§0 INTRODUCTION

In [3], Foreman, Magidor and Shelah introduced a forcing axiom, called Martin's Maximum, which says that if P is a forcing notion preserving stationary subsets of ω_1, and $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ is a sequence of dense subsets, then there exists a filter $G \subseteq P$ meeting every D_α. They showed that if the existence of a supercompact cardinal is consistent with the set theory, then Martin's Maximum is also consistent with the set theory. They also showed many consequences of Martin's Maximum. For example, Martin's Maximum implies that the Singular Cardinal Hypothesis holds, the nonstationary ideal on ω_1 is saturated, and every stationary set on every countable space reflects.

In this paper, we continue the study of Martin's Maximum initiated in [3]. We define *projective* stationary sets and prove that Martin's Maximum implies that every projective stationary set contains an increasing continuous $\in$–chain of length ω_1. We will also show that the consequences of Martin's Maximum in [3] follow from the assumption that every projective stationary set contains an increasing continuous $\in$–chain of length ω_1. We will call this assumption the Projective Stationary Reflection principle. It turns out that this principle is in fact equivalent to the Strong Reflection Principle of Todorcevic [9] (see next section for details). It differs from the known stationary reflection principles in two respects. First, it is not applicable to every stationary set, but only to the projective stationary sets. Basically, a projective stationary set is a stationary set which projects to every stationary subset of ω_1. Hence there are many stationary sets which are not projective stationary. Secondly, it claims that every projective stationary set reflects to a closed unbounded subset, not just a stationary subset , of some countable space $[X]^\omega$ with X has cardinality of $\aleph_1$.

It is a strong reflection principle because it implies the known stationary reflection principles, not the other way around. A typical stationary reflection principle is the following reflection principle from [3]:

(RP) *If $S \subseteq [H_\kappa]^\omega$ is stationary, then there exists an X of size $\aleph_1$ such that $\omega_1 \subseteq X$ and $S \cap [X]^\omega$ is stationary in $[X]^\omega$.*

This stationary reflection principle has many interesting consequences. For example, RP implies that every ω_1–stationary preserving partially ordered set is semiproper; RP implies that the nonstationary ideal on ω_1 is presaturated (see later for definitions), theorems of Foreman, Magidor and Shelah [3]; RP implies that $2^{\aleph_0} \leq \aleph_2$, a theorem of Todorcevic [8], etc. (See [1,2,3,8,9,10] for more results on this direction.)

However, it is also proved in [3] that when a supercompact cardinal is Levy collapsed to ω_2, in the resulting model, every stationary set reflects. But in such a model, the strong reflection principle does not hold.

In section 1, we study the Projective Stationary Reflection Principle. We will prove that it follows from Martin's Maximum and it is equivalent to the Strong Reflection Principle of Todorcevic.

In section 2, we present some natural examples of projective stationary sets.

In section 3, we study some special projective stationary sets, which form a filter. It turns out that this filter is closely connected to the saturation of the nonstationary ideal on ω_1. Namely, the saturation of the nonstationary ideal on ω_1 itself is a kind of reflection.

We refer to [4,5] for all terms used but not explicitly defined in the text.

§1 PROJECTIVE STATIONARY SETS

Let κ be a regular cardinal $\geq \omega_2$. Let H_κ be the set of all sets hereditarily of size less than κ. We assume that H_κ is endowed with a well ordering. We use $[H_\kappa]^\omega$ to denote the set of all countable subsets of H_κ. Notice that $[H_\kappa]^\omega \subseteq H_\kappa$. A subset $C \subseteq [H_\kappa]^\omega$ is **closed** and **unbounded** (a **club**, in short) if there is a function $f : [H_\kappa]^{<\omega} \to H_\kappa$ such that for every countable $x \in [H_\kappa]^\omega$, $x \in C$ if and only if x is closed under f, i.e., if $e \in [x]^{<\omega}$, then $f(e) \in x$, where $[A]^{<\omega}$ denotes the set of all finite subsets of A. A subset $S \subseteq [H_\kappa]^\omega$ is **stationary** if for every club $C \subseteq [H_\kappa]^\omega$ the intersection $C \cap S$ is not empty. A basic fact that is used frequently is the normality of the club filter: if S is stationary, $f : S \to H_\kappa$ is a choice function, i.e., $f(x) \in x$ for all $x \in S$, then there is a stationary subset on which f is a constant. In particular, the intersection of countably many clubs contains a club.

We use $N \prec M$ to denote that N is an elementary submodel of M as usual. Also if $f : X \to Y$ and $A \subseteq X$, we use $f''A$ to denote the set $\{f(a) \mid a \in A\}$.

A sequence $\langle N_\alpha \mid \alpha < \theta\rangle$ of countable submodels of H_κ is an **increasing continuous $\in$–chain** of length θ if for all $\alpha < \theta$, $(N_\alpha \prec H_\kappa)$ and for all $\alpha < \beta < \theta$, $(N_\alpha \in N_\beta)$, and if $\alpha < \theta$ is a limit ordinal then $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$. It is a **strongly increasing continuous $\in$–chain** of length θ if , in addition, $\alpha + 1 < \theta$ implies that $\langle N_\beta \mid \beta < \alpha\rangle \in N_{\alpha+1}$ for all limit ordinals $\alpha < \theta$.

A sequence $\langle N_\alpha \mid \alpha < \theta\rangle$ of countable submodels of H_κ is an **increasing continuous $\subseteq$–chain** of length θ if for all $\alpha < \theta$, $(\alpha \subseteq N_\alpha \prec H_\kappa)$ and for all $\alpha < \beta < \theta$, $(N_\alpha \subseteq N_\beta)$, and if $\alpha < \theta$ is a limit ordinal then $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$.

We will frequently use a simple fact without mentioning it. This fact says that if $N \prec H_\kappa$ and $x \in N$ is countable then $x \subseteq N$. It follows that every increasing continuous $\in$–chain is an increasing continuous $\subseteq$–chain.

Note that for closed unbounded many $X \in [H_\kappa]^\omega$, $X \cap \omega_1$ is an ordinal.

DEFINITION 1.1 The **projection** of a set $S \subseteq [H_\kappa]^\omega$ is the set $\mathrm{Proj}(S) = \{X \cap \omega_1 : X \in S\}$. We say that $S \subseteq [H_\kappa]^\omega$ is **projective stationary** if for every club $C \subseteq [H_\kappa]^\omega$, $\mathrm{Proj}(S \cap C)$ contains a club in ω_1. Equivalently, S is projective stationary if and only if the set $S_T = \{X \in S \mid X \cap \omega_1 \in T\}$ is stationary for every stationary subset T of ω_1.

It follows from the definition that every projective stationary set is stationary. But the converse is not true. Also, there are disjoint projective stationary sets. We will see some natural examples later.

We are interested in projective stationary sets primarily because we are interested in the following Projective Stationary Reflection principle:

Projective Stationary Reflection: *For every regular cardinal $\kappa \geq \omega_2$, if $S \subseteq [H_\kappa]^\omega$ is a projective stationary set, then there exists an increasing continuous $\in$–chain $\langle N_\alpha \mid \alpha < \omega_1\rangle$ of countable elementary submodels of H_κ of length ω_1 such that $N_\alpha \in S$ for all $\alpha < \omega_1$.*

We now show that the Projective Stationary Reflection Principle follows from Martin's Maximum.

Let us recall Martin's Maximum:

A forcing P is ω_1–stationary preserving if every stationary subset of ω_1 in the ground model remains stationary in the generic extension. Martin's Maximum is the following statement:

If P is a ω_1–stationary preserving partially ordered set, if $\{D_\alpha \mid \alpha < \omega_1\}$ is a sequence of dense subsets of P of size $\aleph_1$, then there exists a filter $G \subseteq P$ which meets every D_α.

It is proved in [3] that if the existence of a supercompact cardinal is consistent with the axioms of set theory, then Martin's Maximum is also consistent with the axioms of set theory. It is shown also in [3] that Martin's Maximum has many interesting consequences.

To show that Martin's Maximum implies the Projective Stationary Reflection Principle, we need a lemma saying that every stationary subset $S \subseteq [H_\kappa]^\omega$ contains strongly increasing continuous $\in$–chains of any countable length. This is a generalization of the following lemma (see [4], Exercise 7.12, Page 60).

Lemma 1.1 If $S \subseteq \omega_1$ is stationary, then for any countable ordinal α, for any countable ordinal γ, there exists an $x \subseteq S - \gamma$ such that x is closed and has order type $\alpha + 1$.

The proof of this lemma is by induction on α. We will use this lemma to prove the following:

Lemma 1.2 If $S \subseteq [H_\kappa]^\omega$ is stationary, then for every countable ordinal α, there exists a strongly increasing continuous $\in$–chain $\langle N_\gamma \mid \gamma \leq \alpha\rangle$ of length $\alpha + 1$ such that $N_\gamma \in S$ for all $\gamma \leq \alpha$.

Proof Let $S \subseteq [H_\kappa]^\omega$ be stationary. Let $\alpha < \omega_1$ be fixed. We will show that in some generic extension by a σ–closed forcing the conclusion of the lemma holds for this S and α. That will be enough.

Consider the following forcing P.

A condition p is a strongly increasing continuous $\in$–chain of length $\theta + 1$ for some $\theta < \omega_1$ such that $p(\beta) \prec H_\kappa$ is countable for all $\beta < \theta + 1$.

The ordering is by extension.

Claim P is σ–closed.

To see this, let $\langle p_n \mid n < \omega \rangle$ be a decreasing sequence of conditions. Let θ_n be the largest countable ordinal in the domain of p_n. Let $\theta = \bigcup_{n<\omega} \theta_n$ and let $N = \bigcup_{n<\omega} p_n(\theta_n)$. Then $N \prec H_\kappa$ is a countable elementary submodel. Define $q(\theta) = N$ and $q(\beta) = p_n(\beta)$ for $\beta < \theta$ with n the least such that $\beta \leq \theta_n$. q is then a condition extending all the p_n.

Hence P is a σ–closed forcing.

Let $G \subseteq P$ be a generic filter over V. In $V[G]$, let $f = \bigcup G$. Then f is an increasing continuous $\in$–chain of length ω_1 which enumerates a closed unbounded subset of $([H_\kappa]^\omega)^V$. Since S remains stationary in the generic extension, the following set is a stationary subset of ω_1:

$$T = \{\beta < \omega_1 \mid f(\beta) \in S\}.$$

By the quoted lemma above, let $x \subseteq T$ be a closed set of order type $\alpha + 1$. Let $x = \{\gamma_\beta \mid \beta \leq \alpha\}$ be the canonical enumeration. Let $N_\beta = f(\gamma_\beta)$ for $\beta \leq \alpha$. Then$\langle N_\beta \mid \beta \leq \alpha \rangle \in V$ is the desired strongly increasing continuous $\in$–chain of length $\alpha + 1$.

This proves the lemma.

□

THEOREM 1.1 Assume the Martin's Maximum. Then the Projective Stationary Reflection principle holds. In fact, if $S \subseteq [H_\kappa]^\omega$ is projective stationary, then there exists a strongly increasing continuous $\in$–chain of length ω_1 through S.

Proof Let $S \subseteq [H_\kappa]^\omega$ be projective stationary.

The idea is to shoot a strongly increasing continuous $\in$–chain of length ω_1 through S. Thus, a condition $p = \langle N_\alpha \mid \alpha \leq \theta \rangle$ is a strongly increasing continuous $\in$–chain of length $\theta + 1$ for some $\theta < \omega_1$ such that for all $\alpha \leq \theta$ $(N_\alpha \in S)$. The ordering is by extension.

Let P be the set of all conditions. We are going to show that forcing with P preserves stationary subsets of ω_1.

For $\alpha < \omega_1$, let $D_\alpha = \{p \in P \mid \alpha \in dom(p)\}$. For $x \in H_\kappa$, let $D_x = \{p \in P \mid \exists\, \alpha \in dom(p)\ (x \in p(\alpha))\}$.

We claim that each D_α and each D_x is dense. To see this, let $\alpha < \omega_1$ and $x \in H_\kappa$. Let $p = \langle N_\beta \mid \beta \leq \theta \rangle$ be a condition.

Then $S_1 = \{N \in S \mid \{p, x\} \subseteq N\}$ is stationary. Applying the lemma above, let $\langle M_\gamma \mid \gamma \leq \alpha \rangle$ be a strongly increasing continuous $\in$–chain from S_1. We then define $q(\beta) = M_\beta$ for all $\beta \leq \alpha$ such that $\beta > \theta$ and define $q(\beta) = p(\beta)$ for all $\beta \leq \theta$. It follows that $q \leq p$ and $q \in D_\alpha \cap D_x$.

We can now show that the forcing preserves stationary subsets of ω_1.

Let $T \subseteq \omega_1$ be stationary. Let $\dot{C}$ be a name for a club subset of ω_1. We would like to show that $\Vdash \check{T} \cap \dot{C} \neq \emptyset$.

Let $p \in P$ be a condition.

Let $S_T = \{N \in S \mid N \cap \omega_1 \in T\}$. Then S_T is stationary.

Let $\lambda \geq (2^{2^{|P|}})^+$ be a regular cardinal. Consider the structure

$$\mathcal{H} = \langle H_\lambda, \in, \triangle, P, \{\dot{C}\}, S_T, \cdots \rangle,$$

where $\triangle$ is a well ordering of H_λ.

Let $N \prec \mathcal{H}$ be countable such that $N \cap H_\kappa \in S_T$, and $\{\dot{C}, S, T, S_T, p\} \subseteq N$. Let $\delta = N \cap \omega_1$. Let $\langle D_n \mid n < \omega \rangle$ be a list of all dense subsets of P which are in N. Let $p_0 = p$. Inductively, pick $p_{n+1} \in D_n \cap N$ so that $p_{n+1} \leq p_n$. Let θ_n be the largest countable ordinal in $dom(p_n)$. By elementarity and a density argument, we have $\delta = \bigcup_{n<\omega} \theta_n$ and $N \cap H_\kappa = \bigcup_{n<\omega} p_n(\theta_n)$. Therefore,

$$q = \bigcup_{n<\omega} p_n \cup \{(\delta, N \cap H_\kappa)\}$$

is a condition stronger than all p_n. Since $\dot{C} \in N$, for each $\beta < \delta$, there is a name $\dot{\gamma} \in N$ such that

$$\Vdash \dot{\gamma} \in \dot{C} \ \&\ \check{\beta} < \dot{\gamma}.$$

Each such name corresponds to a dense subset in N. Hence, $q \Vdash \dot{\gamma} \in \check{\delta}$. It follows that $q \Vdash$ "$\dot{C} \cap \check{\delta}$ is unbounded in $\check{\delta}$.". Thus, $q \Vdash \check{\delta} \in \dot{C}$. Therefore, $q \Vdash \dot{C} \cap \check{T} \neq \emptyset$.

It follows then that forcing with P preserves stationary subsets of ω_1.

Now let $G \subseteq P$ be a filter meeting all the dense subsets D_α for $\alpha < \omega_1$ defined above. Let

$$\langle N_\alpha \mid \alpha < \omega_1 \rangle = \bigcup G.$$

Then this is a strongly increasing continuous $\in$–chain of length ω_1 with each $N_\alpha \in S$.

This finishes the proof.

□

From the proof above, we have seen that for a given stationary $S \subseteq [H_\kappa]^\omega$, there is a natural forcing notion P_S associated with S to shoot an increasing continuous

$\in$–chain of length ω_1. Then S is projective stationary if and only if this forcing notion P_S preserves stationary sets of ω_1. (We have proved the harder direction. To see the other direction, assume that S is not projective stationary. Let $T \subseteq \omega_1$ be stationary such that $S_T = \{N \in S \mid N \cap \omega_1 \in T\}$ is not stationary. Then T is not stationary in the extension by P_S.)

After the work of Foreman, Magidor and Shelah [3], Todorcevic in a circulated hand written note [9] in September 1987 formulated the following Strong Reflection Principle (SRP) (See also [1], page 57–60):

Strong Reflection Principle: *For every κ, every $S \subseteq [\kappa]^\omega$ and for every regular $\theta > \kappa$ there is an increasing continuous $\in$–chain $\{N_\alpha \mid \alpha < \omega_1\}$ of countable elementary models of H_θ (with N_0 containing a prescribed element of H_θ) such that for all $\alpha < \omega_1$, $N_\alpha \cap \kappa \in S$ if and only if there exists a countable elementary submodel M of H_θ such that $N_\alpha \subseteq M$, $M \cap \omega_1 = N_\alpha \cap \omega_1$, and $M \cap \kappa \in S$.*

In the following, we show that the Strong Reflection Principle and the Projective Stationary Reflection principle are equivalent.

THEOREM 1.2 The Strong Reflection Principle holds if and only if the Projective Stationary Reflection principle holds.

Proof First we show that the Strong Reflection Principle implies the Projective Reflection principle.

Let $S \subseteq [H_\kappa]^\omega$ be a projective stationary set. Let $\{N_\alpha \mid \alpha < \omega_1\}$ be a continuous $\in$–chain of countable elementary submodels of H_θ with $\theta > \kappa$, given by SRP for S.

We would like to show that $\{\alpha < \omega \mid N_\alpha \cap H_\kappa \in S\}$ contains a club in ω_1.

Assume not. Let $T = \{\alpha < \omega_1 \mid N_\alpha \cap H_\kappa \notin S \text{ and } N_\alpha \cap \omega_1 = \alpha\}$. Then T is stationary.

Define D to be the following set

$$D = \{N \in [H_\theta]^\omega \mid H_\kappa \in N \;\&\; \forall \beta \in N \cap \omega_1 \; N_\beta \in N\}.$$

By normality, D contains a club on $[H_\theta]^\omega$.

Since S is projective stationary, there exists an $N \in D$ such that $N \cap \omega_1 \in T$, $N \cap H_\kappa \in S$ and N is an elementary submodel of H_θ. Let $\alpha = N \cap \omega_1$. Since if $\beta < \alpha$ then $N_\beta \subseteq N$, $N_\alpha \cap \omega_1 = \alpha = N \cap \omega_1$ and $N_\alpha \subseteq N$. Hence $N_\alpha \cap H_\kappa \in S$ by the property of N_α. This is a contradiction.

This proves that $\{\alpha < \omega_1 \mid N_\alpha \cap H_\kappa \in S\}$ contains a club in ω_1.

We prove now that the Projective Stationary Reflection principle implies the Strong Reflection Principle.

Let $\kappa \geq \omega_1$ and $\theta > \kappa$. Assume that $S \subseteq [\kappa]^\omega$ and θ is regular. Define S^* to be the following set: for $N \in [H_\theta]^\omega$, let $N \in S^*$ if and only if $N \prec (H_\theta, \in, \triangle)$ and that there exists a countable $M \prec (H_\theta, \in, \triangle)$ such that $N \subseteq M$, $N \cap \omega_1 = M \cap \omega_1$ and $M \cap \kappa \in S$ implies that $N \cap \kappa \in S$.

Claim S^* is projective stationary in $[H_\theta]^\omega$.

Let $g : [H_\theta]^{<\omega} \to H_\theta$ and $T \subseteq \omega_1$ be stationary in ω_1.

Let λ be a regular cardinal larger than the cardinality of H_θ.

Let $N' \prec (H_\lambda, \in, \triangle)$ be countable such that $N' \cap \omega_1 \in T$ and $\{\kappa, \theta, S, g\} \subseteq N'$. Assume that there exists a countable $M \prec (H_\theta, \in, \triangle)$ such that $M \cap \omega_1 = N' \cap \omega_1$, $N' \cap H_\theta \subseteq M$ and $M \cap \kappa \in S$. (If there are no such M, then $N' \cap H_\theta \in S^*$. We have what we want.) Let N be the skolem hull of $N' \cup (M \cap \kappa)$ in the structure $(H_\lambda, \in, \triangle)$.

We claim that $N \cap \kappa = M \cap \kappa$. Hence $N \cap H_\theta \in S^*$ and we finish the proof.

Let $\alpha \in N \cap \kappa$. Let τ be a skolem term. Let $a \in N'$ and $\alpha_1, \cdots, \alpha_m \in M \cap \kappa$ be such that $\alpha = \tau(a, \alpha_1, \cdots, \alpha_m)$.

Define $h : [\kappa]^m \to \kappa$ by

$$h(x_1, \cdots, x_m) = \begin{cases} \tau(a, x_1, \cdots, x_m), & \text{if } \tau(a, x_1, \cdots, x_m) < \kappa, \\ 0, & \text{otherwise.} \end{cases}$$

Then $h \in N'$. Hence $h \in N' \cap H_\theta \subseteq M$. Therefore, $\alpha = h(\alpha_1, \cdots, \alpha_m) \in M \cap \kappa$.

This finishes the proof that S^* is projective stationary in $[H_\theta]^\omega$.

Applying the Projective Stationary Reflection principle, let $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing continuous $\in$–chain of length ω_1 such that $N_\alpha \in S^*$ for all $\alpha < \omega_1$. Certainly, this is what the Strong Reflection Principle needs for the given S.

□

§2 SOME EXAMPLES

In this section, we investigate some natural examples of projective stationary sets. These examples have been used previously by others, and in some cases we are unsure of the authorship of these examples.

First let us consider the saturation of the nonstationary ideal on ω_1.

Let F be a set of stationary sets on ω_1. F is an antichain if $A \cap B$ is nonstationary for $A \neq B$ in F. F is a maximal antichain if F is an antichain and for every stationary subset $T \subseteq \omega_1$ there exists some $A \in F$ such that $T \cap A$ is stationary. The nonstationary ideal on ω_1 is saturated if every maximal antichain has size at most $\aleph_1$.

In [7], Steel and Van Wesep showed that the nonstationary ideal may be saturated using determinancy. In [3], Foreman, Magidor and Shelah showed that

Martin's Maximum implies that the nonstationary ideal on ω_1 is saturated. In [9], Todorcevic showed that the Strong Reflection Principle is sufficient.

Example 2.1

Let F be a maximal antichain of stationary subsets of ω_1. Consider the following set:

$$S = \{N \in [H_{\omega_2}]^\omega \mid N \prec H_{\omega_2}, F \in N, \text{and } \exists\, A \in F \cap N\ (N \cap \omega_1 \in A)\}.$$

We now check that S is projective stationary.

Let $T \subseteq \omega_1$ be stationary. Let $A \in F$ be such that $A \cap T$ is stationary. Let C be a club on $[H_{\omega_2}]^\omega$. Then we can find an elementary submodel N of H_{ω_2} with the property that $N \cap \omega_1 \in A \cap T$ and $N \in C$. Hence S is projective stationary.

Applying the Strong Reflection Principle, let $\{N_\alpha \mid \alpha < \omega_1\}$ be an increasing continuous $\subseteq$–chain of elementary submodels of H_{ω_2} such that $\alpha \subseteq N_\alpha$ for every α and there exists a club $C \subseteq \omega_1$ such that if $\alpha \in C$ then $\alpha = N_\alpha \cap \omega_1$ and $N_\alpha \in S$.

We proceed to check that $F \subseteq X = \bigcup\{N_\alpha \mid \alpha < \omega_1\}$.

Let $A \in F$. Assume that this $A \notin X$. Let Y be the skolem hull of $X \cup \{A\}$. Let M_α be the skolem hull of $N_\alpha \cup \{A\}$. Let $D \subseteq C$ be a club such that for every $\alpha \in D$ we have that $M_\alpha \cap \omega_1 = N_\alpha \cap \omega_1 = \alpha$.

Let $\alpha \in D \cap A$. Then $\alpha = N_\alpha \cap \omega_1 = M_\alpha \cap \omega_1$. There must be some $B \in F \cap N_\alpha$ such that $\alpha \in B$. This B must be different from A. Hence $A \cap B$ must be nonstationary. But by elementarity, as both A and B are in M_α, there must be some closed and unbounded subset $E \in M_\alpha$ of ω_1 such that $E \cap A \cap B$ is empty. We then have a contradiction because $E \in M_\alpha$ implies that $M_\alpha \cap \omega_1 \in E$ and $M_\alpha \cap \omega_1 \in A \cap B$.

Therefore $F \subseteq X$. Hence the size of F is at most $\aleph_1$.

Hence we have the theorem of Todorcevic [1,9] that the Strong Reflection Principle implies that the nonstationary ideal on ω_1 is saturated.

□

Example 2.2

Our next example is to show that assuming the Strong Reflection Principle, every stationary set $E \subseteq \omega_2$ of ordinals of cofinality ω contains a closed copy of ω_1. This is a theorem of Todorcevic [1,9].

Fix a regular cardinal $\kappa \geq \omega_2$.

Let E be a stationary subset of κ such that every $\alpha \in E$ has cofinality ω.

Consider the following set:

$$S = \{N \in [H_\kappa]^\omega \mid N \prec H_\kappa, E \in N, \text{and } \sup(N \cap \kappa) \in E\}.$$

We check that S is projective stationary.

Let $T \subseteq \omega_1$ be stationary. Let $f : [H_\kappa]^{<\omega} \to H_\kappa$. Let $M \prec H_\kappa$ be such that $\omega_1 \subseteq M$, M has cardinality $\aleph_1$, M is closed under f and $\sup(M \cap \kappa) \in E$. Then there is a countable $N \prec M$ such that N is closed under f and $N \cap \omega_1 \in T$ and $\sup(N \cap \kappa) = \sup(M \cap \kappa)$.

This shows that S is projective stationary.

Applying the Strong Reflection Principle, let $\{N_\alpha \mid \alpha < \omega_1\}$ be an increasing continuous $\in$–chain such that $N_\alpha \in S$ for $\alpha \in \omega_1$. Then for each $\alpha < \omega_1$, $\sup(N_\alpha \cap \kappa) \in E$. We are done.

□

EXAMPLE 2.3

Let $E = \{\alpha < \kappa \mid cf(\alpha) = \omega\}$.

Let $\{E_\alpha \mid \alpha < \omega_1\}$ be a family of disjoint stationary subsets of E. Let $\{T_\alpha \mid \alpha < \omega_1\}$ be a partition of ω_1 so that each T_α is stationary for $\alpha < \omega_1$ and for every stationary subset $X \subseteq \omega_1$, there is some $\alpha < \omega_1$ such that $X \cap T_\alpha$ is stationary.

Then the following set S is projective stationary:

$$S = \{N \in [H_\kappa]^\omega \mid \forall\, \alpha < N \cap \omega_1 \; N \cap \omega_1 \in T_\alpha \Rightarrow \sup(N \cap \kappa) \in E_\alpha\}.$$

To see this, let $T \subseteq \omega_1$ be stationary. Let α be the least countable ordinal such that $T \cap T_\alpha$ is stationary. Then the following set is stationary:

$$\{N \in [H_\kappa]^\omega \mid \alpha < N \cap \omega_1 \in T \cap T_\alpha \,\&\, \sup(N \cap \kappa) \in E_\alpha\}.$$

This can be used to show, assuming the Strong Reflection Principle, that if $\kappa \geq \omega_2$ is regular, then $\kappa^{\aleph_1} = \kappa$. By Silver's theorem [6], this in turn implies the Singular Cardinal Hypothesis holds. Here, by the Singular Cardinal Hypothesis we mean the following statement:

For every singular cardinal λ, if $2^{cf(\lambda)} < \lambda$, then $\lambda^{cf(\lambda)} = \lambda^+$.

Thus, we have that the Strong Reflection Principle implies that the Singular Cardinal Hypothesis holds. In fact the Singular Cardinal Hypothesis follows from a weaker reflection principle as shown by Velickovic [10].

□

EXAMPLE 2.4

We now prove that the Strong Reflection Principle implies the Reflection Principle. In fact, if $S \subseteq [H_\kappa]^\omega$ is stationary then there exists an increasing continuous $\in$–chain $\langle N_\alpha \mid \alpha < \omega_1\rangle$ such that $\{\alpha < \omega_1 \mid N_\alpha \in S\}$ is stationary in ω_1. (This is the version of the reflection principle used by Velickovic in [10] to show the Singular Cardinal Hypothesis.) Notice that the Reflection Principle is strictly weaker than the Strong Reflection Principle, because it does not imply that the nonstationary ideal on ω_1 is saturated.

Let $S \subseteq [H_\kappa]^\omega$. There is another projective stationary set $p(S) \subseteq [H_\kappa]^\omega$ naturally associated with S.

Let $T \subseteq \omega_1$. Define $S_T = \{N \in S \mid N \cap \omega_1 \in T\}$.

Let $F \subseteq \{T \subseteq \omega_1 \mid T \text{ is stationary and } S_T \text{ is not stationary }\}$ be a maximal antichain of the smallest cardinality.

Define $p(S)$ to be the following set:

$$\{N \prec H_\kappa \mid N \in [H_\kappa]^\omega \text{ and } (\exists\, A \in N \cap F\, (N \cap \omega_1 \in A) \iff N \notin S)\}.$$

Then $p(S)$ is projective stationary in $[H_\kappa]^\omega$.

If we assume that $\langle N_\alpha \mid \alpha < \omega_1\rangle$ is an increasing continuous $\in$–chain from $p(S)$, then S is stationary if and only if $\{\alpha < \omega_1 \mid N_\alpha \in S\}$ is stationary.

Let us assume that S is stationary. Toward a contradiction, let us assume that $\{\alpha < \omega_1 \mid N_\alpha \in S\}$ is not stationary. Then the complement of this set contains a club. It follows that for every $A \in F$ there is some $\alpha < \omega_1$ such that $A \in N_\alpha$. Let $f : \omega_1 \to F$ be a surjective mapping. For each $\alpha < \omega_1$ let $C_\alpha \subseteq [H_\kappa]^\omega$ be a club to witness that $f(\alpha) \in F$. Let $C = \Delta_{\alpha<\omega_1} C_\alpha$. Then $C \cap S$ is stationary and the following set T is stationary in ω_1:

$$T = \{N \cap \omega_1 \mid N \in C \cap S\}.$$

Let $\alpha \in T$ be such that for all $A \in N_\alpha \cap F$ there is a $\beta < \alpha$ with $A = f(\beta)$, for all $\beta < \alpha$, $f(\beta) \in N_\alpha$, and $\alpha = N_\alpha \cap \omega_1$. Then $N_\alpha \in S$. This is a contradiction. Hence $\{\alpha < \omega_1 \mid N_\alpha \in S\}$ is stationary in ω_1.

To see the other direction, assume that S is not stationary. By minimality, the cardinality of F must be one and the member of F must contain a club. By elementarity, there must be such a club in N_0. Therefore, for all $\alpha < \omega_1$, $N_\alpha \cap \omega_1$ must be in this club. Hence no N_α will be in S.

Furthermore, if we require the ω_1 sequences to be strongly increasing continuous, then we can have that if $S \subseteq [H_\kappa]^\omega$ is stationary and $T \subseteq [H_\kappa]^{\omega_1}$ is ω_1–closed and unbounded, then there exists an $X \in T$ such that $S \cap [X]^\omega$ is stationary in $[X]^\omega$.

□

EXAMPLE 2.5

Let C be a sequence defined on all the limit ordinals $< \kappa$ such that for all limit ordinals $\alpha < \kappa$, $C_\alpha \subseteq \alpha$ is a club in α and for all limit ordinals $\beta < \alpha$, if $\beta \in C_\alpha$ is a limit point of C_α, then $C_\beta = \beta \cap C_\alpha$. C is a $\square^*_\kappa$ sequence if there is no club $D \subseteq \kappa$ such that for every limit point α of D, $C_\alpha = \alpha \cap D$.

Let S_C be the set of all countable elementary submodels N of H_κ such that, letting $\alpha = \sup(N \cap \kappa)$, $C_\alpha \cap N$ is bounded in α.

The following theorem is essentially due to Velickovic [10].

THEOREM 2.1 The following are equivalent:

(1) S_C is projective stationary.

(2) S_C is stationary.

(3) There is no club $D \subseteq \kappa$ such that for all limit points α of D, $C_\alpha = \alpha \cap D$. That is, C is a $\square^*_\kappa$ sequence.

Proof

(2) $\Rightarrow$ (3). Assume that (3) is false. Let D be a witness. Let $\theta > \kappa$ be a sufficiently large regular cardinal. Let $N \prec H_\theta$ be a countable elementary submodel containing all the relevant objects. Then $N \cap H_\kappa \notin S_C$.

(3) $\Rightarrow$ (1). This is essentially due to Velickovic [10].

For an $f : [H_\kappa]^{<\omega} \to H_\kappa$, and a countable ordinal α, let us consider the following game $G(\alpha, f)$, due to Velickovic [10]. Two players, Player One and Player Two, take in turn playing in ω many steps. Player One plays an interval $I_n \subseteq \kappa$ of size less than κ and an ordinal $\alpha_n \in I_n$ at his nth move. Player Two plays an ordinal β_n in his responding move. At the end of the play, Player One wins the play if and only if for every $n < \omega$, $\beta_n < \inf(I_{n+1})$ and the elementary submodel N of H_κ, which is generated by $\alpha \cup \{\alpha_n \mid n < \omega\}$ and which is also closed under f, has the property that $N \cap \omega_1 = \alpha$ and $N \cap \kappa \subseteq \bigcup_{n<\omega} I_n$.

Notice that if Player One loses a play, Player One loses at some stage even before the game is over. Hence this game is a determined game.

LEMMA 2.1 (Velickovic) There exists a club $E_f \subseteq \omega_1$ such that for every $\alpha \in E_f$, Player One has a winning strategy σ^α_f for the game $G(\alpha, f)$.

Let $f : [H_\kappa]^{<\omega} \to H_\kappa$. Let $T \subseteq \omega_1$ be stationary.

Assume (3), we need to find a countable elementary submodel $N \prec H_\kappa$ with the property that N is closed under f, $N \cap \omega_1 \in T$ and $N \in S_C$.

Let $\alpha \in T \cap E_f$. Let σ be a winning strategy for Player One in the game $G(\alpha, f)$.

Let $\theta = (2^\kappa)^+$. Let $\langle M_\alpha \mid \alpha < \kappa \rangle$ be an increasing continuous $\in$–chain of elementary submodels of H_θ of size $< \kappa$ such that all the relevant objects are in M_0 and $M_\alpha \cap \kappa \in \kappa$ for all $\alpha < \kappa$. Let $D = \{M_\alpha \cap \kappa \mid \alpha < \kappa\}$. Then D is a club in κ.

Let $M \prec H_\theta$ be an elementary submodel of size less than κ, containing all the relevant objects, such that $M \cap \kappa$ is an ordinal of cofinality ω_1. Let $\delta = M \cap \kappa$.

Assume that there is a $X \in M$ such that $\delta \cap X \subseteq C_\delta$ and $M \models$ "X is a club in κ ". Let Y be the set of all the limit points of X. Then $Y \in M$. Let $\gamma < \beta$ be in M such that both are in Y. Then γ and β are limit points of C_δ. Hence $C_\gamma = \gamma \cap C_\beta$. Therefore, for every pair $\gamma < \beta$ from Y, we have $C_\gamma = \gamma \cap C_\beta$. This contradicts to our assumption (3).

Thus, for every club $X \subseteq \kappa$, if $X \in M$ then there is some ordinal $\gamma \in X \cap \delta$ which is not in C_δ. It follows that for every $\gamma < \delta$, there is some $\xi < \delta$ such that

$\eta_\xi = M_\xi \cap \kappa$ is not in C_δ and $\eta_\xi > \gamma$.

Inductively pick ξ_n so that $\eta_{\xi_n} \notin C_\delta$ and there is some ordinal $\beta \in C_\delta$ with $\eta_{\xi_n} < \beta < \eta_{\xi_{n+1}}$.

Let $\beta = \bigcup\{\eta_{\xi_n} \mid n < \omega\}$. Then $C_\beta = \beta \cap C_\delta$.

Consider the following play of the game $G(\alpha, f)$. Player One follows his winning strategy σ which is in M_0. Player Two plays $\mu_n = \max(\eta_{\xi_n} \cap C_\delta)$, which is in M_{ξ_n}, in his nth move. Let I_n and β_n be the nth move of Player One following his winning strategy σ. Then $\beta_n \in I_n \subseteq (\mu_n, \eta_{\xi_n})$ for all $n < \omega$. Let $N \prec M$ be generated by $\alpha \cup \{\beta_n \mid n < \omega\}$ which is closed under f. Then $N \cap \omega_1 \in T$ and $\sup N \cap \kappa = \beta$ and $N \cap C_\beta \subseteq I_0$. Hence $N \cap H_\kappa \in S_C$.

This shows that (3) implies (1).

□

COROLLARY 2.1 If for every stationary $S \subseteq [H_\kappa]^\omega$ there is an increasing continuous $\in$–chain $\langle N_\alpha \mid \alpha < \omega_1\rangle$ such that $\{\alpha < \omega_1 \mid N_\alpha \in S\}$ is stationary, then there is a club $C \subseteq \kappa$ such that for every limit point $\alpha \in C$, $C_\alpha = \alpha \cap C$. In particular, there is no $\square^*_\kappa$ sequence.

Proof By the previous theorem, we need only to prove that S_C is not stationary.

Toward a contradiction, let us assume that S_C is stationary. Let $\langle N_\alpha \mid \alpha < \omega_1\rangle$ be an increasing continuous $\in$–chain be such that $T = \{\alpha < \omega_1 \mid N_\alpha \in S_C\}$ is stationary. Let $\gamma_\alpha = \sup(N_\alpha \cap \kappa)$ for $\alpha < \omega_1$. Let $\delta = \sup\{\gamma_\alpha \mid \alpha < \omega_1\}$. Let $\alpha \in T$ be such that $\gamma_\alpha \in C_\delta$ and γ_α is a limit point of $C_\delta \cap \{\gamma_\beta \mid \beta < \omega_1\}$. We get a contradiction, since $\alpha \in T$ implies that $N_\alpha \in S_C$ and $C_{\gamma_\alpha} = \gamma_\alpha \cap C_\delta$ whose intersection with N_α is not bounded in γ_α.

□

§3 SATURATION AND REFLECTION

In this section, we take a closer look at the saturation of the nonstationary ideal on ω_1. It turns out that it itself is a kind of reflection property.

To start with, let us define first certain filters. For a regular cardinal $\kappa \geq \omega_2$, for $X \subseteq [H_\kappa]^\omega$, let X be in $\mathcal{F}_\kappa$ if and only if for every stationary subset $A \subseteq \omega_1$ there exist a stationary subset $B \subseteq A$ and a closed and unbounded subset $C \subseteq [H_\kappa]^\omega$ such that $\{N \in C \mid N \cap \omega_1 \in B\} \subseteq X$.

THEOREM 3.1 The following are equivalent:

(1) The nonstationary ideal NS_{ω_1} on ω_1 is saturated.

(2) For every regular cardinal $\kappa \geq \omega_2$, for every stationary set $S \subseteq [H_\kappa]^\omega$, there exists a stationary set $A \subseteq \omega_1$ such that S is A–projective stationary (i.e., for every stationary $B \subseteq A$, the set $\{N \in S \mid N \cap \omega_1 \in B\}$ is stationary).

(3) For every regular cardinal $\kappa \geq \omega_2$, for every $X \subseteq [H_\kappa]^\omega$, $X \in \mathcal{F}_\kappa$ if and only if X contains a closed and unbounded subset. Namely, the filter $\mathcal{F}_\kappa$ is just the club filter on $[H_\kappa]^\omega$.

(4) For every regular cardinal $\kappa \geq \omega_2$, for every $X \in \mathcal{F}_\kappa$, there exists an increasing continuous $\in$–chain $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ of countable elementary submodels of H_κ of length ω_1 such that $N_\alpha \in X$ for all $\alpha < \omega_1$.

Proof (1) $\Rightarrow$ (2)

Let $S \subseteq [H_\kappa]^\omega$ be a stationary set. If S is projective stationary, then there is nothing needed to be proved. So we assume that S is not projective stationary.

For a stationary $T \subseteq \omega_1$, we let $T \in F$ if and only if there exists a club $C \subseteq [H_\kappa]^\omega$ such that

$$T \cap \{\, N \cap \omega_1 \mid N \in C \cap S\} = \emptyset.$$

Since S is not projective stationary, F is not empty.

Because the nonstationary ideal on ω_1 is saturated, we can take a sequence $\{A_\alpha \mid \alpha < \omega_1\}$ from F so that for every $A \in F$, $A - \nabla_{\alpha<\omega_1} A_\alpha$ is nonstationary, where $\nabla_{\alpha<\omega_1} A_\alpha$ is the diagonal union of the sequence $\{A_\alpha \mid \alpha < \omega_1\}$ defined by

$$\nabla_{\alpha<\omega_1} A_\alpha = \{\beta < \omega \mid \exists\, \alpha < \beta\ (\beta \in A_\alpha)\}.$$

For each $\alpha < \omega$, we let $C_\alpha \subseteq [H_\kappa]^\omega$ be a witness for A_α to be in F.

Let $\Delta_{\alpha<\omega_1} C_\alpha$ be the diagonal intersection of the sequence $\{C_\alpha \mid \alpha < \omega_1\}$ defined by

$$\Delta_{\alpha<\omega_1} C_\alpha = \{N \in [H_\kappa]^\omega \mid \forall\, \alpha \in N \cap \omega_1\ N \in C_\alpha\}.$$

Let $A = \omega - \nabla_{\alpha<\omega_1} A_\alpha$. We claim that S is A–projective stationary.

First notice that A is stationary. This is because $\Delta_{\alpha<\omega_1} C_\alpha \cap S$ is stationary. Now let $T \subseteq A$ be stationary and let $C \subseteq [H_\kappa]^\omega$ be a club.

Since $T \cap \nabla_{\alpha<\omega_1} A_\alpha$ is not stationary, $T \notin F$. Therefore the following intersection is not empty:

$$T \cap \{N \cap \omega_1 \mid N \in C \cap \Delta_{\alpha<\omega_1} C_\alpha \cap S\}.$$

Hence $\{N \in S \mid N \cap \omega_1 \in T\}$ is stationary.

(2) $\Rightarrow$ (3)

Since every closed and unbounded subset of $[H_\kappa]^\omega$ is in the filter $\mathcal{F}_\kappa$, and (2) implies that every stationary subset of $[H_\kappa]^\omega$ intersects every member of the filter $\mathcal{F}_\kappa$, the filter $\mathcal{F}_\kappa$ and the club filter on $[H_\kappa]^\omega$ are the same filter.

(3) implies (4) is trivial.

(4) $\Rightarrow$ (1)

Let F be a maximal antichain of stationary subsets of ω_1. Let $S_F = \{N \in [H_\kappa]^\omega \mid \exists\, A \in F \cap N\ (N \cap \omega_1 \in A)\}$. Then $S_F \in \mathcal{F}_\kappa$. Namely, for a given stationary subset $A \subseteq \omega_1$, let $T \in F$ be such that $B = A \cap T$ is stationary. Let C be the club of all countable elementary submodels of H_κ which contains T. Then if $N \in C$ and $N \cap \omega_1 \in B$ then $N \in S_F$. Now by the argument of Example 2.1, assuming (4), F has cardinality at most $\aleph_1$.

This completes the proof.

$\square$

The next thing we want to show is that the presaturation of the nonstationary ideal on ω_1 is also a kind of reflection property, which in turn is equivalent to the filter $\mathcal{F}_\kappa$ being countably closed.

First let us recall that the nonstationary ideal on ω_1 is **presaturated** if for every countable sequence $\{F_n \mid n < \omega\}$ of maximal antichains of the nonstationary ideal on ω_1, for every stationary subset T, there exists a stationary subset $B \subseteq T$ such that for each n the set $\{A \in F_n \mid A \cap B$ is stationary $\}$ has cardinality at most $\aleph_1$ (see [3]).

THEOREM 3.2 The following are equivalent:

(1) The nonstationary ideal NS_{ω_1} on ω_1 is presaturated.

(2) For every regular $\kappa \geq \omega_2$, the filter $\mathcal{F}_\kappa$ is σ–closed.

(3) For every regular cardinal $\kappa \geq \omega_2$, for every countable sequence $\langle X_n \mid n < \omega\rangle$ from $\mathcal{F}_\kappa$, for every stationary subset $T \subseteq \omega_1$, there exists an $M \prec H_\kappa$ such that $\omega_1 \subseteq M$, M has cardinality $\aleph_1$ and $\{N \in [M]^\omega \cap \bigcap_{n<\omega} X_n \mid N \cap \omega_1 \in T\}$ is stationary in $[M]^\omega$.

Proof (1) $\Rightarrow$ (2)

Let $X_n \in \mathcal{F}_\kappa$ for $n < \omega$. Let $X = \bigcap_{n<\omega} X_n$. We need to show that $X \in \mathcal{F}_\kappa$.

For each $n < \omega$, let $\langle B^n_\alpha, C^n_\alpha \mid \alpha < \theta_n\rangle$ be such that all the B^n_α's form a maximal antichain of stationary sets and for each $\alpha < \theta_n$, $N \in C^n_\alpha$ and $N \cap \omega_1 \in B^n_\alpha$ imply that $N \in X_n$.

Let $A \subseteq \omega_1$ be stationary. Applying the presaturation property of the nonstationary ideal, let $B \subseteq A$ be stationary such that for each $n < \omega$, the set $I_n = \{\alpha < \theta_n \mid B \cap B^n_\alpha$ is stationary $\}$ has cardinality at most $\aleph_1$. To simplify the notation, we assume that $I_n = \omega_1$. Let

$$T = B \cap \bigcap_{n<\omega} \nabla_{\alpha<\omega_1} B^n_\alpha$$

and let

$$C = \bigcap_{n<\omega} \Delta_{\alpha<\omega_1} C^n_\alpha.$$

It follows that if $N \in C$ and $N \cap \omega_1 \in T$ then $N \in X$.

Hence the filter $\mathcal{F}_\kappa$ is σ–closed.

(2) $\Rightarrow$ (3)

It will be sufficient to show that every $X \in \mathcal{F}_\kappa$ reflects.

Let $X \in \mathcal{F}_\kappa$. Let T be a stationary subset of ω_1. Let $B \subseteq T$ be stationary and let $C \subseteq [H_\kappa]^\omega$ be a closed and unbounded subset such that for all $N \in C$, $N \cap \omega_1 \in B$ implies that $N \in X$.

Let $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing continuous $\in$–chain from C. Let $M = \bigcup_{\alpha<\omega_1} N_\alpha$. Then $\{N_\alpha \mid \alpha = N_\alpha \cap \omega_1 \in B\}$ is stationary in $[M]^\omega$.

(3) $\Rightarrow$ (1)

Fix a sequence $\{F_n \mid n < \omega\}$ of maximal antichains of the nonstationary ideal on ω_1. For each $n < \omega$, let

$$S_n = \{N \prec H_\kappa \mid N \text{ is countable and } \exists\, A \in F_n \cap N \ (N \cap \omega_1 \in A)\}.$$

Then every $S_n \in \mathcal{F}_\kappa$.

Fix a stationary subset T of ω_1. Applying (3), let $X \prec H_{\omega_2}$ be such that $\omega_1 \subseteq X$, X has cardinality $\aleph_1$ and $\{N \in [X]^\omega \cap \bigcap_{n<\omega} S_n \mid N \cap \omega_1 \in T\}$ is stationary in $[X]^\omega$. Write $X = \bigcup_{\alpha<\omega_1} N_\alpha$, the union of an increasing continuous $\subseteq$–chain of countable elementary submodels with $\alpha \subseteq N_\alpha$. Then

$$B = \{\alpha \in T \mid \alpha = N_\alpha \cap \omega_1 \ \&\ N_\alpha \in \bigcap_{n<\omega} S_n\}$$

is a stationary subset of T.

We claim that for every n the set $\{\ A \in F_n \mid A \cap B \text{is stationary } \}$ has cardinality at most $\aleph_1$.

Assume not. Let n be the least counterexample. Let $A \in F_n - X$ be such that $A \cap B$ is stationary. Let M_α be the skolem hull of $N_\alpha \cup \{A\}$ for each $\alpha < \omega_1$. Then

$$C = \{\alpha < \omega_1 \mid \alpha = N_\alpha \cap \omega_1 = M_\alpha \cap \omega_1\}$$

is a club. Let $\alpha \in C \cap A \cap B$. Let $Z \in N_\alpha \cap F_n$ be such that $\alpha \in Z$. Since both A and Z are in M_α which is an elementary submodel of H_κ, and $M_\alpha \cap \omega_1 = \alpha \in A \cap Z$, we conclude that $A \cap Z$ is stationary. This is a contradiction.

Hence for every $n < \omega$, for every $A \in F_n$, if $A \cap B$ is stationary, then $A \in X$. We are done since X has cardinality $\aleph_1$. $\square$

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Projective Stationary Sets and A Strong Reflection Principle

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ABSTRACT[1] *

We study projective stationary sets. The Projective Stationary Reflection principle is the statement that every projective stationary set contains an increasing continuous $\in$–chain of length ω_1. We show that if Martin's Maximum holds, then the Projective Stationary Reflection Principle holds. Also it is equivalent to the Strong Reflection Principle. We show that the saturation of the nonstationary ideal on ω_1 is equivalent to a certain kind of reflection.

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