Appendix C
The gamma function

For any complex number s not equal to a non-positive integer we define the gamma function by its Weierstrass product,

$$\Gamma(s) = \frac{e^{-C_0 s}}{s} \prod_{n=1}^{\infty} \frac{e^{s/n}}{1 + s/n}. \tag{C.1}$$

Here C_0 is Euler's constant, and we recall from Corollary 1.14 or Exercise B.15 that this constatnt is determined by the relation

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + C_0 + O(1/N). \tag{C.2}$$

From (C.1) it is evident that $1/\Gamma(s)$ is an entire function with simple zeros at the non-positive integers, which is to say that $\Gamma(s)$ is a non-vanishing meromorphic function with simple poles at the non-positive integers as depicted in Figure C.1. On considering the N^{th} partial product in (C.1) and appealing to (C.2), we obtain *Gauss's formula*,

$$\Gamma(s) = \lim_{N\to\infty} \frac{N^s N!}{s(s+1)\cdots(s+N)}. \tag{C.3}$$

By taking $s = 1$ we see that $\Gamma(1) = 1$. Moreover, from (C.3) it is also immediate that

$$s\Gamma(s) = \Gamma(s+1). \tag{C.4}$$

Hence by induction we find that

$$\Gamma(n+1) = n! \tag{C.5}$$

for non-negative integers n. As will become apparent, the gamma function not only interpolates the values of the factorial, but does so quite smoothly.

The function $\Gamma(s)\Gamma(1-s)$ has a simple pole at every integer. Since the same can be said for $1/\sin \pi s$, it is reasonable to investigate the relation between these

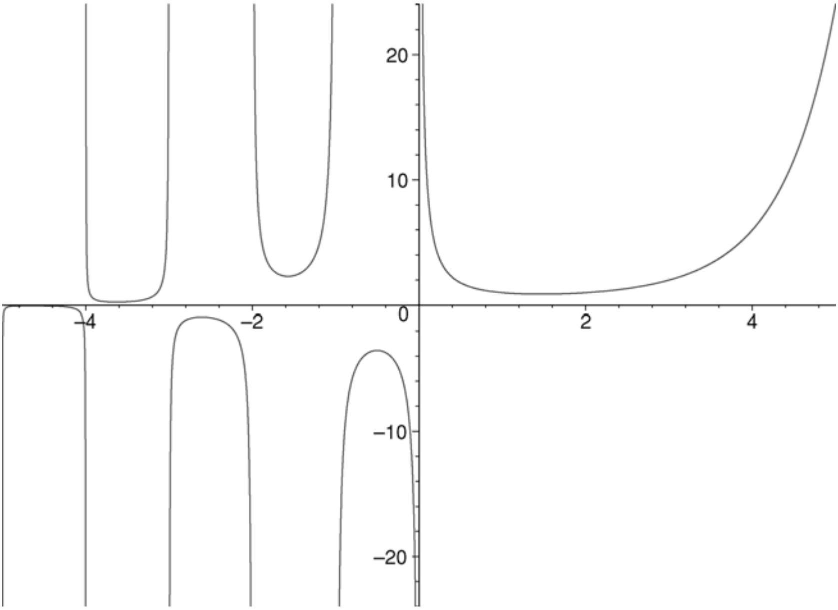

Figure C.1 Graph of $\Gamma(s)$ for $-5 < s \le 5$.

two functions. To this end we let $p_N(s)$ denote the expression on the right in (C.3), and note that

$$p_N(s)p_N(1-s) = \frac{N}{s(N+1-s)} \prod_{n=1}^{N} (1-(s/n)^2)^{-1}.$$

On the other hand, we recall that the Weierstrass product for the sine function may be written

$$\sin s = s \prod_{n=1}^{\infty} \Big(1 - \frac{s^2}{(\pi n)^2}\Big).$$

On comparing these formulæ we conclude that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{C.6}$$

We take $s = 1/2$ to see that $\Gamma(1/2)^2 = \pi$. But from (C.1) it is clear that $\Gamma(1/2) > 0$, so we have

$$\Gamma(1/2) = \sqrt{\pi}. \tag{C.7}$$

From (C.1) we see that $\Gamma(s)$ never takes the value 0, and that it has simple poles at the non-positive integers. Let k be a non-negative integer. Since

$\sin \pi s \sim (-1)^k \pi(s+k)$ as $s \to -k$, and since $\Gamma(k+1) = k!$, it follows from (C.6) that

$$\Gamma(s) \sim \frac{(-1)^k}{k!(s+k)} \tag{C.8}$$

as $s \to -k$.

Similarly we observe that $\Gamma(s)\Gamma(s+1/2)$ has a simple pole at $0, -1/2, -1, -3/2, -2, \ldots$, and that the same is true of $\Gamma(2s)$. We now establish a relation between these two functions by observing that

$$\frac{p_N(s)p_N(s+1/2)}{p_{2N}(2s)} = 2^{1-2s}\frac{N+1/2}{N+s+1/2}p_N(1/2).$$

On letting $N \to \infty$ and using (C.7) we obtain *Legendre's duplication formula*,

$$\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s). \tag{C.9}$$

On taking logarithmic derivatives in (C.1) we find that the *digamma function* $\frac{\Gamma'}{\Gamma}(s)$ can be written

$$\frac{\Gamma'}{\Gamma}(s) = -\frac{1}{s} - C_0 - \sum_{n=1}^{\infty}\Big(\frac{1}{s+n} - \frac{1}{n}\Big). \tag{C.10}$$

Setting $s = 1$, we see in particular that

$$\frac{\Gamma'}{\Gamma}(1) = -C_0. \tag{C.11}$$

Since $\Gamma(1) = 1$, this is equivalent to

$$\Gamma'(1) = -C_0. \tag{C.12}$$

We write $z = re(\theta)$ in the power series expansion $\log(1-z)^{-1} = \sum_{n=1}^{\infty} z^n/n$, let $r \to 1^-$, and apply Abel's theorem to see that

$$\sum_{n=1}^{\infty}\frac{e(n\theta)}{n} = -\log(1-e(\theta)) \tag{C.13}$$

provided that $\theta \notin \mathbb{Z}$. By applying this formula for various rational values of θ we can express the series in (C.10) in closed form, for any rational value of s. For example, by taking $\theta = 1/2$ we find that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2,$$

which with (C.10) gives

$$\frac{\Gamma'}{\Gamma}(1/2) = -C_0 - 2\log 2. \tag{C.14}$$

Also, since

$$\frac{-1-i}{4}e(n/4) - \frac{1}{2}e(n/2) + \frac{-1+i}{4}e(3n/4) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{otherwise,} \end{cases}$$

by taking $\theta = 1/4, 1/2, 3/4$ in (C.13) we deduce via (C.10) that

$$\frac{\Gamma'}{\Gamma}(1/4) = -C_0 - 3\log 2 - \pi/2. \tag{C.15}$$

Similarly,

$$\frac{\Gamma'}{\Gamma}(3/4) = -C_0 - 3\log 2 + \pi/2. \tag{C.16}$$

We now consider the asymptotic behaviour of the gamma function.

Theorem C.1 *Let $\delta > 0$ be given, and let $\mathcal{R} = \mathcal{R}(\delta)$ be the set of those complex numbers s for which $|s| \ge \delta$ and $|\arg s| < \pi - \delta$. Then*

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(1/|s|) \tag{C.17}$$

and

$$\Gamma(s) = \sqrt{2\pi}\, s^{s-1/2} e^{-s}(1 + O(1/|s|)) \tag{C.18}$$

uniformly for $s \in \mathcal{R}$.

The second estimate here is *Stirling's formula* for the gamma function, which generalizes his estimate (B.26) for $n!$. From this we see that

$$|\Gamma(s)| \asymp \tau^{\sigma-1/2} e^{-\pi\tau/2} \tag{C.19}$$

as $|t| \to \infty$ with σ uniformly bounded.

Proof From (C.2) and (C.10) we see that if $N > |s|$, then

$$\frac{\Gamma'}{\Gamma}(s) = \log N - \sum_{n=0}^{N} \frac{1}{n+s} + O(|s|/N).$$

By the Euler–MacLaurin summation formula (Theorem B.5) with $f(x) = 1/(x+s)$, $a = 0^-$, $b = N$, $K = 2$ we find that

$$\sum_{n=0}^{N} \frac{1}{n+s} = \log(N+s) - \log s + \frac{1}{2s} + \frac{1}{2(s+N)} + O(|s|^{-2}).$$

On combining these estimates and letting N tend to infinity we find that

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O(|s|^{-2}). \tag{C.20}$$

This estimate is more precise than (C.17), and still greater accuracy can be obtained by choosing a larger value of K.

To derive (C.18) we begin by taking logarithms in (C.3) and applying the Euler–MacLaurin summation formula, or we integrate (C.20) from s to $s+\infty$ along a ray parallel to the real axis. In either case we find that

$$\log\Gamma(s) = s\log s - s - \frac{1}{2}\log s + c + O(1/|s|),$$

and it remains to determine the value of the constant c. This may be done in a number of ways. For example, we could appeal to (C.5) and (B.26). Alternatively, we can take logarithms in (C.9) and apply the above to see that $c = (\log 2\pi)/2$. Then (C.18) follows by exponentiating. □

The gamma function can be expressed as a definite integral in various ways. We now establish two important integral representations for the gamma function.

Theorem C.2 (Euler's integral) *If* $\Re s > 0$, *then*

$$\int_0^\infty e^{-x}x^{s-1}\,dx = \Gamma(s). \tag{C.21}$$

Proof By integrating by parts repeatedly it is easy to verify that

$$\frac{N!}{s(s+1)\cdots(s+N)} = \int_0^1 (1-y)^N y^{s-1}\,dy.$$

We make the change of variable $x = Ny$ and recall Gauss's formula (C.3) to find that

$$\Gamma(s) = \lim_{N\to\infty}\int_0^\infty f_N(x)\,dx$$

where

$$f_N(x) = \begin{cases} (1-x/N)^N x^{s-1} & \text{for } 0\le x\le N,\\ 0 & \text{for } x > N.\end{cases}$$

To complete the proof we employ the dominated convergence theorem. Put $f(x) = e^{-x}x^{\sigma-1}$. Then $\int_0^\infty f(x)\,dx < \infty$ when $\sigma > 0$, and $|f_N(x)| \le f(x)$ uniformly in N and x. Since

$$\lim_{N\to\infty} f_N(x) = e^{-x}x^{s-1}$$

for each fixed x, the formula (C.21) now follows. □

Let $\mathcal{C}(\rho)$ denote the circular arc $\{z = \rho e(\theta) : 0 \le \theta \le 1/4\}$. It is easy to verify that

$$\int_{\mathcal{C}(\rho)} |e^{-z} z^{s-1}|\,|dz| \to 0$$

as $\rho \to \infty$. Thus by Cauchy's theorem the formula (C.21) still holds if x is replaced by a complex variable z that goes to infinity along a ray from the origin, $z = \rho e(\theta)$, $0 \le \rho < \infty$, provided that $-1/4 \le \theta \le 1/4$.

For $r > 0$ we let $\mathcal{H} = \mathcal{H}(r)$ denote the Hankel contour, which consists of a path that passes from $-ir - \infty$ to $-ir$ along the ray $x - ir$, $-\infty < x \le 0$, and then from $-ir$ to ir along the semicircle $re(\theta)$, $-1/4 \le \theta \le 1/4$, and then from ir to $ir - \infty$ along the ray $x + ir$, $-\infty < x \le 0$.

Theorem C.3 (Hankel) *For any complex number s,*

$$\frac{1}{2\pi i} \int_{\mathcal{H}} e^z z^{-s}\, dz = \frac{1}{\Gamma(s)}. \tag{C.22}$$

Here z^{-s} is assumed to have its principal value.

As in the preceding theorem, the contour of integration may be altered substantially without changing the value of the integral. For example, the ray from ir to $-\infty + ir$ may be replaced by a ray in the direction $e(\theta)$, provided that $1/4 < \theta < 1/2$.

Proof It is clear that the left-hand side is an entire function of s. Thus it suffices to prove the identity when $\sigma < 1$. For such s we let $r \to 0^+$, and note that the integral along the semicircle tends to 0. The remaining integrals tend to

$$e^{i\pi s} \int_0^\infty e^{-x} x^{-s}\, dx - e^{-i\pi s} \int_0^\infty e^{-x} x^{-s}\, dx = 2i(\sin \pi s)\Gamma(1-s)$$

by (C.21). To complete the proof it suffices to appeal to (C.6). □

Euler's formula asserts that the gamma function is the Mellin transform of the function e^{-x}. We now establish the inverse.

Theorem C.4 (Mellin) *If $\Re z > 0$ and $c > 0$, then*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s}\, ds = e^{-z}.$$

Proof From Stirling's formula we see that

$$\int_{-K+iK}^{c+iK} |\Gamma(s) z^{-s}|\,|ds| \longrightarrow 0$$

as $K \to \infty$, and similarly for the integral from $-K - iK$ to $c - iK$. Moreover,

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if we first apply (C.6) and then Stirling's formula, we find that

$$\int_{-K-iK}^{-K+iK} |\Gamma(s)z^{-s}|\,|ds| \longrightarrow 0$$

as $K \to \infty$ through values of the form $K = n + 1/2, n \in \mathbb{Z}$. (We are assuming here that the path of integration is a line segment joining the two endpoints.) Thus by the calculus of residues

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)z^{-s}\,ds = \sum_{k=0}^{\infty} Res\Big(\Gamma(s)z^{-s}\Big|_{s=-k}.$$

From (C.8) we see that the above is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^k = e^{-z}.$$

□

The digamma function can be examined in a similar way. In view of (C.17), this function is not absolutely integrable on the line $\sigma = c$, and thus we cannot define its Fourier transform in the classical manner. We now formulate a useful substitute.

Theorem C.5 *Let $a > 0$ and $b > 0$ be fixed. If $x < 0$ and $T \geq 1$, then*

$$\int_{-T}^{T} \frac{\Gamma'}{\Gamma}(a+ibt)e(-xt)\,dt = -\frac{\Gamma'}{\Gamma}(a+ibT)\frac{e(-xT)}{2\pi ix} + \frac{\Gamma'}{\Gamma}(a-ibT)\frac{e(xT)}{2\pi ix} - 2\pi b^{-1}e^{2\pi ax/b}(1-e^{2\pi x/b})^{-1} + O(x^{-2}T^{-1}),$$

while if $x > 0$ and $T \geq 1$, then

$$\int_{-T}^{T} \frac{\Gamma'}{\Gamma}(a+ibt)e(-xt)\,dt = -\frac{\Gamma'}{\Gamma}(a+ibT)\frac{e(-xT)}{2\pi ix} + \frac{\Gamma'}{\Gamma}(a-ibT)\frac{e(xT)}{2\pi ix} + O(x^{-2}T^{-1}).$$

Proof We write the integral as

$$\frac{1}{i}\int_{-iT}^{iT} \frac{\Gamma'}{\Gamma}(a+bs)e^{-2\pi xs}\,ds.$$

Suppose that $x < 0$. Let $\mathcal{C}$ be the contour passing by line segment from $-\infty - iT$ to $-iT$ to iT to $-\infty + iT$. By the calculus of residues and (C.10) we find that

$$\int_{\mathcal{C}} \frac{\Gamma'}{\Gamma}(a+bs)e^{-2\pi xs}\,ds = -\frac{2\pi i}{b}\sum_{n=0}^{\infty} e^{2\pi x(n+a)/b} = -\frac{2\pi i}{b}e^{2\pi ax/b}\big(1-e^{2\pi x/b}\big)^{-1}.$$

We parametrize the integral $\int_{-\infty-iT}^{-iT}$, and integrate by parts, to see that it is

$$\int_{-\infty}^{0} \frac{\Gamma'}{\Gamma}(a+b\sigma-ibT)e(xT)e^{-2\pi x\sigma}\,d\sigma$$
$$= -\frac{\Gamma'}{\Gamma}(a-ibT)\frac{e(xT)}{2\pi x}+\frac{be(xT)}{2\pi x}\int_{-\infty}^{0}\left(\frac{\Gamma'}{\Gamma}\right)'(a+b\sigma-ibT)e^{-2\pi x\sigma}\,d\sigma.$$

But

$$\left(\frac{\Gamma'}{\Gamma}\right)'(s)=\sum_{n=0}^{\infty}(n+s)^{-2}\ll 1/|t|$$

for $|t|\geq 1$, and hence the last integral above is $\ll x^{-2}T^{-1}$. Similarly,

$$\int_{iT}^{-\infty+iT}\frac{\Gamma'}{\Gamma}(a+bs)e^{-2\pi xs}\,ds=\frac{\Gamma'}{\Gamma}(a+ibT)\frac{e(-xT)}{2\pi x}+O(x^{-2}T^{-1}).$$

We obtain the stated result on combining these estimates. The case $x>0$ is treated similarly, but with a contour from $+\infty-iT$ to $-iT$ to iT to $+\infty+iT$. □

Exercises

1. Show:
   (a) $$|\Gamma(it)|^2=\frac{\pi}{t\sinh \pi t};$$
   (b) $$|\Gamma(1/2+it)|^2=\frac{\pi}{\cosh \pi t};$$
   (c) $$\Im\frac{\Gamma'}{\Gamma}(s)>0 \text{ if } t>0;$$
   (d) $$\frac{\partial}{\partial t}\log|\Gamma(s)|<0 \text{ when } t>0;$$
   (e) For any given σ, $|\Gamma(s)|$ is a strictly decreasing function of t on the interval $0<t<\infty$.
2. (Gauss 1812) Prove *Gauss's multiplication formula*:
$$\prod_{a=0}^{q-1}\Gamma(s+a/q)=(2\pi)^{(q-1)/2}q^{1/2-qs}\Gamma(qs).$$
3. Show:
   (a) $$\frac{\Gamma'}{\Gamma}(1-s)-\frac{\Gamma'}{\Gamma}(s)=\pi\cot\pi s;$$
   (b) $$\frac{\Gamma'}{\Gamma}(s+1)=\frac{1}{s}+\frac{\Gamma'}{\Gamma}(s);$$
   (c) If n is an integer, $n>1$, then
$$\frac{\Gamma'}{\Gamma}(n)=-C_0+\sum_{k=1}^{n-1}\frac{1}{k}.$$

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4. (Gauss 1812) Using additive characters (as discussed in Chapter 4), or otherwise, show that if $0 < a \le q$, then

$$\frac{\Gamma'}{\Gamma}(a/q) = -C_0 - \log q + \sum_{h=1}^{q-1} e(-ah/q)\log(1 - e(h/q)).$$

5. Show that $\frac{\Gamma'}{\Gamma}(1/3) = -C_0 - \frac{3}{2}\log 3 - \pi\sqrt{3}/6$.
6. Show that

$$\frac{\Gamma'}{\Gamma}(s) = -C_0 + \sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)(s-1)^n$$

for $|s-1| < 1$.
7. Show:
(a)
$$\left(\frac{\Gamma'}{\Gamma}\right)'(s) = \sum_{n=0}^{\infty}(s+n)^{-2};$$
(b)
$$\frac{\Gamma''(s)}{\Gamma(s)} = \frac{\Gamma'}{\Gamma}(s)^2 + \sum_{n=0}^{\infty}(s+n)^{-2};$$
(c) The functions $\Gamma(\sigma)$, $\Gamma''(\sigma)$ have the same sign for all real σ.
8. Show that if $x > 0$ and $y \ge 1$, then

$$\frac{\Gamma(x+y)}{\Gamma(x)} \ge x^y.$$

9. (Hermite 1881) Let x_n denote the unique critical point of $\Gamma(\sigma)$ in the interval $(-n, -n+1)$. Show that $x_n = -n + (\log n)^{-1} + O((\log n)^{-2})$ for $n \ge 2$.
10. Show that $\left(\frac{\Gamma'}{\Gamma}\right)'(s) = s^{-1} + \frac{1}{2}s^{-2} + O(|s|^{-3})$ uniformly in the region $\mathcal{R}$ of Theorem C.1.
11. (a) Show that $\int_1^\infty e^{-x}x^{s-1}\,dx$ is an entire function.

(b) Show that if $\sigma > 0$, then

$$\int_0^1 e^{-x}x^{s-1}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(s+n)}.$$

(c) Show that if s is not a non-positive integer, then

$$\Gamma(s) = \int_1^\infty e^{-x}x^{s-1}\,dx + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(s+n)}.$$

12. (a) Show that if $\sigma > 0$, then

$$\Gamma^{(k)}(s) = \int_0^\infty e^{-x}x^{s-1}(\log x)^k\,dx.$$

(b) Show that

$$\int_0^\infty e^{-x} \log x \, dx = -C_0.$$

13. (Cauchy 1827; Saalschütz 1887, 1888) Show that if $-1 < \sigma < 0$, then

$$\Gamma(s) = \int_0^\infty (e^{-x} - 1)x^{s-1} \, dx.$$

14. Let s be fixed with $\sigma > 0$, and let $f_N(x)$ be the function defined in the proof of Theorem C.2. Show that

$$\int_0^\infty f_N(x) \, dx = \Gamma(s) - \Gamma(s+2)/(2N) + O(N^{-2}).$$

15. (Mellin 1883a, b) Let $P(z)$ and $Q(z)$ be relatively prime polynomials over $\mathbb{C}$, with roots $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$, respectively, and suppose that none of these roots is a positive integer.

 (a) Suppose that $\prod_{k=1}^\infty \frac{P(k)}{Q(k)}$ converges. Show:

  (i) $m = n$;

  (ii) P and Q have the same leading coefficient;

  (iii) $\sum \alpha_i = \sum \beta_i$.

 (b) Show conversely that if conditions (i)–(iii) hold, then the product converges, and has the value

$$\prod_{i=1}^m \frac{\Gamma(1-\beta_i)}{\Gamma(1-\alpha_i)}.$$

 (c) Show that if a and b are complex numbers such that none of $a, b, a+b$ is a negative integer, then

$$\prod_{n=1}^\infty \frac{n(n+a+b)}{(n+a)(n+b)} = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}.$$

16. (Liouville 1852) Show that if q is an integer, $q > 1$, then

$$\prod_{n=1}^\infty (1 - (z/n)^q)^{-1} = -z^q \prod_{a=1}^q \Gamma(-ze(a/q)).$$

17. (Mellin 1891, p. 324)

 (a) Show that

$$\frac{\Gamma(\sigma)^2}{|\Gamma(s)|^2} = \prod_{n=0}^\infty \Big(1 + \frac{t^2}{(n+\sigma)^2}\Big).$$

 (b) Give a second derivation of the assertion of Exercise 1(e).

18. (Gram 1899) Show that

$$\prod_{n=2}^\infty \frac{(n^3-1)}{(n^3+1)} = \frac{2}{3}.$$

19. Show that if $\sigma > 0$, then

$$\Gamma(s) = \int_0^1 (\log 1/x)^{s-1}\, dx,$$

and

$$\Gamma(s) = \int_{-\infty}^{\infty} e^{-e^x} e^{sx}\, dx.$$

20. (Euler 1794)
(a) Show that if $-1 < \sigma < 1$, then

$$\int_0^\infty (\sin x) x^{s-1}\, dx = \Gamma(s) \sin \frac{1}{2}\pi s.$$

(b) Show that if $0 < \sigma < 1$, then

$$\int_0^\infty (\cos x) x^{s-1}\, dx = \Gamma(s) \cos \frac{1}{2}\pi s.$$

21. For $\Re a > 0$, $\Re b > 0$ let the *beta function* $B(a, b)$ be defined to be

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx.$$

(a) Write

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty e^{-u-v} u^{a-1} v^{b-1}\, du\, dv$$

and make the change of variables $u = rx$, $v = r(1-x)$ to show that

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

(b) Show that if $\Re a > 0$ and $\Re b > 0$, then

$$\int_0^\infty x^{2a-1}(1-x^2)^{b-1}\, dx = \frac{1}{2} B(a, b).$$

(c) Show that if $\Re a > 0$ and $\Re b > 0$, then

$$\int_0^{\pi/2} (\sin\theta)^{2a-1}(\cos\theta)^{2b-1}\, d\theta = \frac{1}{2} B(a, b).$$

(d) By writing $t = \tan^2\theta$, or otherwise, show that if $\Re a > 0$ and $\Re b > 0$, then

$$\int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\, dt = B(a, b).$$

22. (Dirichlet 1839; Liouville 1839) Let $f(x)$ be a continuous function defined on $[0, 1]$. Let $\mathcal{R}$ denote that portion of $\mathbb{R}^n$ for which $x_i \geq 0$ and $\sum x_i \leq 1$.

Show that

$$\int_{\mathcal{R}} f(x_1 + \cdots + x_n) x_1^{a_1 - 1} \cdots x_n^{a_n - 1}\, dx_1 \cdots dx_n = \frac{\Gamma(a_1) \cdots \Gamma(a_n)}{\Gamma(a_1 + \cdots + a_n)} \int_0^1 f(x) x^{a-1}\, dx$$

where $a = \sum a_i$ and $\Re a_i > 0$ for all i.

23. (Mellin 1902) Suppose that z lies in the slit plane formed by deleting the negative real axis. Show that if $0 < c < \Re a$, then

$$\frac{\Gamma(a)}{(1+z)^a} = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \Gamma(s)\Gamma(a - s) z^{-s}\, ds.$$

(This is the inverse of the Mellin transform in Exercise 21(d).)

24. (Raabe 1844) Show that if s is not a negative real number or 0, then

$$\int_s^{s+1} \log \Gamma(z)\, dz = s \log s - s + \frac{1}{2} \log 2\pi.$$

25. (Barnes 1900) Let

$$G(s+1) = (2\pi)^{s/2} \exp\left(-\frac{1}{2}(C_0 + 1)s^2 - \frac{1}{2}s\right) \prod_{n=1}^{\infty} \left(\left(1 + \frac{s}{n}\right)^n e^{-s - s^2/(2n)}\right).$$

Show:

(a) $G(s)$ is an entire function.

(b) $G(1) = 1$.

(c) $G(s+1) = \Gamma(s)G(s)$.

(d)

$$G(n+1) = \frac{(n!)^n}{1^1 2^2 3^3 \cdots n^n}.$$

26. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3 + 1} = \frac{1}{3} \ln 2 - \frac{1}{3} - \frac{\pi}{3 \cosh(\pi \sqrt{3}/2)}.$$

C.1 Notes

Euler, in a letter of 1729 to Goldbach (cf. Fuss 1843, p. 3) gave the formula

$$\Gamma(s) = \frac{1}{s} \prod_{n=1}^{\infty} \left(\left(1 + \frac{1}{n}\right)^s \left(1 + \frac{s}{n}\right)^{-1}\right).$$

This is substantially the same as the formula (C.3) that Gauss (1812) took to be fundamental. Based on the above definition of the gamma function, the formula

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(C.1) was proved by Schlömilch (1844) and Newman (1848). Weierstrass (1856) took (C.1) to be the definition of the gamma function. Euler had given the special value (C.7) already in his letter to Goldbach. Euler (1771) also discovered the reflection formula (C.6). The duplication formula (C.9) of Legendre (1809) is a special case of the multiplication formula of Gauss (1812), given in Exercise C.3. Stirling (1730, p. 135) gave the series expansion

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + \sum_{n=2}^{\infty} \frac{B_n}{n(n-1)s^{n-1}}.$$

This series diverges, but a partial sum provides an asymptotic expansion.The approximation (C.17) is a weak form of this. To calculate $\Gamma(s)$ numerically, it suffices to consider $\sigma \geq 1/2$, in view of (C.6). If $|s|$ is small then (C.4) should be used repeatedly. Thus it remains to evaluate $\Gamma(s)$ when $\sigma \geq 1/2$ and $|s|$ is large, and this is quickly achieved by using the expansion above. By these means it may be found that the sole minimum of $\Gamma(\sigma)$ for $\sigma > 0$ is at $\sigma_0 = 1.4616321\ldots$, and that $\Gamma(\sigma_0) = 0.88560319\ldots$. The convenient estimate (C.19) was noted by Pincherle (1888). Theorems C.1 and C.2 may be established in several ways. An instructive collection of such proofs is found in Sections 8.4, 8.5, 11.1, 11.11, and 12.12 of Henrici (1977). Euler (1730) gave the formula of Theorem C.2, expressed in the form $n! = \int_0^1 (\log 1/y)^n \, dy$, and subsequently found many other integral formulæ involving the gamma function. Thus Euler was led in quite a different direction than Gauss (1812), whose independent investigations were more directly related to Gauss's formula (C.3). Legendre (1809) called the formula (C.21) the 'Euler integral of the second kind', and introduced the notation $\Gamma(z)$. The 'Euler integral of the first kind' is known today as the beta function (see Exercise C.21). Theorem C.3 is due to Hankel (1864), and Theorem C.4 to Mellin (1896, p. 76, 1899, p. 39).

Simple proofs of Stirling's formula for $n!$, using a minimum of tools, have been given by Robbins (1955) and Feller (1965).

For more extensive expositions of the subject the reader is referred to Artin (1964), Henrici (1977), Jensen (1916), Nielsen (1906), and to Whittaker & Watson (1950, Chapter 12). The related Mellin–Barnes integrals are discussed in Section 8.8 of Henrici (1977).

Gauss and Binet established several useful formulæ for $\log \Gamma(s)$ and for $\frac{\Gamma'}{\Gamma}(s)$. Kummer (1847) proved that if $0 < \sigma < 1$, then

$$\log \Gamma(\sigma) = (C_0 + \log 2)\left(\frac{1}{2} - \sigma\right) + (1 - \sigma) \log \pi - \frac{1}{2} \log \sin \pi\sigma + \sum_{n=1}^{\infty} \frac{\log n}{\pi n} \sin 2\pi n\sigma.$$

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In conjunction with the analysis of Chapter 9, this gives

$$\sum_{a=1}^{q} \chi(a) \log \Gamma(a/q) = -(C_0 + \log 2\pi) \sum_{a=1}^{q} a\chi(a) - \frac{\sqrt{q}}{\pi} L'(1, \chi)$$

where χ is a primitive character (mod q) for which $\chi(-1) = -1$.

Artin (1931, 1964; p. 14) showed that if $f(x)$ is positive and $\log f(x)$ is convex for $x > 0$, if $xf(x) = f(x+1)$ for all $x > 0$, and $f(1) = 1$, then $f(x) = \Gamma(x)$.

Hölder (1886) showed that $\Gamma(s)$ does not satisfy an algebraic differential equation. Additional proofs of this have been given by Moore (1897), Jensen (1916, pp. 103–112) and Ostrowski (1919).

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