DIOPHANTINE APPROXIMATION BY PRIME NUMBERS, II

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1. Introduction

We continue the study of the problems discussed in the introduction to [**22**]. We suppose, as before, that $\lambda_1, \ldots, \lambda_s$ are s non-zero real numbers not all of the same sign and not all in rational ratio, and k is a natural number. Davenport and Heilbronn ([**11**]) have shown that the inequality

$$\left| \sum_{j=1}^{s} \lambda_j n_j{}^k \right| < \varepsilon \tag{1.1}$$

has infinitely many solutions in natural numbers n_j provided that $s \geqslant 2^k + 1$. Davenport has given an account of this in [**9**]. Davenport and Roth ([**12**]) reduced the permissible size of s for large k by showing that $s > Ck \log k$ will suffice, where C is an absolute constant. Their method shows that if $s_0(k)$ is the least s for which (1.1) has an infinity of solutions, then $\overline{\lim}\, s_0(k)/(k \log k) \leqslant 6$. They mention that for small values of k it does not seem altogether easy to obtain results which correspond exactly to those found by Davenport ([**5**], [**6**], and [**8**]) for the classical Waring's problem, but they do show that $s_0(3) \leqslant 8$. Also, Danicic ([**3**]) has shown that $s_0(4) \leqslant 14$.

Let $s_1(k)$ be the smallest s for which the inequality

$$\left| \sum_{j=1}^{s} \lambda_j p_j{}^k \right| < \varepsilon \tag{1.2}$$

has infinitely many solutions in prime numbers p_j. Schwarz ([**20**]) has obtained $s_1(k) \leqslant 2^k + 1$ $(k < 12)$ and $s_1(k) \leqslant 2k^2(2 \log k + \log\log k + \frac{5}{2}) - 1$ $(k \geqslant 12)$. Danicic ([**4**]), Baker ([**1**]), and Ramachandra ([**18**]) have made further contributions to this subject (see the introduction to [**22**]). We show that $s_1(k) \leqslant Ck \log k$ also holds in this problem, and we obtain results which correspond exactly to those found by Hua ([**16**], Chapter 9) and based on Davenport's methods, for small values of k in the Waring–Goldbach problem. In fact, our bounds for $s_1(k)$ improve all known bounds when $k \geqslant 4$, and C can be taken arbitrarily close to 4 when k is large. We also obtain bounds for $s_0(k)$ which, except when $k = 5$, are the same as those obtained for $G(k)$, when k is small, by Davenport's methods. The

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difficulty when $k = 5$ arises because we have not been able to obtain an analogue of Davenport's Theorem 4 of [7].

It would be nice to show that $\overline{\lim}\, s_0(k)/(k\log k) < 4$, for instance as is obtained for $G(k)$ in Vinogradov's Chapter 4 of [24], but again there is a difficulty, not dissimilar to the one that occurs above. We content ourselves by giving a bound just 1 smaller than that given for $s_1(k)$.

2. The main theorems

Throughout, $\lambda_1, \ldots, \lambda_s$ are s non-zero real numbers such that λ_1, λ_2, and λ_3 are not all of the same sign and λ_1/λ_2 is irrational, and η is a real number. The symbols $\mathscr{U}(X)$ and $\mathscr{V}(X)$, with or without suffices or superfices, are used to denote finite sets of distinct real numbers such that no element exceeds X in absolute value and each pair of different elements, say u, u', satisfies $|u-u'| \geqslant 1$. We use $|\mathscr{A}|$ to denote the cardinality of the set $\mathscr{A}$, and we say that $\mathscr{U}(X)$ has *density* ν if $|\mathscr{U}(X)| > X^{\nu}$. We further assume that $k \geqslant 4$, and put

$$\theta = 2^{1-k}\ (k \leqslant 12), \quad \theta = (2k^2(2\log k + \log\log k + 3))^{-1}\ (k > 12) \tag{2.1}$$

and

$$\alpha = (2^{2k+2}(k+1))^{-1}. \tag{2.2}$$

The theorems enunciated here depend on a technical lemma which embodies the principal new idea and therefore also merits the title of theorem. They are thus numbered 2 and 3.

THEOREM 2. *Suppose that* $r \geqslant \frac{1}{2}k+1$, $\nu > 1-2r\theta/k$, $s \geqslant 2r+2m+1$, $0 < \sigma < \frac{1}{5}\alpha$, *and that for every sufficiently large* X *there are sets* $\mathscr{U}_t(X)$ $(t = 1, 2)$ *with density* ν *and such that every element can be written in the form*

$$\sum_{j=0}^{m-1} \lambda_{2r+2j+t+1} p_j^{\,k}$$

with $p_j^{\,k} \leqslant X$. *Then there are infinitely many solutions of the inequality*

$$\left|\eta + \sum_{j=1}^{s} \lambda_j p_j^{\,k}\right| < (\overset{*}{\max} p_j)^{-\sigma}. \tag{2.3}$$

Let $\mathscr{D}(k)$ be the least s for which (2.3) has infinitely many solutions.

COROLLARY 2.1. *Let*

$$\kappa = 1/k \tag{2.4}$$

and

$$N = [(-\log 2\theta + \log(1-2\kappa))/(-\log(1-\kappa))]. \tag{2.5}$$

Then

$$\mathscr{D}(k) \leqslant 2k + 2N + 7. \tag{2.6}$$

COROLLARY 2.2. *We have* $\mathscr{D}(4) \leqslant 15$, $\mathscr{D}(5) \leqslant 25$, $\mathscr{D}(6) \leqslant 37$, $\mathscr{D}(7) \leqslant 55$, $\mathscr{D}(8) \leqslant 75$, $\mathscr{D}(9) \leqslant 97$, *and* $\mathscr{D}(10) \leqslant 123$.

These bounds correspond exactly to those found by Hua ([**16**], Chapter 9) in the Waring–Goldbach problem.

THEOREM 3. *Suppose that* $r \geqslant k+3$, $\nu > 1-r\theta/k$, $s \geqslant r+2m$, *and that for every sufficiently large* X *there are sets* $\mathscr{U}_t(X)$ $(t = 1, 2)$ *with density* ν *and such that every element can be written in the form*

$$\sum_{j=0}^{m-1} \lambda_{r+2j+t} n_j^{\,k}$$

with $n_j^{\,k} \leqslant X$. *Then there is a positive number* δ *such that there are infinitely many solutions of the inequality*

$$\left|\eta + \sum_{j=1}^{s} \lambda_j n_j^{\,k}\right| < (\max n_j)^{-\delta}. \tag{2.7}$$

Let $D(k)$ be the least s for which (2.7) has infinitely many solutions.

COROLLARY 3.1. *Suppose that* (2.4) *and* (2.5) *hold. Then*

$$D(k) \leqslant 2k+2N+6.$$

COROLLARY 3.2. *We have* $D(5) \leqslant 24$, $D(6) \leqslant 36$, $D(7) \leqslant 53$, $D(8) \leqslant 73$, $D(9) \leqslant 96$, *and* $D(10) \leqslant 121$.

This compares with the bounds $G(5) \leqslant 23$, $G(6) \leqslant 36$, $G(8) \leqslant 73$, $G(9) \leqslant 96$, and $G(10) \leqslant 121$ due to Davenport ([**8**]), Davenport ([**8**]) and Sambasiva Rao ([**19**]), Narasimhamurti ([**17**]), Cook ([**2**]), and Cook ([**2**]) respectively. When $k = 7$ Sambasiva Rao's claim $G(7) \leqslant 52$, or rather his Lemma 5, of [**19**] cannot be substantiated by the method described there. It seems that $G(7) \leqslant 53$ is the best that can be done.

3. Theorem 1

We consider the numbers $\lambda_1, \ldots, \lambda_s$, η, s, k, ν, and σ as constant and let δ denote a sufficiently small positive number. P and ε are positive numbers which are respectively large and small in terms of δ. In the proof of Theorems 2 and 3 we shall take $\varepsilon = P^{-\sigma}$ and $\varepsilon = P^{-\delta}$ respectively. The implied constants in Vinogradov's forms of the O notation, $\ll$ and $\gg$, depend at most on δ. We write $e(x) = e^{2\pi i x}$,

$$f(x) = \sum_{\delta P < n \leqslant P} e(xn^k) \tag{3.1}$$

and

$$K_\varepsilon(x) = \pi^{-2}x^{-2}\sin^2 \pi\varepsilon x \ (x \neq 0), \quad K_\varepsilon(0) = \varepsilon^2. \tag{3.2}$$

THEOREM 1. *Let* $R = P^{k-\delta}$, *and let* $\mathscr{V} = \mathscr{V}(R)$ *denote a set with density* ν. *Suppose that*

$$F(x) = \sum_{v \in \mathscr{V}} e(xv), \tag{3.3}$$

$$(l+1)\theta > k-k\nu, \tag{3.4}$$

and

$$\theta > \theta_1 > k - k\nu - l\theta. \tag{3.5}$$

Then

$$\int_{-\infty}^{\infty} |f(\lambda_j x)^{l+1} F(x)^2| K_\varepsilon(x)\, dx \ll \varepsilon P^{l+1-k} |\mathscr{V}|^2 \quad (l \geqslant k+1) \tag{3.6}$$

and

$$\int_{-\infty}^{\infty} |f(\lambda_j x)^{l} F(x)^2| K_\varepsilon(x)\, dx \ll \varepsilon P^{l+\theta_1-k} |\mathscr{V}|^2 \quad (l \geqslant k+2). \tag{3.7}$$

4. Proof of Theorem 1

We proceed by lemmas.

Lemma 1. *For every real* y,

$$\int_{-\infty}^{\infty} e(xy) K_\varepsilon(x)\, dx = \max(0, \varepsilon - |y|).$$

This follows easily from Lemma 4 of Davenport and Heilbronn ([**11**]).

Let

$$Q = P^{1-\delta}, \tag{4.1}$$

$$L(y) = \int_{\delta P}^{P} e(yz^k)\, dz, \tag{4.2}$$

and

$$S(q,a) = \sum_{n=1}^{q} e(an^k/q). \tag{4.3}$$

Lemma 2. *Suppose that* $q \leqslant Q$, $(q,a) = 1$, $x = y + a/q$, *and* $|y| \leqslant q^{-1}QP^{-k}$. *Then*

$$f(x) - q^{-1}S(q,a)L(y) \ll P^{1-1/k+\delta} \ll P^{1-\theta}.$$

Proof. By Lemma 7.11 of Hua's book [**16**] the left side is $\ll q^{1-1/k+\delta}$. The stated result follows at once.

Lemma 3. *If* $(q_1, q_2) = (q_1, a_1) = (q_2, a_2) = 1$, *then*

$$S(q_1 q_2, a_1 q_2 + a_2 q_1) = S(q_1, a_1) S(q_2, a_2), \tag{4.4}$$

and if $(q,a) = 1$, *then*

$$S(q,a) \ll q^{1-1/k}. \tag{4.5}$$

Furthermore, if $p \nmid ak$, *then*

$$|S(p,a)| < k\sqrt{p}, \tag{4.6}$$

$$S(p^h, a) = p^{h-1} \quad (1 < h \leqslant k), \tag{4.7}$$

and

$$S(p^h, a) = p^{k-1}S(p^{h-k}, a) \quad (h > k). \tag{4.8}$$

This is due to Hardy and Littlewood; (4.4) is essentially (3.11) of [**13**], (4.5) is (3.41) of [**15**], (4.6) follows from Lemma 13 of [**14**], (4.7) from Lemma 12 of [**14**], and (4.8) from (4.11) and (4.12) of [**13**]. Vinogradov has collected all these results together in Chapter 2 of [**24**].

LEMMA 4. *Suppose that* $|x-a/q| \leqslant q^{-2}$ *with* $(q,a) = 1$ *and* $Q < q \leqslant P^kQ^{-1}$. *Then*

$$f(x) \ll P^{1-\theta+\delta}.$$

When $k \leqslant 12$ this is due to Weyl; see, for instance, Lemma 3.6 of [**16**]. When $k > 12$ this is shown in the same way as Theorem 9 of [**16**]. The slight extension of the range of q makes no essential difference to the argument.

LEMMA 5. *Suppose that* $r \geqslant 2$. *Then*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |L(y)|^r\, dy \ll P^{r-k}.$$

Proof. By (4.2) and a partial integration, $L(y) \ll P(P^k|y|)^{-1}$ $(y \neq 0)$, and trivially $L(y) \ll P$.

Let

$$A_r(q) = \sum_{\substack{a=1\\(q,a)=1}}^{q} |S(q,a)|^r q^{-r}. \tag{4.9}$$

LEMMA 6. *Suppose that* $r \geqslant k+2$. *Then*

$$\sum_{q\leqslant Q} A_r(q) \ll 1.$$

Proof. By (4.4) and (4.9), $A_r(q)$ is multiplicative, and clearly $A_r(1) = 1$. Hence

$$\sum_{q\leqslant Q} A_r(q) \leqslant \prod_{p\leqslant Q}\left(1+\sum_{h=1}^{\infty} A_r(p^h)\right)$$

providing that all of the infinite series on the right converge. It suffices, therefore, to show that

$$\sum_{h=1}^{\infty} A_r(p^h) \ll p^{-2}. \tag{4.10}$$

If $p\,|\,k$, then by (4.5) and (4.9), $A_r(p^h) \ll p^{-2h/k}$, so that

$$\sum_{h=1}^{\infty} A_r(p^h) \ll 1 \ll p^{-2}.$$

If $p \nmid k$, then, by (4.6), (4.7), and (4.8), for $b \geqslant 0$ we have

$$\sum_{j=1}^{k} A_r(p^{bk+j}) \ll k^r p^{1-\frac{1}{2}r-b(r-k)} + \sum_{j=2}^{k} p^{j-r-b(r-k)}$$
$$\ll p^{-2-b(r-k)},$$

which gives (4.10).

We dissect the real line into basic and supplementary intervals as follows. When $1 \leqslant q \leqslant Q$ and $(q,a) = 1$ we use $\mathscr{M}_j(q,a)$ to denote the closed interval with endpoints $(a-QP^{-k})\lambda_j^{-1}q^{-1}$ and $(a+QP^{-k})\lambda_j^{-1}q^{-1}$. These intervals are disjoint. We use $\mathscr{M}_j$ to mean their union and we write

$$\mathscr{N}_j = \mathbf{R} \backslash \mathscr{M}_j. \tag{4.11}$$

Let $x \in \mathscr{N}_j$. By a well-known elementary theorem we may choose q, a so that $|\lambda_j x - a/q| \leqslant q^{-1}QP^{-k}$, $1 \leqslant q \leqslant P^k Q^{-1}$ and $(q,a) = 1$. Since $x \notin \mathscr{M}_j$ we have $q > Q$. Hence, by Lemma 4,

$$f(\lambda_j x) \ll P^{1-\theta+\delta} \quad (x \in \mathscr{N}_j). \tag{4.12}$$

On hypothesis, the elements of $\mathscr{V}$ are well spaced and $|\mathscr{V}| > R^\nu$. Hence, by (3.3) and Lemma 1,

$$\int_{-\infty}^{\infty} |F(x)|^2 K_\varepsilon(x)\,dx < \varepsilon R^{-\nu} |\mathscr{V}|^2. \tag{4.13}$$

Therefore, by (3.4), (3.5), and (4.12),

$$\int_{\mathscr{N}_j} |f(\lambda_j x)^{l+1} F(x)^2| K_\varepsilon(x)\,dx \ll \varepsilon P^{l+1-k} |\mathscr{V}|^2 \tag{4.14}$$

and

$$\int_{\mathscr{N}_j} |f(\lambda_j x)^{l} F(x)^2| K_\varepsilon(x)\,dx \ll \varepsilon P^{l+\theta_1-k} |\mathscr{V}|^2. \tag{4.15}$$

Suppose that $r \geqslant k+2$. By Lemma 2,

$$\int_{\mathscr{M}_j} |f(\lambda_j x)^{r} F(x)^2| K_\varepsilon(x)\,dx \ll |\mathscr{V}|^2 H_1 + P^{r-r\theta} H_2, \tag{4.16}$$

where

$$H_1 = \sum_{q \leqslant Q} \sum_{\substack{a=-\infty \\ (q,a)=1}}^{\infty} |S(q,a)|^r q^{-r} \int_{\mathscr{M}_j(q,a)} |L(\lambda_j x - a/q)|^r K_\varepsilon(x)\,dx \tag{4.17}$$

and

$$H_2 = \int_{\mathscr{M}_j} |F(x)|^2 K_\varepsilon(x)\,dx. \tag{4.18}$$

We split the sum over a in (4.17) into two parts according as $|a| \leqslant q/\varepsilon$ or $|a| > q/\varepsilon$, so that

$$H_1 \ll H_1' + H_2'', \tag{4.19}$$

where, by Lemma 5 and (3.2),

$$H_1' = \varepsilon^2 P^{r-k} \sum_{q \leqslant Q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \sum_{\substack{0 \leqslant a \leqslant q/\varepsilon \\ a \equiv b \pmod q}} |S(q,b)|^r q^{-r}$$

and

$$H_1'' = P^{r-k} \sum_{q \leqslant Q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} |S(q,b)|^r q^{-r} \sum_{\substack{a > q/\varepsilon \\ a \equiv b \pmod q}} q^2 a^{-2}.$$

Therefore, by Lemma 6,

$$H_1' + H_1'' \ll \varepsilon P^{r-k}. \tag{4.20}$$

By (4.18), (4.13), (3.4), and (3.5),

$$P^{r-r\theta} H_2 \ll \varepsilon P^{r-k} |\mathscr{V}|^2 \quad (r = l+1)$$

and

$$P^{r-r\theta} H_2 \ll \varepsilon P^{r+\theta_1-k} |\mathscr{V}|^2 \quad (r = l).$$

Thus, by (4.16), (4.19), and (4.20) we have

$$\int_{\mathscr{M}_j} |f(\lambda_j x)^{l+1} F(x)^2| K_\varepsilon(x)\, dx \ll \varepsilon P^{l+1-k} |\mathscr{V}|^2$$

and

$$\int_{\mathscr{M}_j} |f(\lambda_j x)^{l} F(x)^2| K_\varepsilon(x)\, dx \ll \varepsilon P^{l+\theta_1-k} |\mathscr{V}|^2.$$

This with (4.14), (4.15), and (4.11) completes the proof of Theorem 1.

5. Proof of Theorem 2

Without loss of generality we may assume that $s = 2r+2m+1$. Since λ_1/λ_2 is irrational there are infinitely many pairs of integers q, a with

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a}{q}\right| \leqslant q^{-2}, \tag{5.1}$$

$(q,a) = 1$, $q > 0$, and $a \neq 0$. We choose $q > q_0(\delta)$ and let

$$P = q^{2/k}, \tag{5.2}$$

$$\sigma\alpha^{-1} < \sigma_1 < \tfrac{1}{5}, \tag{5.3}$$

$$W = P^{\sigma_1}, \tag{5.4}$$

$$\varepsilon = P^{-\sigma}, \tag{5.5}$$

$$\tau = WP^{-k}, \tag{5.6}$$

and

$$T = P^{\frac{1}{3}}. \tag{5.7}$$

We further define

$$g(x) = \sum_{\delta P < p \leqslant P} e(xp^k) \tag{5.8}$$

and

$$I(x) = \int_{\delta P}^{P} \frac{e(xy^k)}{\log y}\, dy. \tag{5.9}$$

We use $\rho = \beta + i\gamma$ (β, γ real) to denote a typical zero of the Riemann zeta function and $\sum'$ to denote summation over all those ρ with $|\gamma| \leqslant T$ and $\beta \geqslant \frac{2}{3}$. We then let

$$\Xi_\rho(x) = \sum_{\delta^k P^k < n \leqslant P^k} (\log n)^{-1} n^{-1+\rho/k} e(xn), \tag{5.10}$$

$$J(x) = \sum{}' \, \Xi_\rho(x), \tag{5.11}$$

$$B(x) = g(x) - I(x) + J(x), \tag{5.12}$$

and for any function of a real variable, $\Phi(x)$,

$$\Phi_j(x) = \Phi(\lambda_j x). \tag{5.13}$$

We use C to denote a positive absolute constant, not necessarily the same one on each occurrence.

LEMMA 7. *We have*

$$B(x) \ll P^{\frac{2}{3}} (\log P)^C (1 + P^k |x|).$$

This can be shown in the same way as Lemma 5 of [**22**].

LEMMA 8. *We have*

$$I(x) \ll P \min(1, P^{-k} |x|^{-1}), \tag{5.14}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |J(x)|^2\, dx \ll P^{2-k} \exp(-2(\log P)^{\frac{1}{6}}), \tag{5.15}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |I(x)|^2\, dx \ll P^{2-k}, \tag{5.16}$$

$$\int_{-\tau}^{\tau} |B_j(x)|^2\, dx \ll P^{2-k} \exp(-2(\log P)^{\frac{1}{6}}), \tag{5.17}$$

and

$$\int_{-\tau}^{\tau} |g_j(x)|^2\, dx \ll P^{2-k}. \tag{5.18}$$

Proof. The inequality (5.14) follows from (5.9) by partial integration, and (5.15) is shown by the same kind of argument that is used to prove (29) of [**22**]. To prove (5.16) we simply apply (5.14). The inequality (5.17) is a consequence of (5.13), Lemma 7, (5.6), (5.4), and (5.3). Now (5.18) follows from (5.13) and (5.12).

For convenience we write

$$\Psi(x) = \prod_{j=1}^{2r+1} g_j(x) \tag{5.19}$$

and

$$\Psi^*(x) = \prod_{j=1}^{2r+1} I_j(x), \tag{5.20}$$

and we dissect the real line as follows;

$$E_1 = \{x: |x| \leqslant \tau\}, \tag{5.21}$$

$$E_2 = \{x: \tau < |x| \leqslant W, |g_1(x)| \leqslant |g_2(x)|\}, \tag{5.22}$$

$$E_3 = \{x: \tau < |x| \leqslant W, |g_1(x)| > |g_2(x)|\}, \tag{5.23}$$

$$E_4 = \{x: |x| > W\}. \tag{5.24}$$

LEMMA 9. *Let E_1 be given by* (5.21). *Then*

$$\int_{E_1} |\Psi(x) - \Psi^*(x)| K_\varepsilon(x)\, dx \ll \varepsilon^2 P^{2r+1-k} \exp(-(\log P)^{\frac{1}{8}}).$$

Proof. We first note that

$$g_j(x), I_j(x) \ll P \tag{5.25}$$

and, by (5.19), (5.20), and (5.12),

$$\Psi(x) - \Psi^*(x) = \sum_{j=1}^{2r+1} \left(\prod_{h=1}^{j-1} g_h(x)\right)(B_j(x) - J_j(x))\left(\prod_{h=j+1}^{2r+1} I_h(x)\right). \tag{5.26}$$

We recall that, by (3.2), $K_\varepsilon(x) \ll \varepsilon^2$. We then replace all but one of the $2r$ terms in the products in (5.26) by their bounds (5.25), so that

$$\int_{E_1} |\Psi(x) - \Psi^*(x)| K_\varepsilon(x)\, dx$$

$$\ll \varepsilon^2 P^{2r-1} \sum_{j=1}^{2r+1} \sum_{h=1}^{2r+1} \int_{-\tau}^{\tau} |B_j(x) - J_j(x)| (|g_h(x)| + |I_h(x)|)\, dx.$$

The proof is completed by using Schwarz's inequality and Lemma 8.

LEMMA 10. *Suppose that* $|x - m/n| \leqslant n^{-2}$, $(m, n) = 1$,

$$Y = \min(X^{\frac{1}{3}}, n, X^k n^{-1}),$$

α *is given by* (2.2), *and*

$$\log Y \geqslant 2^{6k-2}(2k+1)\log\log X.$$

Then

$$\sum_{p \leqslant X} e(xp^k) \ll XY^{-\alpha}.$$

This is a theorem of Vinogradov ([**23**]).

LEMMA 11. *Let $h = 2$ or 3 and suppose that $x \in E_h$. Then*

$$g_{h-1}(x) \ll \varepsilon P (\log P)^{-2r-2}.$$

Proof. Let $x \in E_2 \cup E_3$, so that by (5.22) and (5.23), $\tau < |x| \leqslant W$. Let $V = \delta W$ and choose a_j/q_j so that $q_j \leqslant P^k V^{-1}$ and $|\lambda_j x - a_j/q_j| \leqslant q_j^{-1} V P^{-k}$. By the method of Davenport and Heilbronn in [**11**], Lemma 13, we have $\max(q_1, q_2) \geqslant W$. Hence, by Lemma 10, (5.3), and (5.5),

$$\min(|g_1(x)|, |g_2(x)|) \ll P^{1-\alpha\sigma_1} \ll \varepsilon P (\log P)^{-2r-2}.$$

The lemma is an immediate consequence of this, (5.22), and (5.23).

Let

$$F_t(x) = \sum_{u \in \mathscr{U}_t(R)} e(xu), \tag{5.27}$$

where $R = P^{k-\delta}$ agrees with the definition in Theorem 1, and

$$U_t = |\mathscr{U}_t(R)|. \tag{5.28}$$

LEMMA 12. *Let $h = 2$ or 3. Then*

$$\int_{E_h} |\Psi(x) F_1(x) F_2(x)| K_\varepsilon(x)\, dx \ll \varepsilon^2 P^{2r+1-k} (\log P)^{-2r-2} U_1 U_2.$$

Proof. In view of Lemma 11 it suffices to show that for $j = 1, 2, \ldots, 2r+1$ and $t = 1, 2$,

$$\int_{-\infty}^{\infty} |g_j(x)^r F_t(x)|^2 K_\varepsilon(x)\, dx \ll \varepsilon P^{2r-k} U_t^2. \tag{5.29}$$

By Lemma 1 we can write the left side as a finite sum, which is clearly bounded by the integral in (3.6) with F replaced by F_t and $l+1 = 2r$. This gives (5.29).

LEMMA 13. *Let $\Omega(x) = \sum e(x\omega(y_1, \ldots, y_n))$, where ω is any real function and the summation is over any finite set of values of $y_1, \ldots, y_n$. Then, for any $X > 4/\varepsilon$ we have*

$$\int_{|x|>X} |\Omega(x)|^2 K_\varepsilon(x)\, dx \leqslant \frac{16}{X\varepsilon} \int_{-\infty}^{\infty} |\Omega(x)|^2 K_\varepsilon(x)\, dx.$$

Proof. The left side is

$$\varepsilon \int_{|x|>X\varepsilon} |\Omega_\varepsilon(x)|^2 K_1(x)\, dx$$

where $\Omega_\varepsilon(x) = \sum e(x\omega(y_1, \ldots, y_n)/\varepsilon)$, and the right side is

$$\frac{16}{X} \int_{-\infty}^{\infty} |\Omega_\varepsilon(x)|^2 K_1(x)\, dx.$$

The lemma now follows from Davenport and Roth's Lemma 2 of [**12**].

LEMMA 14. *Let E_4 be given by* (5.24). *Then*

$$\int_{E_4} |\Psi(x)F_1(x)F_2(x)|\, K_\varepsilon(x)\, dx \ll \varepsilon^2 P^{2r+1-k}(\log P)^{-2r-2}U_1U_2.$$

Proof. By (5.29), (5.4), (5.3), (5.5), and Lemma 13,

$$\int_{E_4} |g_j(x)^r F_t(x)|^2\, K_\varepsilon(x)\, dx \ll \varepsilon^2 P^{2r-k}(\log P)^{-2r-2}U_t^2.$$

The lemma now follows easily.

LEMMA 15. *We have*

$$\int_{|x|>\tau} |\Psi^*(x)|\, K_\varepsilon(x)\, dx \ll \varepsilon^2 P^{2r+1-k}(\log P)^{-2r-2} \tag{5.30}$$

and

$$\int_{-\infty}^{\infty} \Psi^*(x)F_1(x)F_2(x)e(x\eta)K_\varepsilon(x)\, dx \gg \varepsilon^2 P^{2r+1-k}(\log P)^{-2r-1}U_1U_2. \tag{5.31}$$

Proof. The inequality (5.30) follows easily from (5.14). To prove (5.31) we use Lemma 1 to write the integral on the left as

$$\sum_{u_1}\sum_{u_2}\int_{\mathscr{B}} \frac{z_1^{-1+1/k}\ldots z_{2r+1}^{-1+1/k}}{(\log z_1)\ldots(\log z_{2r+1})} \max\left(0, \varepsilon - \left|\eta + u_1 + u_2 + \sum_{j=1}^{2r+1}\lambda_j z_j\right|\right) dz_1\ldots dz_{2r+1}, \tag{5.32}$$

where the box $\mathscr{B}$ is the cartesian product of the intervals $\delta^k P^k \leqslant z_j \leqslant P^k$. Suppose that

$$|\xi| \leqslant 3R \tag{5.33}$$

and h and l are chosen so that $\lambda_h\lambda_l < 0$. If

$$2\delta\left|\frac{\lambda_h}{\lambda_l}\right| P^k \leqslant z_l \leqslant 3\delta\left|\frac{\lambda_h}{\lambda_l}\right| P^k$$

and

$$\delta^2 P^k \leqslant z_j \leqslant 2\delta^2 P^k \quad (1 \leqslant j \leqslant 2r+1,\, j \neq h,\, j = l),$$

then, by (5.33),

$$\delta P^k + \tfrac{1}{2}\varepsilon|\lambda_h|^{-1} \leqslant -\left(\xi + \sum_{\substack{j=1\\ j\neq h}}^{2r+1}\lambda_j z_j\right)\lambda_h^{-1} \leqslant P^k - \tfrac{1}{2}\varepsilon|\lambda_h|^{-1}.$$

Hence the box $\mathscr{B}$ contains a region $\mathscr{B}'$ with volume $\gg \varepsilon P^{2rk}$ and such that if $(z_1, \ldots, z_{2r+1}) \in \mathscr{B}'$ and $u_t \in U_t$ $(t = 1, 2)$, then

$$\left|\eta + u_1 + u_2 + \sum_{j=1}^{2r+1}\lambda_j z_j\right| \leqslant \tfrac{1}{2}\varepsilon.$$

Thus the expression (5.32) is

$$\gg (P^{1-k})^{2r+1}(\log P)^{-2r-1}\varepsilon^2 P^{2rk}U_1U_2.$$

This completes the proof of Lemma 15.

It is now a straightforward matter to deduce Theorem 2 from Lemmas 1, 9, 12, 14, and 15.

6. Proof of Corollaries 2.1 and 2.2

The next lemma is an analogue of Davenport's Theorem 2 of [**7**].

LEMMA 16. *For each large X let $\mathscr{U}(X)$ be a set with density $\nu-\delta$, where $1/k < \nu < 1$, and let*

$$\mu = \max_{1\leqslant h\leqslant k-2} \frac{1}{k}\left(1+(k-1)\nu+\frac{h+1-\nu(k-1)}{2^h-1+\nu}\nu\right). \tag{6.1}$$

Suppose further that λ is a non-zero real number and

$$\varphi = \frac{\mu k-1}{\nu k}. \tag{6.2}$$

Then, for every sufficiently large Y, there is a set $\mathscr{V}(Y)$ with density $\mu-\delta$ and such that every element v can be written in the form λp^k+u with $p^k < Y$ and $u \in \mathscr{U}(Y^\varphi)$.

Proof. Let $Y > Y_0(\delta)$, so that, in particular, if

$$X = Y^\varphi \tag{6.3}$$

then

$$|\mathscr{U}(X)| > X^{\nu-\delta}. \tag{6.4}$$

Let

$$Z = \delta Y^{1/k} \tag{6.5}$$

and Γ_n be the number of solutions of

$$-\tfrac{1}{2} < \lambda p^k+u-n \leqslant \tfrac{1}{2} \tag{6.6}$$

with

$$Z < p \leqslant 2Z \tag{6.7}$$

and

$$u \in \mathscr{U}(X). \tag{6.8}$$

To prove the lemma it suffices to show that when $1 \leqslant h \leqslant k-2$ we have

$$\sum_{n=-\infty}^{\infty} \Gamma_n{}^2 \ll Z^{1+\delta}|\mathscr{U}(X)|(1+XZ^{-k+1-2^{1-h}} + X^{(1-2^{-h})}Z^{-(1-2^{-h})(k-1)-(h+1)2^{-h}}|\mathscr{U}(X)|^{2^{-h}}). \tag{6.9}$$

For then the argument of Davenport's Theorem 2 of [**7**] gives

$$\sum_{\substack{n=-\infty\\ \Gamma_n>0}}^{\infty} 1 \gg Z^{1+k\varphi\nu-3\delta}$$

so that, by (6.5) and (6.2),

$$\sum_{\substack{n=-\infty \\ \Gamma_n>0}}^{\infty} 1 > 2Y^{\mu-\delta}. \tag{6.10}$$

We are thus able to take a representative v from each interval $(n-\frac{1}{2}, n+\frac{1}{2}]$ for which $\Gamma_n > 0$, so that, by (6.7), (6.8), (6.5), and (6.3), $v = \lambda p^k + u$, $u \in \mathscr{U}(Y^{\varphi})$, $p^k \leqslant Y$, and $|v| \leqslant Y$. The worst that can happen is that all of these v are in consecutive intervals and we have to discard half of them. By (6.10) the remainder form a set $\mathscr{V}(Y)$ as required.

To prove (6.9) we formally follow the argument of Davenport's Theorem 1 of [7]. Clearly $\sum_n \Gamma_n{}^2$ is at most the number of solutions of the inequality

$$-1 \leqslant \lambda(n_1{}^k - n_2{}^k) + u_1 - u_2 \leqslant 1$$

with $Z < n_1, n_2 \leqslant 2Z$ and $u_1, u_2 \in \mathscr{U}(X)$. We use the notation

$$\Delta_t(\Phi(x)) = \Phi(x+t) - \Phi(x), \quad \Delta_{t,t_1,\ldots,t_q}(\Phi(x)) = \Delta_{t_q}(\Delta_{t,t_1,\ldots,t_{q-1}}(\Phi(x))),$$

and let N_q $(1 \leqslant q \leqslant k-2)$ be the number of solutions of

$$-2^{q-1} \leqslant \lambda\Delta_{t,t_1,\ldots,t_{q-1}}(n^k) + u_1 - u_2 \leqslant 2^{q-1} \tag{6.11}$$

with $Z < n \leqslant 2Z$, $u_1, u_2 \in \mathscr{U}(X)$, $0 < t \ll XZ^{1-k}$, and $0 < t_j \leqslant Z$. Then

$$\sum_{n=-\infty}^{\infty} \Gamma_n{}^2 \ll Z|\mathscr{U}(X)| + N_1. \tag{6.12}$$

Let $\Theta(\mathbf{t}, u)$ be the number of solutions of (6.11) with $(t, t_1, \ldots, t_{q-1}) = \mathbf{t}$ and $u_2 = u$. Then, by Cauchy's inequality,

$$N_q{}^2 \ll XZ^{-k+q}|\mathscr{U}(X)| \sum_{\mathbf{t},u} \Theta(\mathbf{t}, u)^2. \tag{6.13}$$

The sum $\sum_{\mathbf{t},u} \Theta(\mathbf{t}, u)^2$ does not exceed the number of solutions in $t, t_1, \ldots, t_{q-1}, n_1, n_2, u_1, u_2$, and u of

$$-2^q \leqslant \lambda\Delta_{t,t_1,\ldots,t_{q-1}}(n_1{}^k) + u_1 - \lambda\Delta_{t,t_1,\ldots,t_{q-1}}(n_2{}^k) - u_2 \leqslant 2^q \tag{6.14}$$

with

$$-2^{q-1} \leqslant \lambda\Delta_{t,t_1,\ldots,t_{q-1}}(n_1{}^k) + u_1 - u \leqslant 2^{q-1}. \tag{6.15}$$

For each set of numbers $t, t_1, \ldots, t_{q-1}, n_1, n_2, u_1$, and u_2 the number of choices for u is at most $2^q + 1$. Hence the number of solutions of (6.14) and (6.15) with $n_1 \neq n_2$ is $\ll N_{q+1}$. Also, for each set of numbers t, t_1, $\ldots$, t_{q-1}, n_1, n_2, u_1, and u the number of choices for u_2 is at most $2^{q+1} + 1$. Hence the number of solutions of (6.14) and (6.15) with $n_1 = n_2$ is $\ll N_q$. Thus, by (6.13),

$$N_q{}^2 \ll XZ^{-k+q}|\mathscr{U}(X)|(N_q + N_{q+1}). \tag{6.16}$$

Since $q \leqslant k-2$, the number of integers in an interval of length $2^q/|\lambda|$ is $\ll 1$. Thus, for each pair u_1, u_2 the number of choices for $t, t_1, \ldots, t_{q-1}$

and n which satisfy (6.11) with $Z < n \leqslant 2Z$ and

$$0 < tt_1 \ldots t_q \leqslant XZ^{-k+q},$$

is $\ll Z^{\delta}$. Thus

$$N_q \ll Z^{\delta} |\mathscr{U}(X)|^2. \tag{6.17}$$

The inequality (6.9), and thus the lemma, now follows from (6.12), (6.16), and (6.17).

We next prove Corollary 2.1. Let $\lambda_1^{(t)}, \ldots, \lambda_m^{(t)}$ be m ($\geqslant 1$) non-zero real numbers and X be a large real number. It is trivial that there is a set $\mathscr{U}_t^{(1)}(X)$ with density $(1/k) - \frac{1}{2}\delta$ and every element well spaced and of the form $\lambda_1^{(t)} p_1{}^k$ with $p_1{}^k \leqslant X$. We apply Lemma 16 iteratively, noting that

(i) if $\nu = (1/k) + \frac{1}{2}\delta$, then $\mu > 2/k$,

(ii) $\mu > (1/k)(1 + \nu(k-1)) + \delta$ whenever $\nu > 1/k$.

This enables us to assert the existence of a $\mathscr{U}_t^{(m)}(X)$ with density $\nu^{(m)}$, where

$$\nu^{(m)} = 1 - (1 - 2\kappa)(1 - \kappa)^{m-2} \quad (m \geqslant 3), \tag{6.18}$$

and such that every $u \in \mathscr{U}_t^{(m)}(X)$ can be written in the form

$$u = \sum_{j=1}^{m} \lambda_j^{(t)} p_j{}^k$$

with $p_j{}^k \leqslant X$. We use this in Theorem 2 with $r = k$, $m = N + 3$, and $\lambda_j^{(t)} = \lambda_{2r+2j+t+1}$. This with (2.1), (2.4), and (2.5) ensures that

$$\nu^{(m)} > 1 - 2r\theta/k.$$

This completes the proof of (2.6).

The estimate (6.18) is in essence Hardy and Littlewood's Lemma 22 of [**15**], and with $\nu^{(m)}$ replaced by $\nu^{(m)} + \delta$ it could have been shown in much the same way. However, we require the full strength of Lemma 16 for Corollary 2.2. We begin as in the proof of Corollary 2.1 by noting the existence of a $\mathscr{U}_t^{(2)}(X)$ with $\nu^{(2)} = 2\kappa - \delta$. When $5 \leqslant k \leqslant 9$ we apply Lemma 16 iteratively. For convenience we follow the calculations of Hua ([**16**], Chapter 9) when $5 \leqslant k \leqslant 8$ and Cook ([**2**]) when $k = 9$ (we extend the calculations one step further when $k = 5$, 6, and 7). We obtain

(a) $\nu^{(8)} = 0{\cdot}911 \quad (k = 5)$,

(b) $\nu^{(13)} = 0{\cdot}9482 \quad (k = 6)$,

(c) $\nu^{(19)} = 0{\cdot}9668 \quad (k = 7)$,

(d) $\nu^{(28)} = 0{\cdot}9838 \quad (k = 8)$,

(e) $\nu^{(40)} = 0{\cdot}9933 \quad (k = 9)$.

The hypothesis of Theorem 2 is now satisfied if we take $r = 4$, 5, 8, 9, and 8 respectively. This gives the asserted upper bounds for $\mathscr{D}(k)$ when $k = 5$, 6, 7, 8, and 9.

When $k = 10$, we use the method of Cook ([**2**]) to obtain

(f) $\nu^{(51)} = 0{\cdot}9963 \quad (k = 10)$.

Note that the estimate given by Davenport and Erdös ([**10**]) can be adapted to our needs in a straightforward manner; for generalizations in a different direction see [**21**]. The hypothesis of Theorem 2 is now satisfied with $r = 10$. Thus $\mathscr{D}(10) \leqslant 123$.

When $k = 4$ there is an extra difficulty. We should like to take $m = 5$, $\nu^{(m)} = (5539/6268) - \delta$, and $r = 2$, but this violates the requirement $r \geqslant \frac{1}{2}k + 1$. This can be overcome by replacing the expression $F(x)$ in (3.6) by

$$f_j(x) \sum_{u \in \mathscr{U}_t(P^{4\varphi})} e(xu),$$

where

$$\varphi = \frac{1236}{1567} \tag{6.19}$$

and $\mathscr{U}_t(P^{4\varphi})$ has density $\nu - \delta$ with

$$\nu = \frac{331}{412}. \tag{6.20}$$

On the supplementary intervals, $\mathscr{N}_j$, we note that by the method of Lemma 16 the number of solutions of

$$|\lambda_j(p_1^4 - p_2^4) + u_1 - u_2| < \varepsilon$$

with $\delta P < p_h \leqslant P$ and $u_1, u_2 \in \mathscr{U}_t(P^{4\varphi})$ is

$$\ll P^{2-4\mu+4\delta} |\mathscr{U}_t(P^{4\varphi})|^2,$$

where

$$\mu = \frac{5539}{6268}. \tag{6.21}$$

The estimation over these intervals then proceeds as before. On the basic intervals, $\mathscr{M}_j$, we use Lemma 2 to replace the factor $|f_j(x)|^6$ by $|q^{-1}S(q, a)L(\lambda_j x - a/q)|^6 + P^{\frac{9}{2}+6\delta}$. The contribution over $\mathscr{M}_j$ of the first term can be estimated as in the proof of (3.6), and the second term contributes

$$\ll P^{\frac{9}{2}+6\delta} \int_{-\infty}^{\infty} \Big| \sum_{u \in \mathscr{U}_t(P^{4\varphi})} e(xu) \Big|^2 K_\varepsilon(x)\, dx$$

$$\ll \varepsilon P^{\frac{9}{2}+6\delta} |\mathscr{U}_t(P^{4\varphi})|$$

$$\ll \varepsilon P^{\frac{9}{2}+10\delta-4\nu\varphi} |\mathscr{U}_t(P^{4\varphi})|^2.$$

By (6.19) and (6.20), $\frac{9}{2} - 4\nu\varphi = \frac{6159}{3134} < 2$. We thus obtain

$$\int_{-\infty}^{\infty} \Big| f_j(x)^3 \sum_{u \in \mathscr{U}_t(P^{4\varphi})} e(xu) \Big|^2 K_\varepsilon(x)\, dx \ll \varepsilon P^2 |\mathscr{U}_t(P^{4\varphi})|^2.$$

The method used to deduce Theorem 2 from Theorem 1 now shows that when $k = 4$ and $0 < \sigma < (25600)^{-1}$ there are infinitely many solutions of (2.3) with $s = 15$. This gives $\mathscr{D}(4) \leqslant 15$, and completes the proof of Corollary 2.2.

7. Proof of Theorem 3

We give only the briefest outline. Let $P^{k/2}$ be the denominator of a convergent to the continued fraction for λ_1/λ_2 and dissect the real line into the regions

$$\begin{aligned}
\mathscr{E}_1 &= \{x: |x| \leqslant P^{1-k-\delta}\},\\
\mathscr{E}_2 &= \{x: P^{1-k-\delta} < |x| \leqslant P,\ |f_1(x)| \leqslant |f_2(x)|\},\\
\mathscr{E}_3 &= \{x: P^{1-k-\delta} < |x| \leqslant P,\ |f_2(x)| < |f_1(x)|\},\\
\mathscr{E}_4 &= \{x: |x| > P\}.
\end{aligned}$$

Let $\varepsilon = P^{-\delta}$. The region $\mathscr{E}_1$ can be treated in a straightforward manner. On $\mathscr{E}_2$ and $\mathscr{E}_3$ the method of Davenport and Roth's Lemma 6 of [**12**] and the bound (3.7) with $l = r-1$ give a suitable estimate. If r is even we use Lemma 13 and (3.6) to majorize

$$\int_{\mathscr{E}_4} |f_j(x)^r F_l(x)^2|\, K_\varepsilon(x)\, dx.$$

If r is odd we replace this by

$$P \int_{\mathscr{E}_4} |f_j(x)^{r-1} F_l(x)^2|\, K_\varepsilon(x)\, dx$$

and by Lemma 13 and (3.7) this is

$$\ll P^\delta \int_{-\infty}^{\infty} |f_j(x)^{r-1} F_l(x)^2|\, K_\varepsilon(x)\, dx \ll \varepsilon^2 P^{r-k-\delta} |\mathscr{V}_l|^2.$$

8. Proof of Corollaries 3.1 and 3.2

Corollary 3.1 follows from Theorem 3 in the same way that Corollary 2.1 follows from Theorem 2.

We deduce Corollary 3.2 from Theorem 3 by noting that the computations giving rise to (a), (b), ..., (f) in § 6 enable us to assert the existence of suitable sets $\mathscr{U}_l(X)$ such that in Theorem 3 we may take

(a) $m = 8,\ r = 8 \quad (k = 5)$,
(b) $m = 13,\ r = 10 \quad (k = 6)$,
(c) $m = 19,\ r = 15 \quad (k = 7)$,
(d) $m = 28,\ r = 17 \quad (k = 8)$,
(e) $m = 40,\ r = 16 \quad (k = 9)$,
(f) $m = 51,\ r = 19 \quad (k = 10)$.

This establishes Corollary 3.2.

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