MEAN VALUE THEOREMS IN PRIME NUMBER THEORY

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1. *Introduction*

In 1965, Bombieri [1] and A. I. Vinogradov [18] proved a theorem which states, in the slightly stronger form given by Bombieri, that if A is a given positive number and $Q \leqslant x^{\frac{1}{2}}(\log x)^{-B}$ where $B = 3A+23$, then

$$\sum_{q \leqslant Q} \max_{\substack{a,\, y \\ (a,q)=1,\, y \leqslant x}} \left| \psi(y,q,a) - \frac{y}{\phi(q)} \right| \ll_A x(\log x)^{-A}. \tag{1}$$

Davenport [4] has given a proof with $B = 4A+40$ and Gallagher [7] has given a simple proof with $B = 16A+103$. More recently Montgomery [14; Chapter 15] and Huxley [11; Chapter 24] have given $B = A+13$ and $B = A+10$ respectively. All these proofs, except that of Gallagher, depend substantially on zero density estimates which (apart from that of Vinogradov) are deduced in turn from estimates of the large sieve type. The more refined proofs of Montgomery and Huxley also require an approximate functional equation for $L(s,\chi)^2$ of the form given by Lavrik [13; Theorem 1]. Gallagher, in his proof, appeals directly to the large sieve, but with some loss of efficiency.

We follow in the spirit of Gallagher's method, but with a substantial modification. For a given Dirichlet L-function Gallagher writes $L'/L = (1-LG)^2 L'/L + 2L'G - LL'G^2$, where G is a partial sum of the Dirichlet series for $1/L$. This line of approach goes back to Heilbronn [9], who gave an improvement on the work of Hoheisel [10] concerning prime numbers in short intervals, and Fogels [3] who gave a different proof of Ingham's result [12] on the same subject. Even with Lavrik's approximate functional equation, Gallagher's method will apparently give nothing better than $B = 2A+10$. Instead, we write

$$L'/L = (L'/L+F)(1-LG) + (L'+LF)G - F,$$

where F is a partial sum of the Dirichlet series for $-L'/L$. This idea appears to be completely new, and may have applications to other problems. We obtain (1) with $B = A+7/2$ simply and directly from (i) the large sieve, (ii) a theorem given by Montgomery as a straightforward consequence of Lavrik's approximate functional equation and (in common with all the other proofs) (iii) the Siegel–Walfisz theorem.

Our first mean value theorem is as follows.

THEOREM 1. *Suppose that* $\xi \geqslant 1$ *and* $x \geqslant 2$. *Then*

$$\sum_{q \leqslant \xi} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \max_{y \leqslant x} |\psi(y,\chi)| \ll x\mathscr{L}^3 + x^{3/4}\xi^{5/4}\mathscr{L}^{23/8} + x^{1/2}\xi^2\mathscr{L}^{7/2},$$

where $\sum^*$ *denotes summation over all the primitive characters modulo* q, $\psi(y,\chi) = \sum_{n \leqslant y} \Lambda(n)\chi(n)$, Λ *is von Mangoldt's function and* $\mathscr{L} = \log x\xi$.

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When $\xi \leqslant x^{1/2}$ we may compare this with the bound given on page 140 of [14], which when summed over $Q = 2^k$ with $2 \leqslant 2^k \leqslant \xi$ implies that

$$\sum_{q \leqslant \xi} \sum_{\chi}{}^{*} \max_{y \leqslant x} |\psi(y, \chi)| \ll (x\xi^{2/3} + x^{1/2}\xi^2)(\log x)^{11}.$$

There are applications, e.g., [16] and [17], in which inequalities of this kind have been more useful than the Bombieri–Vinogradov theorem.

COROLLARY 1.1. *Suppose that* $\sum'$ *denotes summation over all the non-principal characters modulo* q. *Then*

$$\sum_{q \leqslant \xi} \frac{1}{\phi(q)} \sum_{\chi}{}' \max_{y \leqslant x} |\psi(y, \chi)| \ll_A x(\log x)^{-A} + x^{1/2}\xi\mathscr{L}^{7/2}.$$

Note that in both this corollary and Theorem 1 the bounds are sharp even if ξ is enormous compared with x, and for large ξ are as good, apart from the logarithmic factor, as one can obtain on the assumption of the generalized Riemann hypothesis.

The next corollary is our form of the Bombieri–Vinogradov theorem.

COROLLARY 1.1.1. *Suppose that*

$$\psi(y, q, a) = \sum_{n \leqslant y,\ n \equiv a (\mathrm{mod}\ q)} \Lambda(n).$$

Then

$$\sum_{q \leqslant \xi} \max_{\substack{a, y \\ (a, q) = 1,\ y \leqslant x}} \left| \psi(y, q, a) - \frac{y}{\phi(q)} \right| \ll_A x(\log x)^{-A} + x^{1/2}\xi\mathscr{L}^{7/2}.$$

In 1937, I. M. Vinogradov [19–22] (see also [23–28]) obtained, by elementary methods, estimates for sums of the form

$$\sum_{p \leqslant x} e(\alpha p) \qquad \left(e(z) = e^{2\pi i z}\right).$$

In particular, [28; Theorem 3, Chapter IX] implies that if

$$x^{1/2} \leqslant \tau \leqslant x \exp\left(-(\log x)^{\varepsilon_0}\right), \quad |\alpha - a/q| \leqslant 1/(q\tau), \quad (a, q) = 1$$

and

$$\exp\left((\log x)^{\varepsilon_0}\right) \leqslant q \leqslant \tau,$$

then

$$\sum_{p \leqslant x} e(\alpha p) \ll_{\varepsilon, \varepsilon_0} (xq^{-1/2+\varepsilon/2} + x^{4/5+\varepsilon} + x^{1/2+\varepsilon/2} q^{1/2-\varepsilon/2}). \tag{2}$$

Vinogradov also gives inequalities valid in the regions $1 \leqslant q \leqslant \exp\left((\log x)^{\varepsilon_0}\right)$ and $x\exp\left(-(\log x)^{\varepsilon_0}\right) < q \leqslant x$, but with a much weakened middle term.

Recently, Montgomery [14; Chapter 16] has used zero density estimates to show that if $(a, q) = 1$, then

$$\sum_{n \leqslant x} \Lambda(n)\, e(an/q) \ll (xq^{-1/2} + x^{5/7} q^{3/14} + x^{1/2} q^{1/2})(\log xq)^{17}.$$

In fact he obtains an inequality of the kind given in Theorem 2 below, but with the right hand side replaced by $(x + x^{5/7} q^{5/7} + x^{1/2} q)(\log xq)^{17}$.

We use the methods developed for our proof of Theorem 1 to give a mean value theorem for characters modulo q. This implies an inequality which is always superior to that of Montgomery, although the middle term is still not quite as good as that in (2).

THEOREM 2. *Suppose that* $q \geqslant 1$ *and* $x \geqslant 2$. *Then*

$$\sum_{\chi \bmod q} \max_{y \leqslant x} |\psi(y,\chi)| \ll xl^3 + x^{3/4} q^{5/8} l^{23/8} + x^{1/2} q l^{7/2}$$

where $l = \log xq$.

The remark after Corollary 1.1 applies here also.

COROLLARY 2.1. *Suppose that* $(a,q) = 1$. *Then*

$$\sum_{n \leqslant x} \Lambda(n)\, e(an/q) \ll (xq^{-1/2} + x^{3/4} q^{1/8} + x^{1/2} q^{1/2})\, l^4. \tag{3}$$

The logarithmic factor here could be slightly improved.

COROLLARY 2.1.1. *Suppose that* $1 \leqslant \eta \leqslant x^{1/3}$, $\eta < q \leqslant x\eta^{-1}$, $(a,q) = 1$ $|\alpha - a/q| \leqslant 2\eta q^{-1} x^{-1}$. *Then*

$$\sum_{n \leqslant x} \Lambda(n)\, e(\alpha n) \ll x\eta^{-1/2} l^4.$$

COROLLARY 2.1.2. *Suppose that* $(a,q) = 1$ *and* $|\alpha - a/q| \leqslant q^{-2}$. *Then*

$$\sum_{n \leqslant x} \Lambda(n)\, e(\alpha n) \ll (xq^{-1/2} + x^{7/8} q^{-1/8} + x^{3/4} q^{1/8} + x^{1/2} q^{1/2})\, l^4.$$

The 'x' term on the right in both Theorem 1 and Theorem 2 occurs not only because of the pole of $\zeta(s)$ at 1 (we consider the trivial character modulo 1 to be primitive) but because there may also be a zero of an L-function very near to $\sigma = 1$. As we have remarked before, the last terms in our theorems are what we would expect if we assumed the generalized Riemann hypothesis. The middle terms we should like to remove altogether. In Theorem 2, q plays precisely the same role as ξ^2 in Theorem 1 and we think of these quantities as being essentially the number of terms over which we are summing.

In a private communication, Professor Gallagher has observed that if one introduces a smoothing factor $\log(x/n)$ into the left hand side of (3), the right hand side can be replaced, apart from a logarithmic factor, by $xq^{-1/2} + x^{1/2} q^{1/2+c}$, where c is such that $L(\tfrac{1}{2}+it,\chi) \ll q^c(|t|+1)$. The method of Burgess [2] gives $c = \tfrac{3}{16} + \varepsilon$ (see also [3]) and the modified bound is then slightly better than that of Vinogradov, (2), in the range $x^{2/5} < q < x^{3/(5+10c)}$.

2. *Fundamental lemmas*

Our first lemma is from the theory of the large sieve.

LEMMA 1. *Suppose that* $T \geqslant 2$ *and* $M \geqslant 0$. *Then*

$$\sum_{q \leqslant \xi} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \int_{-T}^{T} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) n^{-it} \right|^2 \frac{dt}{1+|t|} \ll \sum_{n=M+1}^{M+N} (n + \xi^2 \log T)|a_n|^2 \tag{4}$$

and

$$\sum_{\chi}{}'' \int_{-T}^{T} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) n^{-it} \right|^2 \frac{dt}{1+|t|} \ll \sum_{n=M+1}^{M+N} (n+q \log T) |a_n|^2, \tag{5}$$

where $\sum_{\chi}''$ is used to mean summation over all those primitive characters to moduli d with $1 < d \mid q$.

Proof. We first of all prove (5). We define $a_n = 0$ if $n \leqslant M$ or $n > M+N$. By Theorem 1 of Gallagher [**8**] we have

$$\int_{-T}^{T} \left| \sum_{n} a_n \chi(n) n^{-it} \right|^2 dt \ll T^2 \int_{0}^{\infty} \left| \sum_{y \leqslant n \leqslant \tau y} a_n \chi(n) \right|^2 \frac{dy}{y}, \tag{6}$$

where $\tau = \exp(1/T)$. By Theorem 6.3 of Montgomery [**14**] we have

$$\sum_{\chi}{}'' \left| \sum_{y \leqslant n \leqslant \tau y} a_n \chi(n) \right|^2 \ll ((\tau-1) y + q) \sum_{y \leqslant n \leqslant \tau y} |a_n|^2. \tag{7}$$

This with (6) gives

$$\sum_{\chi}{}'' \int_{-T}^{T} \left| \sum_{n} a_n \chi(n) n^{-it} \right|^2 dt \ll \sum_{n} (n+qT)|a_n|^2,$$

which is in fact essentially Theorem 2 of Gallagher [**8**]. A partial integration then completes the proof of (5).

The proof of (4) is similar, using, instead of (7), the inequality

$$\sum_{q \leqslant \xi} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \left| \sum_{y \leqslant n \leqslant \tau y} a_n \chi(n) \right|^2 \ll ((\tau-1) y + \xi^2) \sum_{y \leqslant n \leqslant \tau y} |a_n|^2$$

which is an immediate consequence of (5) and (3′) of Gallagher [**6**].

For each non-principal character χ, we use $L(s, \chi)$ to denote that Dirichlet L-function defined for $\sigma > 1$ by $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$.

The next lemma is a consequence of an approximate functional equation of Lavrik [**13**] for $L(s, \chi)^2$. The exponent of the logarithm in (9) could be reduced to 4, but makes no difference to our end products.

LEMMA 2. *Suppose that $T \geqslant 2$. Then*

$$\sum_{\chi}{}^{*} \int_{-T}^{T} |L(\tfrac{1}{2}+it, \chi)|^4 \frac{dt}{1+|t|} \ll \phi(q)(\log qT)^5 \tag{8}$$

and

$$\sum_{\chi}{}^{*} \int_{-T}^{T} |L'(\tfrac{1}{2}+it, \chi)|^2 \frac{dt}{1+|t|} \ll \phi(q)(\log qT)^5. \tag{9}$$

Proof. The inequality (8) follows at once by partial integration from Theorem 10.1 of Montgomery [**14**] with $\sigma = \frac{1}{2}$. To prove (9) we note that

$$\sum_{\chi}{}^{*} \int_{-T}^{T} |L'(\tfrac{1}{2}+it, \chi)|^4 dt \ll \phi(q)\, T(\log qT)^8$$

can be shown in the same way as Corollary 10.2 of Montgomery [14]. Cauchy's inequality and a partial integration then give the required result.

We here introduce two sums F and G which approximate to $-L'/L$ and $1/L$ respectively when $\sigma = 1+(\log x)^{-1}$. We write, for each character χ,

$$F(s,\chi) = \sum_{n\leqslant u} \chi(n)\,\Lambda(n)\,n^{-s} \qquad (u \geqslant 1) \tag{10}$$

and

$$G(s,\chi) = \sum_{n\leqslant v} \chi(n)\,\mu(n)\,n^{-s} \qquad (v \geqslant 1). \tag{11}$$

For brevity, we let

$$\theta = 1+(\log x)^{-1}. \tag{12}$$

LEMMA 3. *Suppose that $2 \leqslant y \leqslant x$, $T \geqslant x^2$, $u \geqslant 1$ and χ is a character modulo q. Then*

$$\psi(y,\chi)+\frac{1}{2\pi i}\int_{\theta-iT}^{\theta+iT}\left(\frac{L'}{L}(s,\chi)+F(s,\chi)\right)\frac{y^s}{s}\,ds \ll u+\log x.$$

Proof. Clearly

$$\psi(y,\chi)-\sum_{u<n\leqslant y}\Lambda(n)\,\chi(n) \ll u.$$

We complete the proof by noting that by the methods of Chapter 17 of Davenport [4], or by a slightly amended form of Lemma 3.12 of Titchmarsh [15], we have

$$\sum_{u<n\leqslant y}\Lambda(n)\,\chi(n)+\frac{1}{2\pi i}\int_{\theta-iT}^{\theta+iT}\left(\frac{L'}{L}(s,\chi)+F(s,\chi)\right)\frac{y^s}{s}\,ds \ll xT^{-1}(\log x)^2+\log x.$$

The next lemma follows easily from the inequality

$$\left|\sum_{n=1}^{N}\chi(n)\right| \leqslant q \qquad (\chi \neq \chi_0)$$

and some Abel summation.

LEMMA 4. *Suppose that $\sigma \geqslant \frac{1}{2}$ and χ is a non-principal character modulo q. Then*

$$L(s,\chi) \ll (q|s|)^{1/2} \tag{13}$$

and

$$L'(s,\chi) \ll (q|s|)^{1/2}\log\big(q(1+|t|)\big). \tag{14}$$

3. *Description of the proofs*

Our main idea is as follows. We write

$$H(s,\chi) = \left(\frac{L'}{L}(s,\chi)+F(s,\chi)\right)\big(1-L(s,\chi)\,G(s,\chi)\big) \tag{15}$$

and

$$I(s,\chi) = \big(L'(s,\chi)+L(s,\chi)\,F(s,\chi)\big)\,G(s,\chi), \tag{16}$$

so that

$$\frac{L'}{L}(s,\chi)+F(s,\chi) = H(s,\chi)+I(s,\chi). \tag{17}$$

We then use Lemma 1 to estimate directly the contribution from H, averaged over χ, to the integral in Lemma 3. The function I is regular for $\sigma > 0$. We are thus able to move the path of integration to the line $\sigma = \frac{1}{2}$ and combine Lemmas 1 and 2 to obtain a suitable estimate.

Henceforth we assume that

$$\text{either } T = (x\xi)^{10} \quad \text{or} \quad T = (xq)^{10}, \tag{18}$$

and

$$1 \leqslant u \leqslant x \quad \text{and either} \quad 1 \leqslant v \leqslant \xi^2 \quad \text{or} \quad 1 \leqslant v \leqslant q, \tag{19}$$

according as we are considering sums of the form

$$\sum_{q\leqslant\xi} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \quad \text{or} \quad \sum_{\chi}{}''.$$

4. *The line $\sigma = \theta$*

LEMMA 5. *Let H be given by* (15). *Then*

$$\sum_{q\leqslant\xi} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \int_{-T}^{T} |H(\theta+it,\chi)| \frac{dt}{|\theta+it|} \ll (1+\xi^2 u^{-1})^{1/2}(1+\xi^2 v^{-1})^{1/2} \mathscr{L}^3 \tag{20}$$

and

$$\sum_{\chi}{}'' \int_{-T}^{T} |H(\theta+it,\chi)| \frac{dt}{|\theta+it|} \ll (1+qu^{-1})^{1/2}(1+qv^{-1})^{1/2} l^3. \tag{21}$$

Proof. Clearly $\sum_{n\leqslant x} \Lambda(n)^2 \ll x\log x$ and $\sum_{n\leqslant x} d(n)^2 \ll x(\log x)^3$. Hence, by (10), (11) and (4),

$$\sum_{q\leqslant\xi} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \int_{-T}^{T} \left|\frac{L'}{L}(s,\chi)+F(s,\chi)\right|^2 \frac{dt}{|\theta+it|}$$

$$\ll \sum_{n>u} (n+\xi^2\log T)\,\Lambda(n)^2\, n^{-2\theta}$$

$$\ll (1+\xi^2 u^{-1})\,\mathscr{L}^2$$

and

$$\sum_{q\leqslant\xi} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \int_{-T}^{T} |1-L(s,\chi)\,G(s,\chi)|^2 \frac{dt}{|\theta+it|}$$

$$\ll \sum_{n>v} (n+\xi^2\log T) \Big(\sum_{m|n,\, m\leqslant v} \mu(m)\Big)^2 n^{-2\theta}$$

$$\ll (1+\xi^2 v^{-1})\,\mathscr{L}^4.$$

The inequality (20) now follows by Cauchy's inequality, and (21) is shown in the same way, using (5) in place of (4).

5. *The line* $\sigma = \frac{1}{2}$

LEMMA 6. *Let* I *be given by* (16). *Then*

$$\sum_{q \leqslant \xi} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \int_{1/2}^{\theta} |I(\sigma \pm iT, \chi)| \frac{d\sigma}{|\sigma \pm iT|} \ll x^{-1} \tag{22}$$

and

$$\sum_{\chi}{}'' \int_{1/2}^{\theta} |I(\sigma \pm iT, \chi)| \frac{d\sigma}{|\sigma \pm iT|} \ll x^{-1}. \tag{23}$$

This lemma is a trivial consequence of (10), (11), (16) and Lemma 4.

LEMMA 7. *Let* I *be given by* (16). *Then*

$$\sum_{q \leqslant \xi} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \int_{-T}^{T} |I(\tfrac{1}{2}+it, \chi)| \frac{dt}{|\frac{1}{2}+it|} \ll (\xi^2 \mathscr{L}^5)^{1/4} (u^2 \mathscr{L}^2 + \xi^2 \mathscr{L}^5)^{1/4} (v + \xi^2 \mathscr{L}^2)^{1/2} \tag{24}$$

and

$$\sum_{\chi}{}'' \int_{-T}^{T} |I(\tfrac{1}{2}+it, \chi)| \frac{dt}{|\frac{1}{2}+it|} \ll (ql^5)^{1/4} (u^2 l^2 + ql^5)^{1/4} (v + ql^2)^{1/2}. \tag{25}$$

Proof. For an arbitrary function f let

$$\mathscr{A}(f) = \sum_{q \leqslant \xi} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \int_{-T}^{T} |f(\tfrac{1}{2}+it, \chi)| \frac{dt}{|\frac{1}{2}+it|}.$$

By Lemma 2,

$$\mathscr{A}(L^4) \ll \xi^2 \mathscr{L}^5 \tag{26}$$

and

$$\mathscr{A}((L')^2) \ll \xi^2 \mathscr{L}^5. \tag{27}$$

Let

$$g(n) = \sum_{m|n,\ n/u \leqslant m \leqslant u} \Lambda(m) \Lambda(n/m).$$

Clearly

$$\sum_n g(n)^2 \ll \sum_{p \leqslant u} \sum_{m \leqslant (2 \log u)/\log p} m^2 (\log p)^4 + \sum_{p_1{}^m \leqslant u} \sum_{p_2{}^n \leqslant u} (\log p_1)^2 (\log p_2)^2$$

$$\ll u(\log u)^3 + u^2 (\log u)^2.$$

Hence

$$\sum_n g(n)^2 \ll u^2 \mathscr{L}^2. \tag{28}$$

Similarly $\sum_n g(n)^2 n^{-1} \ll \mathscr{L}^4$. Hence, by (28), (10) and (4),

$$\mathscr{A}(F^4) \ll u^2 \mathscr{L}^2 + \xi^2 \mathscr{L}^5. \tag{29}$$

By (4) and (11),

$$\mathscr{A}(G^2) \ll v + \xi^2 \mathscr{L}^2. \tag{30}$$

By (16) and Cauchy's inequality applied several times,

$$\mathscr{A}(I) \ll \left(\mathscr{A}((L')^2)^{1/2} + \mathscr{A}(L^4)^{1/4}\,\mathscr{A}(F^4)^{1/4}\right)\mathscr{A}(G^2)^{1/2}.$$

This with (26), (27), (29) and (30) gives (24). The proof of (25) is identical except that we use (5) instead of (4).

6. *Proofs of Theorem* 1 *and its corollaries*

Let Γ consist of the three line segments $\{\sigma - iT; \theta \geqslant \sigma \geqslant \frac{1}{2}\}$, $\{\frac{1}{2}+it;\ -T \leqslant t \leqslant T\}$ and $\{\sigma + iT; \frac{1}{2} \leqslant \sigma \leqslant \theta\}$. Then, by (15), (16) and (17),

$$\int_{\theta - iT}^{\theta + iT} \left(\frac{L'}{L}(s,\chi) + F(s,\chi)\right)\frac{y^s}{s}\,ds = \int_{\theta - iT}^{\theta + iT} H(s,\chi)\frac{y^s}{s}\,ds + \int_{\Gamma} I(s,\chi)\frac{y^s}{s}\,ds.$$

Theorem 1 now follows from Lemma 3, (20), (22) and (24) with $v = \xi^2$ and $u = \max(1, x^{1/2}\xi^{-1/2}\mathscr{L}^{1/4})$.

To prove the first corollary we note that if χ modulo q is induced by the primitive character χ^* modulo d, then $d \mid q$ and

$$\psi(y,\chi) - \psi(y,\chi^*) \ll (\log xq)^2. \tag{31}$$

We also make use of the trivial inequality

$$\sum_{q \leqslant \xi,\, d \mid q} \frac{1}{\phi(q)} \ll \left(1 + \log\frac{\xi}{d}\right)\frac{1}{\phi(d)} \tag{32}$$

and the Siegel–Walfisz theorem (see, e.g., Chapter 22 of Davenport **[4]**) in the form

$$\sum_{d \leqslant \eta} \frac{1}{\phi(d)} \sum_{\chi}{}' \max_{y \leqslant x} |\psi(y,\chi)| \ll_A x(\log x)^{-A-1}, \tag{33}$$

where

$$\eta = (\log x)^{A+4}. \tag{34}$$

In view of (33) we can clearly assume that $\xi > \eta$. By (31), (32), (33) and (34) the expression we wish to estimate is

$$\ll \sum_{1 < d \leqslant \xi} \left(\sum_{q \leqslant \xi,\, d \mid q} \frac{1}{\phi(q)}\right) \sum_{\chi}{}^* \max_{y \leqslant x} |\psi(y,\chi)| + \xi\mathscr{L}^2$$

$$\ll_A \sum_{\eta < d \leqslant \xi} \left(1 + \log\frac{\xi}{d}\right)\frac{1}{\phi(d)} \sum_{\chi}{}^* \max_{y \leqslant x} |\psi(y,\chi)| + x(\log x)^{-A-1}\mathscr{L} + \xi\mathscr{L}^2$$

$$\ll J(\xi)\xi^{-1} + \int_{\eta}^{\xi} \lambda^{-2}\left(2 + \log\frac{\xi}{\lambda}\right) J(\lambda)\,d\lambda + x(\log x)^{-A} + \xi\mathscr{L}^2,$$

where

$$J(\lambda) = \sum_{\eta < d \leqslant \lambda} \frac{d}{\phi(d)} \sum_{\chi}{}^* \max_{y \leqslant x} |\psi(y,\chi)|.$$

By Theorem 1 and (34) this is

$$\ll x(\log x)^{-A-4}\mathscr{L}^4 + x^{3/4}\xi^{1/4}\mathscr{L}^{23/8} + x^{1/2}\xi\mathscr{L}^{7/2}$$
$$\ll_A x(\log x)^{-A} + x^{1/2}\xi\mathscr{L}^{7/2}.$$

Corollary 1.1.1 follows easily from Corollary 1.1 and the prime number theorem in the form

$$\sum_{q\leqslant \xi}\frac{1}{\phi(q)}\max_{y\leqslant x}|\psi(y,\chi_0)-y| \ll_A x(\log x)^{-A}+\xi.$$

7. *Proofs of Theorem* 2 *and its corollaries*

To prove Theorem 2 we treat the principal character separately. Trivially, this contributes $\ll x$. In view of (31), the sum over the remaining characters can be replaced by

$$\sum_{\chi}{}'' \max_{y\leqslant x}|\psi(y,\chi)|$$

with an error $\ll ql^2$. The theorem now follows from Lemma 3, (21), (23) and (25) with $v = q$ and $u = \max(1, x^{1/2}q^{-1/4}l^{1/4})$, in an analogous manner to Theorem 1.

To prove the first corollary we write

$$\sum_{n\leqslant x}\Lambda(n)\,e(an/q) = \frac{1}{\phi(q)}\sum_{\chi}\chi(a)\,\tau(\bar{\chi})\,\psi(x,\chi)+O(l^2)$$

where

$$\tau(\bar{\chi}) = \sum_{r=1}^{q}\bar{\chi}(r)\,e(r/q).$$

Since $|\tau(\bar{\chi})| \leqslant q^{1/2}$ we have

$$\sum_{n\leqslant x}\Lambda(n)\,e(an/q) \ll \frac{q^{1/2}}{\phi(q)}\sum_{\chi}|\psi(x,\chi)|+l^2.$$

This with Theorem 2 gives Corollary 2.1.

Corollary 2.1.1 is immediate by partial summation.

If $|\alpha - a/q| \leqslant 2x^{-1}$, then Corollary 2.1.2 is also immediate by partial summation. Hence we may assume that

$$2x^{-1} < |\alpha - a/q| \leqslant q^{-2} \tag{35}$$

and

$$q^2 < \tfrac{1}{2}x. \tag{36}$$

Now choose b, r so that $(b, r) = 1$, $r \leqslant x/q$ and $|\alpha - b/r| \leqslant qx^{-1}r^{-1}$. Then, either $r = q$ or $|b/r - a/q| \geqslant q^{-1}r^{-1}$, and in the latter case, by (35) and (36),

$$q^{-2} \geqslant |\alpha - a/q| \geqslant q^{-1}r^{-1} - |\alpha - b/r| \geqslant r^{-1}(q^{-1} - qx^{-1}) \gg r^{-1}q^{-1}.$$

Hence, in either case $q \ll r \leqslant x/q$ and $|\alpha - b/r| \ll x^{-1}$. Therefore, by partial summation and Corollary 2.1 we have the desired result.

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