (\pi p, \pi q)$, p' satisfies (*) in r and therefore in t.

To complete the verification of (6) we define $F_t(\vec{a},(p,q))$ for those $\vec{a}$ and (p,q) that come from the two different conditions. Let $\vec{a} \in C_r - C_s$ and $(p,q) \in B_s - B_r$ (the other case being analogous) be such that $p \geq a_0$. We let $F_t(\vec{a},(p,q))$ be the least m such that $a_m \notin D$.

Let $(p',q') \in B_t$ be stronger than (p,q); we shall show that $p' \not<_t a_m$. This is clear if $p' \in D$, by (3). If $p' \notin D$ then we claim that p' cannot be in A_r; then it follows that $p' \perp_t a_m$. To prove the claim, note that range$(q) \not\subseteq A_r$ (because $(p,q) \notin B_r$) and hence range$(q') \subseteq A_s$. By (4), $p' \in A_s$ and so $p \notin A_r$.

Therefore t is a condition and is stronger than both r and s.

Let G be a generic filter on R. In V_G, we let $P = \bigcup\{A_r : r \in G\}$, $<_P = \bigcup\{<_r : r \in G\}$, and $Q = \bigcup\{B_r : r \in G\}$. $(P, <_P)$ is a partial ordering and $Q \subset P \times Col$. We shall prove that Q is σ-closed and is dense in $P \times Col$, and that the complete Boolean algebra $B(P)$ does not have a dense σ-closed subset.

Lemma 3. $P = \omega_2$.

Proof. We prove that for every s and every $p \in \omega_2$ there exists an $r <_R s$ such that $p \in A_r$. But this is straightforward: let $A_r = A_s \cup \{p\}$, $B_r = B_s$, $C_r = C_s$, $F_r = F_s$ and $<_r = <_s$; properties (1)–(11) are easily verified. (Note that $p \perp_r a$ for all $a \in A_s$.)

Lemma 4. *Q is dense in $P \times Col$.*

Proof. Let s be a condition and let $p_0 \in A_s$ and $q_0 \in Col$. We shall find an $r <_R s$, $p \in A_r$ and $q \supset q_0$ such that $p <_r p_0$ and $(p,q) \in B_r$: Let p be an ordinal greater than $\sup A_s$, let $q \in Col$ be such that $q \supset q_0$ and $p \in$ range (q), and let $A_r = A_s \cup$ range (q), $B_r = B_s \cup \{(p,q)\}$, $C_r = C_s$, and let $<_r$ be the partial order of A_r that extends $<_s$ by making $p <_r p_0$. Finally, let $F_r(\vec{a},(p,q)) = 0$ for all $\vec{a} \in C_r$.

To see that $r = ((A_r, <_r), B_r, C_r, F_r)$ is a condition, note that for every $\vec{a} \in C_r$, p is not a lower bound of $\vec{a}$ (because p_0 isn't) and hence $p \perp_r \vec{a}$. This implies both (5) and (6). Since adding p does not affect the relation $\perp$ on A_s, we have (8) and so r is stronger than s.

Next we prove that Q is σ-closed.

Lemma 5. *If $u = \{(p_n,q_n)\}_{n=0}^{\infty}$ is a descending chain in Q then u has a lower bound.*

Proof. Let $\dot{u}$ be a name for a descending chain and let s be a condition. By extending s ω times if necessary (R is σ-closed), we may assume that there is a sequence $u = \{(p_n,q_n)\}_{n=0}^{\infty}$ in $\omega_2 \times Col$ such that s forces $\dot{u} = u$, such that for every n, $p_n \in A_s$, $(p_n,q_n) \in B_s$, that $p_0 >_s p_1 >_s \cdots >_s p_n > \cdots$ is a descending chain in $(A_s, <_s)$ and that $q_0 \subset q_1 \subset \cdots \subset q_n \subset \ldots$.

Let p be an ordinal greater than $\sup A_s$, let $q \supseteq \bigcup_{n=0}^{\infty} q_n$ be such that $p \in$ range$(q) \subseteq A_s \cup \{p\}$, let $A_r = A_s \cup \{p\}$, $B_r = B_s \cup \{(p,q)\}$, $C_r = C_s$, and let $<_r$ be the partial order of A_r that extends $<_s$ by making p a lower bound of $\{p_n\}_{n=0}^{\infty}$. Finally, let $F_r(\vec{a},(p,q)) = 0$ for all $\vec{a} \in C_r$ and $r = ((A_r, <_r), B_r, C_r, F_r)$.

We shall show that for every $\vec{a} \in C_s$, p is not a lower bound of $\vec{a}$. This implies that $p \perp_r \vec{a}$ and (5) and (6) follow. Since making p a lower bound of $\{p_n\}_n$ does not affect the relation $\perp$ on A_s, we'll have (8) and hence $r <_R s$. In r, (p,q) is a lower bound of u.

Thus let $\vec{a} = \{a_k\}_k \in C_s$. We claim that

$$\exists k \, \forall n \, p_n \not<_s a_k .$$

This implies that $p \not<_r a_k$ and hence p is not a lower bound of $\vec{a}$.

If $p_n < a_0$ for all n then we let $k = 0$ because then $p_n \not<_s a_0$ for all n.

Otherwise let N be the least N such that $p_N \geq a_0$, and let $m = F_s(\vec{a}, (p_N, q_N))$. Either $p_n \not<_s a_m$ for all n and we are done (with $k = m$) or else $p_M <_s a_m$ for some $M \geq N$. By (*) there exists some k such that $p_M \perp_s a_k$ and hence $p_n \not<_s a_k$ for all n.

Finally, we shall prove that $B(P)$ is not countably closed.

Lemma 6. *The complete Boolean algebra $B(P)$ does not have a dense σ-closed subset.*

Proof. Assume that $B(P)$ does have a dense σ-closed subset D. For $a, b \in P$, we define

$$a \prec b \quad \text{if} \quad a <_P b \quad \text{and} \quad \exists\, d \in D \quad \text{such that} \quad a <_{B(P)} d <_{B(P)} b.$$

The relation $\prec$ is a partial ordering of P, $(P, \prec)$ is σ-closed, $a \prec b$ implies $a <_P b$ and for every $a \in P$ there is some $b \in P$ such that $b \prec a$.

Toward a contradiction, let s be a condition and assume that s forces the preceding statement. For each $\alpha < \omega_2$, there exist a condition s_α stronger than s, and a descending chain $\{c_n^\alpha\}_n$ in A_{s_α} such that $c_0^\alpha \geq \alpha$ and that for every n, $s_\alpha \Vdash c_{n+1}^\alpha \prec c_n^\alpha$.

By a Δ-system argument we find among these a countable sequence $r_n = s_{\alpha_n} = ((A_n, <_n), B_n, C_n, F_n)$ and a set E such that for every m and n with $m < n$ we have $E = A_m \cap A_n$, $\sup E < \min(A_m - E)$, $\sup A_m < \min(A_n - E)$, $<_m$ and $<_n$ agree on E, $\perp_m$ and $\perp_n$ agree on E, $B_m \cap (E \times Col(E)) = B_n \cap (E \times Col(E))$, $C_m \cap E^\omega = C_n \cap E^\omega$, and $F_m(\vec{a}, (p,q)) = F_n(\vec{a}, (p,q))$ whenever $\vec{a} \in C_m \cap E^\omega$ and $(p,q) \in B_m \cap (E \times Col(E))$. Moreover, there exists a mapping π_{mn} of A_m onto A_n that is an isomorphism between $(r_m, \{c_k^{\alpha_m}\}_k)$ and $(r_n, \{c_k^{\alpha_n}\}_k)$ and is the identity on E. We also let $\pi_{nm} = {\pi_{mn}}^{-1}$, $\pi_{mm} = id$ and assume that the π_{mn} form a commutative system. Note that for every n and k, $c_k^{\alpha_n} \notin E$.

For each n and k, let $a_k^n = c_{2k}^{\alpha_n}$ and $b_k^n = c_{2k+1}^{\alpha_n}$. Let $\vec{u} = \{u_n\}_n$ be the "diagonal sequence"

$$u_{2n} = a_n^n, \quad u_{2n+1} = b_n^n.$$

We shall find a condition $t = ((A_t, <_t), B_t, C_t, F_t)$ stronger than all r_n such that the diagonal sequence $\vec{u}$ is a descending chain and belongs to C_t. Since $t \Vdash b_n^n \prec a_n^n$ for every n, it forces that $(P, \prec)$ is not σ-closed. This will complete the proof.

To construct t we first let $A_t = \bigcup_{n=0}^\infty A_n$ and $B_t = \bigcup_{n=0}^\infty B_n$. Let $<_t$ be the minimal partial ordering extending $\bigcup_{n=0}^\infty <_n$ such that for every n, $a_{n+1}^{n+1} <_t b_n^n$. Before proceeding to define C_t and F_t we shall prove some properties of $(A_t, <_t)$.

Lemma 7. *(i) Let $m < n$ and let $y \in A_m - E$ and $x \in A_n - E$. If $x <_t y$ then $x \leq_n a^n_n$ and $b^m_m \leq_m y$. If x and y are compatible in $<_t$ then $b^m_m \leq_m y$.*

(ii) For all m and n, if $x \in A_n$ and $y \in A_m$ and if $x <_t y$ then $x <_n \pi_{mn} y$ (and $\pi_{nm} x <_m y$). In particular, if $x, y \in A_n$ then $x <_t y$ if and only if $x <_n y$.

(iii) For all m and n, if $x \in A_n$ and $y \in A_m$ and if x and y are compatible in $<_t$ then x and $\pi_{mn} y$ are compatible in $<_n$ (and $\pi_{nm} x$ and y are compatible in $<_m$). In particular, if $x, y \in A_n$ then $x \perp_t y$ if and only if $x \perp_n y$.

Proof. (i) The first statement is an obvious consequence of the definition of $<_t$, and the second follows because any z such that $z \leq_t x$ is in some $A_k - E$ where $k \geq n$.

(ii) Let $x \in A_n$ and $y \in A_m$ and let $x <_t y$. First assume that $y \notin E$ (and so $x \notin E$.) Necessarily, $m \leq n$ and if $m = n$ then clearly $x <_n y$. Thus consider $m < n$. By (i) $x \leq_n a^n_n <_n b^n_m = \pi_{mn}(b^m_m) \leq_n \pi_{mn} y$.

Now assume that $y \in E$ and proceed by induction on x. If $x \in E$ then $x <_n y$. If $x \notin E$ then either $x <_n y$ or there exists some $z \notin E$ such that $x <_t z <_t y$, and by the induction hypothesis $z <_k \pi_{mk} y$ (where $z \in A_k$). Applying the preceding paragraph to x and z we get $\pi_{nk} x <_k z$ and hence $\pi_{nk} x <_k \pi_{mk} y$. The statement now follows.

(iii) Let $x \in A_n$ and $y \in A_m$ and let $z \in A_k$ be such that $z <_t x$ and $z <_t y$. By (ii) we have $\pi_{kn} z <_n x$ and $\pi_{km} z <_m y$. Hence $\pi_{kn} z = \pi_{mn} \pi_{km} z <_n \pi_{mn} y$. The second statement follows from this and from the second statement of (ii).

Lemma 7 guarantees that t will be stronger than every r_n. Another consequence is that if $\vec{a} \in C_n$ then $\vec{a}$ has no lower bound in $<_t$: if $x \in A_m$ were a lower bound then $\pi_{mn} x$ would be a lower bound in $<_n$.

Let $C_t = \bigcup_{n=0}^{\infty} C_n \cup \{\vec{u}\}$. Every sequence in C_t is a descending chain in $<_t$ without a lower bound (clearly, $\vec{u}$ has no lower bound).

Lemma 8. *For all k and n, if $(p, q) \in B_k - B_n$ and if $(p', q') \in B_t$ is stronger than (p, q) then $(p', q') \in B_k - B_n$.*

Proof. Since $(p, q) \notin B_n$, we have either range$(q) \not\subseteq E$ or $p \notin E$, in which case $p \in$ range(q) by (4) and again range$(q) \not\subseteq E$. Since $q \subseteq q'$ it must be the case that $(p', q') \in B_k - B_n$.

We shall now define F_t so that $F_t \supset \bigcup_{n=0}^{\infty} F_n$ and verify (6). This will complete the proof.

First we let $F_t(\vec{a}, (p, q)) = F_n(\vec{a}, (p, q))$ whenever the right-hand side is defined; we have to show that (6) holds in t. Let $m = F_n(\vec{a}, (p, q))$ and let $(p', q') \in B_k$ be stronger than (p, q). It follows from Lemma 8 that $(p, q) \in B_k$. Now $(\pi_{kn} p', \pi_{kn} q')$ is stronger than $(\pi_{kn} p, \pi_{kn} q) = (p, q)$ and (*) holds for $\pi_{kn} p'$ in r_n. If $p' <_t a_m$ then by Lemma 7 $\pi_{kn} p' <_n a_m$ and hence $\pi_{kn} p' \perp_n \vec{a}$. By Lemma 7 again, $p' \perp_t \vec{a}$.

Next, let $\vec{a}$ and (p, q) be such that $\vec{a} \in C_n - C_k$, $(p, q) \in B_k - B_n$ and $p \geq a_0$. If $k < n$, we have $\pi_{kn} p \geq p \geq a_0$ and we let $F_t(\vec{a}, (p, q)) = F_n(\vec{a}, (\pi_{kn} p, \pi_{kn} q))$. To verify (6), let $m = F_t(\vec{a}, (p, q))$ and let $(p', q') \in B_t$ be stronger than (p, q). By Lemma 8 $(p', q') \in B_k$, and $(\pi_{kn} p', \pi_{kn} q')$ is stronger (in r_n) than $(\pi_{kn} p, \pi_{kn} q)$. If $p' <_t a_m$ then by Lemma 7 $\pi_{kn} p' <_n a_m$ and so $\pi_{kn} p' \perp_n \vec{a}$. By Lemma 7 again, $p' \perp_t \vec{a}$.

If $k > n$, we let $F_t(\vec{a}, (p, q))$ be the least m such that $a_m \notin E$ and that $b^n_n \not\leq_n a_m$ (such m exists as $\vec{a}$ does not have a lower bound in A_n). To verify (6), let $(p', q') \in B_t$

be stronger than (p,q). If $p' \in E$ then $p' \not<_t a_m$ and if $p' \notin E$ then by Lemma 7(i) $p' \perp_t a_m$. In either case, (6) is satisfied.

Finally, we define $F_t(\vec{u},(p,q))$. Thus let $(p,q) \in B_t$ be such that $p \geq u_0$. Since $u_0 = a_0^0 \notin E$, we have $p \notin E$. Let n be the n such that $p \in A_n$. We let $F_t(\vec{u},(p,q)) = 2n+2$. That is, the chosen u_m is $u_{2n+2} = a_{n+1}^{n+1}$. To verify (6), let $(p',q') \in B_t$ be stronger than (p,q). Since $p \in A_n - E$, by Lemma 8 we have $(p',q') \in B_n$ and therefore $p' \in A_n - E$. But $a_{n+1}^{n+1} \in A_{n+1} - E$ and so $p' \not<_t a_{n+1}^{n+1}$. Therefore (6) holds.

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