DIOPHANTINE APPROXIMATION BY PRIME NUMBERS, I

By R. C. VAUGHAN

[Received 11 April 1973]

1. Introduction

In 1946, Davenport and Heilbronn ([**4**]) adapted the Hardy–Littlewood method to prove that if $\lambda_1, \ldots, \lambda_s$ are non-zero real numbers, not all of the same sign, and not all in rational ratio, then for every $\varepsilon > 0$ the inequality

$$\left| \sum_{j=1}^{s} \lambda_j x_j{}^k \right| < \varepsilon$$

has infinitely many solutions in natural numbers x_j provided that $s \geqslant 2^k + 1$. Later, Davenport and Roth ([**5**]) proved that if $k \geqslant 12$, $s > Ck \log k$ will suffice with a suitable absolute constant C.

More recently, Schwarz ([**9**]) has shown that if either $s \geqslant 2^k + 1$ or $s \geqslant 2k^2(2 \log k + \log\log k + \frac{5}{2}) - 1$ $(k \geqslant 12)$, then the inequality

$$\left| \sum_{j=1}^{s} \lambda_j p_j{}^k \right| < \varepsilon$$

has an infinity of solutions with all the p_j prime numbers. In the case $k = 1$, this was rediscovered by Danicic ([**2**]), who had an application in mind.

A. Baker ([**1**]) took things a step further by showing that when $k = 1$, $s = 3$, and n is an arbitrary natural number, the ε may be replaced by $(\log \max p_j)^{-n}$. Ramachandra ([**8**]) has obtained this result for arbitrary k and with the $p_j{}^k$ replaced by arbitrary integer-valued polynomials f_j with prime arguments p_j and leading coefficient positive, provided that s satisfies the same condition as that required by Schwarz. It would be nice if this condition could be replaced by $s > Ck \log k$, but the Davenport–Roth method is surprisingly resistant to attempts to extend it to prime numbers.

In the case $k = 1$, $s = 3$, Ramachandra has refined matters still more by replacing the right side by

$$\exp(-(\log p_1 p_2 p_3)^{1/2}).$$

The object of this paper is to show that when $s = 3$ and $k = 1$ the right side can be replaced by a power of $\max p_j$. The method used is closely related to that of Davenport and Heilbronn, and makes no use of the complicated argument of Baker. There is no difficulty in extending the method to an arbitrary k. However, it should be noted that the trigonometric sum estimates given by Hua ([**6**]) (combining methods of Weyl and I. M. Vinogradov) permit one only to save a power of a logarithm, and the new and extremely elegant method of Montgomery (Chapter 16 of [**7**]) does not appear to combine well with the Weyl method. Instead, it is necessary to use a theorem given by Vinogradov ([**11**]).

2. The main theorem and definitions

THEOREM. *Suppose that $\lambda_1, \lambda_2, \lambda_3$ are non-zero real numbers not all of the same sign, that η is real, and that λ_1/λ_2 is irrational. Then there are infinitely many ordered triples p_1, p_2, p_3 for which*

$$|\eta+\lambda_1 p_1+\lambda_2 p_2+\lambda_3 p_3| < (\max p_j)^{-1/10}(\log \max p_j)^{20}. \tag{1}$$

It is interesting that one can save as much as 1/10. On the generalized Riemann hypothesis, 1/5 may be saved, but it seems to require a new idea to save any more. As with most of the previous work in this field, the basic argument follows, in general principle, that of Davenport and Heilbronn ([**4**]).

Since λ_1/λ_2 is irrational, there are infinitely many different convergents a/q to its continued fraction, with

$$\left|\frac{\lambda_1}{\lambda_2}-\frac{a}{q}\right| \leqslant q^{-2}, \tag{2}$$

$(a,q) = 1$, $q > 0$, and $a \neq 0$. We choose q to be large in terms of λ_1, λ_2, λ_3, and η, and make the following definitions.

$$X = q^{5/3}, \tag{3}$$

$$P = X^{1/5}(\log X)^{-1}, \tag{4}$$

$$\tau = PX^{-1}, \tag{5}$$

$$T = X^{1/3}, \tag{6}$$

$$Q = XP^{-1}(|\lambda_1|^{-1}+|\lambda_2|^{-1}), \tag{7}$$

$$\varepsilon = X^{-1/10}(\log X)^{20}, \tag{8}$$

$$\left.\begin{aligned} K_\varepsilon(x) &= \pi^{-2}x^{-2}\sin^2 \pi\varepsilon x \quad (x \neq 0),\\ K_\varepsilon(0) &= \varepsilon^2, \end{aligned}\right\} \tag{9}$$

$$e(x) = e^{2\pi i x}, \tag{10}$$

$$S(x) = \sum_{p \leqslant X} e(px)\log p, \tag{11}$$

and

$$I(x) = \int_0^X e(xy)\,dy. \tag{12}$$

We use $\rho = \beta + i\gamma$ (β, γ real) to denote a typical zero of the Riemann zeta function, and we write

$$G_\rho(x) = \sum_{n \leqslant X} n^{\rho-1} e(nx). \tag{13}$$

We let

$$\Sigma' = \sum_{|\gamma| \leqslant T,\, \beta \geqslant 2/3} \tag{14}$$

denote summation over all ρ with $|\gamma| \leqslant T$ and $\beta \geqslant 2/3$,

$$J(x) = \Sigma' G_\rho(x), \tag{15}$$

$$\Delta(x) = S(x) - I(x) + J(x) \tag{16}$$

and for any function of a real variable, f,

$$f_j(x) = f(\lambda_j x). \tag{17}$$

Furthermore, we let

$$F(x) = \prod_{j=1}^{3} S_j(x) \tag{18}$$

and

$$H(x) = \prod_{j=1}^{3} I_j(x). \tag{19}$$

Throughout, constants both explicit and implicit, in the O, $\ll$, and $\gg$ notations, depend only on λ_1, λ_2, λ_3, and η. C denotes such a constant, not necessarily the same on each occurrence.

3. Further explanation of the method

The key to the method is the following lemma, which is a trivial corollary of Lemma 4 of Davenport and Heilbronn ([**4**]).

LEMMA 1. *For every real* y,

$$\int_{-\infty}^{\infty} e(xy) K_\varepsilon(x)\,dx = \min(0, \varepsilon - |y|).$$

Our object is to find a lower bound for

$$\int_{-\infty}^{\infty} F(x) e(x\eta) K_\varepsilon(x)\,dx$$

since, by (18), (17), and (11), we may write $F(x)$ as a sum over p_1, p_2, and p_3 not exceeding X, and on interchanging the order of summation and integration we see that there is a non-zero contribution only when

$|\eta+\sum\lambda_j p_j| < \varepsilon$. On heuristic grounds, we expect the integral to be $\varepsilon^2 X^2$ in order of magnitude, and we shall see that this is indeed so. The dominant contribution to this integral is from the region close to the origin, and there H is a good approximation to F. The contribution from the region $|x| > P$ is negligible for comparatively trivial reasons, and the fact that λ_1/λ_2 is irrational enables us to assert that in the intermediate region we cannot approximate to both $\lambda_1 x$ and $\lambda_2 x$ by rational numbers a'/q' and a''/q'' with both q' and q'' smaller than P. Thus one of S_1 and S_2 must be relatively small.

Lemmas 2 to 10 are concerned with the region near the origin, 11 and 12 with the intermediate region, and 13 with the trivial region.

4. Lemmas concerning the distribution of primes

LEMMA 2. *Suppose that $\sigma > 0$ and $t \geqslant 2$, and let $N(\sigma, t)$ denote the number of zeros $\rho = \beta+i\gamma$ of the Riemann zeta function with $|\gamma| \leqslant t$ and $\sigma \leqslant \beta \leqslant 1$. Then*

$$N(\sigma,t) \ll t\log t \tag{20}$$

and

$$N(\sigma,t) \ll t^{5(1-\sigma)/2}(\log t)^{14}. \tag{21}$$

The inequality (20), and even more, is demonstrated by Davenport in Chapter 15 of [**3**]. (21) is a corollary of Montgomery's Theorem 12.1 of [**7**].

LEMMA 3. *Suppose that $2 \leqslant Y \leqslant X$ and*

$$\vartheta(Y) = \sum_{p\leqslant Y} \log p. \tag{22}$$

Then

$$\vartheta(Y) = Y - {\sum}' Y^\rho \rho^{-1} + O(X^{2/3}(\log X)^C). \tag{23}$$

Proof. It is shown by Davenport in Chapter 17 of [**3**] that if $t \geqslant 2$, then

$$\psi(Y) = Y - \sum_{|\gamma|\leqslant t,\, \beta>0} Y^\rho \rho^{-1} + O(Yt^{-1}(\log Yt)^2 + \log Y),$$

where

$$\psi(Y) = \sum_{n\leqslant Y} \Lambda(n)$$

and Λ is von Mangoldt's function. We take $t = T$, and then by (20), (6), and a partial summation we see that the contribution from the zeros ρ with $2/3 > \beta > 0$ is $\ll X^{2/3}(\log X)^C$. The proof of (23) is completed by the observation that $\psi(Y)$ and $\vartheta(Y)$ differ by at most $X^{1/2}\log X$.

LEMMA 4. *There is a positive number C such that if $\zeta(\rho) = 0$, $\rho = \beta+i\gamma$, and $t = |\gamma|+3$, then*

$$\beta \leqslant 1 - C(\log t)^{-2/3}(\log\log t)^{-1/3}.$$

This result is given by Walfisz on p. 226 of [**12**]. See also Montgomery's Corollary 11.4 of [**7**]. For our purposes, any number less than unity would do in place of the 2/3, but then the calculations become more complicated.

5. The neighbourhood of the origin

In the next lemma we show that $I(x)-J(x)$ is a good approximation to $S(x)$ when x is small.

LEMMA 5. *For every real number* x,

$$\Delta(x) \ll X^{2/3}(\log X)^C(1+X|x|).$$

Proof. We begin by noting that if $2 \leqslant Y \leqslant X$, $\frac{2}{3} \leqslant \beta < 1$, and $|\gamma| \leqslant T$, then

$$\begin{aligned}\sum_{n\leqslant Y} n^{\rho-1} &= [Y]Y^{\rho-1} - \int_1^Y (\rho-1)[y]y^{\rho-2}\,dy \\ &= Y^\rho \rho^{-1} + O(T\log Y).\end{aligned}$$

Hence, by Lemma 3, (14), and (20),

$$\begin{aligned}\vartheta(Y) - Y + \sum{}' \sum_{n\leqslant Y} n^{\rho-1} &\ll T^2(\log TY)^2 + X^{2/3}(\log X)^C \\ &\ll X^{2/3}(\log X)^C.\end{aligned}$$

The formula

$$\sum_{n\leqslant X} a_n e(nx) = e(Xx)\sum_{n\leqslant X} a_n - \int_1^X 2\pi i x e(Yx) \sum_{n\leqslant Y} a_n\,dY, \tag{24}$$

with $a_n = \log n + \sum' n^{\rho-1}$ if n is a prime number and $a_n = \sum' n^{\rho-1}$ if n is not a prime number, enables us to deduce from (11), (13), and (15) that

$$S(x)+J(x) = e(Xx)X - \int_1^X 2\pi i x e(Yx)Y\,dY + O(X^{2/3}(\log X)^C(1+X|x|)).$$

By a partial integration we see that the main term on the right differs from the integral in (12) by $O(1)$. This with (16) completes the proof of the lemma.

The next two lemmas are trivial estimates for the functions S, I, and J.

LEMMA 6. *We have*

$$S(x) \ll X \tag{25}$$

and

$$I(x) \ll X. \tag{26}$$

Proof. $S(x) \ll X$ is a consequence of Chebychev's upper bound, and $I(x) \ll X$ is trivial.

LEMMA 7. *If $x \neq 0$, then $I(x) \ll |x|^{-1}$.*

Proof. This result follows at once from (12) by integration.

We expect that the integral

$$\int_{-\infty}^{\infty} F(x)e(xy)K_\varepsilon(x)\,dx$$

is about $\varepsilon^2 X^2$ in order of magnitude. In the next two lemmas we show that $F(x)$ can be replaced by $H(x)$ near the origin with an error of a smaller order.

LEMMA 8. *We have*

$$\int_{-1/2}^{1/2} |I(x)|^2\,dx \ll X, \tag{27}$$

$$\int_{-1/2}^{1/2} |S(x)|^2\,dx \ll X \log X, \tag{28}$$

and

$$\int_{-1/2}^{1/2} |J(x)|^2\,dx \ll X \exp(-(\log X)^{1/5}). \tag{29}$$

Proof. The bound (27) is a consequence of (26) and Lemma 7, and (28) follows easily from Parseval's identity and Chebychev's upper bound.

(29) lies deeper. First of all, note that by (15) the left side is at most

$$\Sigma_1' \Sigma_2' \int_{-1/2}^{1/2} |G_{\rho_1}(x)G_{\rho_2}(x)|\,dx,$$

and, by Schwarz's inequality, this is at most

$$\left(\Sigma' \left(\int_{-1/2}^{1/2} |G_\rho(x)|^2\,dx\right)^{1/2}\right)^2.$$

By (13) and Parseval's identity, the integral in this last expression is simply

$$\sum_{n\leqslant X} n^{2\beta-2}.$$

Hence, on noting that $\beta \geqslant \frac{2}{3}$, we have

$$\int_{-1/2}^{1/2} |J(x)|^2\,dx \ll X^{-1}(\Sigma' X^\beta)^2. \tag{30}$$

By (14) the sum on the right is

$$\Sigma' \left(X^{2/3} + \int_{2/3}^{\beta} X^{\sigma} (\log X)\, d\sigma \right)$$

$$= X^{2/3} N(\tfrac{2}{3}, T) + \int_{2/3}^{1} X^{\sigma} N(\sigma, T)(\log X)\, d\sigma,$$

in the notation of Lemma 2. By Lemma 4 and (6), $N(\sigma, T) = 0$ if $\sigma > 1 - (\log X)^{-3/4}$. Therefore, by (21) and (6), the sum in (30) is

$$\ll (X^{2/3} T^{5/6} + (XT^{-5/2})^{1-(\log X)^{-3/4}} T^{5/2})(\log X)^C$$

$$\ll (X^{17/18} + X \exp(-\tfrac{1}{6}(\log X)^{1/4}))(\log X)^C$$

$$\ll X \exp(-(\log X)^{1/5}).$$

This with (30) gives (29).

LEMMA 9. *We have*

$$\int_{-\tau}^{\tau} | F(x) - H(x) | K_{\varepsilon}(x)\, dx \ll \varepsilon^2 X^2 (\log X)^{-1}.$$

Proof. Note that, by (18), (19), and (16),

$$F(x) - H(x) = \sum_{j=1}^{3} \left(\prod_{k<j} S_k(x) \right) (\Delta_j(x) - J_j(x)) \left(\prod_{k>j} I_k(x) \right). \tag{31}$$

By (5) and (4),

$$| \lambda_j | \tau \ll X^{-4/5}. \tag{32}$$

By (32), (17), and Lemma 5,

$$\int_{-\tau}^{\tau} | \Delta_j(x) |^2\, dx \ll X^{14/15} (\log X)^C. \tag{33}$$

Clearly, by (9),

$$K_{\varepsilon}(x) \ll \varepsilon^2. \tag{34}$$

To estimate our integral, we first of all use the crude inequalities (25) and (26) to replace one of the S_k or one of the I_k in each term in (31). Then, by (34), we obtain as an upper bound the sum of three expressions each having either the form

$$\varepsilon^2 X \int_{-\tau}^{\tau} (| \Delta_j(x) | + | J_j(x) |) | I_k(x) |\, dx$$

or the form

$$\varepsilon^2 X \int_{-\tau}^{\tau} | S_k(x) | (| \Delta_j(x) | + | J_j(x) |)\, dx.$$

Then Schwarz's inequality and an appeal to Lemma 8 and (33) complete the proof.

The next step in our argument is to show that H on its own gives a larger contribution.

LEMMA 10. *We have*

$$\int_{|x|>\tau} |H(x)|\, K_\varepsilon(x)\, dx \ll \varepsilon^2 X^2 (\log X)^{-1} \tag{35}$$

and

$$\int_{-\infty}^{\infty} H(x) e(x\eta) K_\varepsilon(x)\, dx \gg \varepsilon^2 X^2. \tag{36}$$

Proof. To prove (35), we note that $K_\varepsilon(x) \ll \varepsilon^2$ so that, by Lemma 7, the left side is

$$\ll \varepsilon^2 \int_\tau^\infty x^{-3}\, dx = 2\varepsilon^2 \tau^{-2}.$$

This, with (5) and (4), gives (35).

To prove (36), we write the left side as

$$\int_0^X \int_0^X \int_0^X \int_{-\infty}^{\infty} e\left(x\left(\eta + \sum_{j=1}^{3} \lambda_j y_j\right)\right) K_\varepsilon(x)\, dx\, dy_1\, dy_2\, dy_3$$

which, by Lemma 1, is

$$\int_0^X \int_0^X \int_0^X \max\left(0, \varepsilon - \left|\eta + \sum_{j=1}^{3} \lambda_j y_j\right|\right) dy_1\, dy_2\, dy_3. \tag{37}$$

Since λ_1, λ_2, and λ_3 are not all of the same sign (we make no use in the proof of this lemma of the fact that λ_1/λ_2 is irrational) we may assume without loss of generality that $\lambda_1 < 0$, $\lambda_2, \lambda_3 > 0$. For each pair y_2, y_3 with

$$\tfrac{1}{8} X |\lambda_1| \left(\sum_{j=1}^{3} |\lambda_j|\right)^{-1} \leqslant y_2, y_3 \leqslant \tfrac{1}{4} X |\lambda_1| \left(\sum_{j=1}^{3} |\lambda_j|\right)^{-1} \tag{38}$$

we have for large X (i.e. large q),

$$\varepsilon |\lambda_1|^{-1} < -\lambda_1^{-1}(\eta + \lambda_2 y_2 + \lambda_3 y_3) < \tfrac{1}{2} X.$$

Thus for each pair y_2, y_3 satisfying (38), every y_1 with

$$|\lambda_1^{-1}(\eta + \lambda_2 y_2 + \lambda_3 y_3) + y_1| \leqslant \tfrac{1}{2}\varepsilon |\lambda_1|^{-1}$$

satisfies

$$0 < y_1 < \tfrac{1}{2} X + \tfrac{1}{2}\varepsilon |\lambda_1|^{-1} < X.$$

Therefore the multiple integral (37) is greater than

$$\tfrac{1}{64} X^2 |\lambda_1|^2 \left(\sum_{j=1}^{3} |\lambda_j|\right)^{-2} (\varepsilon |\lambda_1|^{-1})(\tfrac{1}{2}\varepsilon)$$
$$\gg \varepsilon^2 X^2.$$

6. The intermediate region

The treatment of the intermediate region depends on the following lemma. The argument we use imitates closely that of Lemma 13 of Davenport and Roth ([**5**]).

LEMMA 11. *For every real number x let*

$$V(x) = \min(|S_1(x)|, |S_2(x)|). \tag{39}$$

Then

$$V(x) \ll XP^{-1/2}(\log X)^{17} \quad (|x| \in (\tau, P]). \tag{40}$$

Proof. Let $|x| \in (\tau, P]$, and for $j = 1, 2$ choose a_j, q_j so that

$$\left|\lambda_j x - \frac{a_j}{q_j}\right| \leqslant Q^{-1} q_j^{-1} \tag{41}$$

with $(a_j, q_j) = 1$ and

$$1 \leqslant q_j \leqslant Q. \tag{42}$$

By (7) and (5), $\tau > |\lambda_j|^{-1} Q^{-1}$. Thus

$$a_1 a_2 \neq 0. \tag{43}$$

We next establish that one of q_1, q_2 is larger than P. To do this, we assume that $q_1, q_2 \leqslant P$ and obtain a contradiction. By (41) and (43),

$$a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 = \frac{a_2/q_2}{\lambda_2 x} q_1 q_2 \left(\lambda_1 x - \frac{a_1}{q_1}\right) - \frac{a_1/q_1}{\lambda_2 x} q_1 q_2 \left(\lambda_2 x - \frac{a_2}{q_2}\right)$$
$$\ll PQ^{-1},$$

so that, by (7), (4), and (3),

$$\left|a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2\right| < \tfrac{1}{2} X^{-3/5} = \tfrac{1}{2} q^{-1}. \tag{44}$$

We recall that q was chosen as the denominator of a convergent to the continued fraction for λ_1/λ_2. Thus, by Legendre's law of best approximation, we have

$$\left|q' \frac{\lambda_1}{\lambda_2} - a'\right| > \tfrac{1}{2} q^{-1}$$

for all integers a', q' with $1 \leqslant q' < q$. Therefore, by (44) and (3),

$$|a_2 q_1| \geqslant q = X^{3/5}.$$

However, by (41) and (4),

$$|a_2 q_1| \ll P q_1 q_2 \ll P^3 = X^{3/5}(\log X)^{-3}.$$

We have thus established that for at least one j, $P < q_j \ll XP^{-1}$. We now make use of a classical result of Vinogradov. The result we require follows easily from [**10**], Chapter 9, Theorem 3. However, for convenience we quote [7], Theorem 16.1, which is in the form we require, except for the trivial replacement of $\Lambda(n)$ by $\log p$. Namely that whenever $(b, r) = 1$ and $Y \geqslant 2$,

$$\sum_{p \leqslant Y} e(bp/r) \log p \ll (r^{1/2}Y^{1/2} + Y^{5/7}r^{3/14} + Yr^{-1/2})(\log Y)^{17}.$$

Hence, by an application of formula (24), we obtain for every real y,

$$S(y) \ll (r^{1/2}X^{1/2} + X^{5/7}r^{3/14} + Xr^{-1/2})(\log X)^{17}(1 + X\,|y - b/r|).$$

This, with $y = \lambda_j x$, $b = a_j$, and $r = q_j$, gives the desired inequality for $V(x)$.

LEMMA 12. *We have*

$$\int_{\tau < |x| \leqslant P} |F(x)|\, K_\varepsilon(x)\, dx \ll \varepsilon^2 X^2 (\log X)^{-1}.$$

Proof. By (18) and (39),

$$|F(x)| \leqslant V(x)(|S_1(x)S_3(x)| + |S_2(x)S_3(x)|),$$

so that

$$|F(x)| \ll V(x) \sum_{j=1}^{3} |S_j(x)|^2. \tag{45}$$

By Lemma 1, (17), and (11),

$$\int_{-\infty}^{\infty} |S_j(x)|^2 K_\varepsilon(x)\, dx = \sum_{p_1 \leqslant X} \sum_{p_2 \leqslant X} (\log p_1)(\log p_2) \max(0, \varepsilon - |\lambda_j(p_1 - p_2)|).$$

Since q is large, $|\lambda_j(p_1 - p_2)| < \varepsilon$ if and only if $p_1 = p_2$. Thus, by Chebychev's upper bound,

$$\int_{-\infty}^{\infty} |S_j(x)|^2 K_\varepsilon(x)\, dx \ll \varepsilon X \log X.$$

Hence, by (45) and Lemma 11,

$$\int_{\tau < |x| \leqslant P} |F(x)|\, K_\varepsilon(x)\, dx \ll \varepsilon X^2 P^{-1/2} (\log X)^{18}.$$

The desired bound now follows from (4) and (8).

7. The trivial region

We dispose of this region in a single lemma.

LEMMA 13. *We have*

$$\int_{|x| > P} |F(x)|\, K_\varepsilon(x)\, dx \ll \varepsilon^2 X^2 (\log X)^{-1}.$$

Proof. By (18) and (25), the integral here is

$$\ll X \sum_{j=1}^{3} \int_{|x|>P} |S_j(x)|^2 K_\varepsilon(x)\,dx.$$

By (17), (11), and (9),

$$\int_{|x|>P} |S_j(x)|^2 K_\varepsilon(x)\,dx \ll \int_{P|\lambda_j|}^{\infty} |S(y)|^2 y^{-2}\,dy$$

$$\ll \sum_{n>P|\lambda_j|} n^{-2} \int_{n-1}^{n} |S(y)|^2\,dy.$$

By Parseval's identity and Chebychev's upper bound this is

$$\ll P^{-1} X \log X.$$

An appeal to (4) and (8) completes the proof of the lemma.

8. Completion of the proof of the main theorem

We conclude the proof of the theorem by collecting the above results. First of all, Lemma 10 gives

$$\int_{-\tau}^{\tau} H(x)e(x\eta)K_\varepsilon(x)\,dx \gg \varepsilon^2 X^2.$$

Moreover, this and Lemmas 9, 12, and 13 together imply that

$$\int_{-\infty}^{\infty} F(x)e(x\eta)K_\varepsilon(x)\,dx \gg \varepsilon^2 X^2.$$

Finally, this, with (18), (17), (11), and Lemma 1, establishes that there are $\gg \varepsilon X^2(\log X)^{-3}$ ordered triples of primes p_1, p_2, p_3 with $p_j \leqslant X$ and $|\eta + \sum \lambda_j p_j| < \varepsilon$. By (8),

$$\varepsilon \leqslant (\max p_j)^{-1/10}(\log \max p_j)^{20}.$$

Thus the fact that q and, hence by (3) and (8), $\varepsilon X^2(\log X)^{-3}$, are large ensures that (1) occurs for an infinity of ordered triples p_1, p_2, p_3.

REFERENCES

1. Alan Baker, 'On some diophantine inequalities involving primes', *J. Reine Angew. Math.* 228 (1967) 166–81.
2. I. Danicic, 'On the integral part of a linear form with prime variables', *Canadian J. Math.* 18 (1966) 621–28.
3. H. Davenport, *Multiplicative number theory* (Markham, Chicago, 1967).
4. —— and H. Heilbronn, 'On indefinite quadratic forms in five variables', *J. London Math. Soc.* 21 (1946) 185–93.
5. —— and K. F. Roth, 'The solubility of certain diophantine inequalities', *Mathematika* 2 (1955) 81–96.
6. L. K. Hua, *Additive theory of prime numbers* (Amer. Math. Soc., Providence, R.I., 1965).

7. HUGH L. MONTGOMERY, *Topics in multiplicative number theory*, Lecture Notes in Mathematics, 227 (Springer-Verlag, Berlin, 1973).
8. K. RAMACHANDRA, 'On the sums $\sum_{j=1}^{K} \lambda_j f_j(p_j)$ (λ_j fixed non-zero real numbers, $f_j(x)$ fixed polynomials and p_j arbitrary primes)', *J. Reine Angew. Math.*, to appear.
9. WOLFGANG SCHWARZ, 'Uber die Lösbarkeit gewisser Ungleichungen durch Primzahlen', ibid. 212 (1963) 150–57.
10. I. M. VINOGRADOV, *The method of trigonometrical sums in the theory of numbers*, translated from the Russian, revised and annotated by K. F. Roth and A. Davenport (Interscience, New York, 1954).
11. —— 'A new estimation of a trigonometric sum, containing primes', *Bull. Acad. Sci. USSR, Sér. Math.* 2 (1938) 1–13.
12. A. WALFISZ, *Weylsche Exponentialsummen in der neuren Zahlentheorie* (VEB Deutscher Verlag der Wiss., Berlin, 1963).

Imperial College
London