Appendix A

The Riemann–Stieltjes integral

We generalize the Riemann integral $\int_a^b f(x)\,dx$ by defining an integral $\int_a^b f(x)\,dg(x)$ as a limit of Riemann sums $\sum_n f(\xi_n)\Delta g(x_n)$. More precisely, for $a < b$ suppose that we have a partition

$$a = x_0 \le x_1 \le \cdots \le x_N = b. \tag{A.1}$$

For ξ_n in the interval $x_{n-1} \le \xi_n \le x_n$ we form the sum

$$S(x_n, \xi_n) = \sum_{n=1}^{N} f(\xi_n)(g(x_n) - g(x_{n-1})).$$

We say that the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists and has the value I if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|S(x_n, \xi_n) - I| < \varepsilon$$

whenever the x_n and the ξ_n are as above and

$$\text{mesh}\{x_n\} = \max_{1\le n\le N}(x_n - x_{n-1}) \le \delta.$$

The values taken on by f and g may be either real or complex. We do not determine precisely the pairs (f, g) for which the Riemann–Stieltjes integral exists. For our purposes it is enough to prove

Theorem A.1 *The Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists if f is continuous on $[a, b]$ and g is of bounded variation on $[a, b]$.*

Proof We recall that by definition

$$\text{Var}_{[a,b]}(g) = \sup \sum_{n=1}^{N} |g(x_n) - g(x_{n-1})|$$

where the supremum is taken over all $\{x_n\}$ satisfying (A.1). Since f is uniformly continuous on $[a, b]$, there is a $\delta > 0$ such that $|f(\xi) - f(\xi')| < \varepsilon$ whenever $|\xi - \xi'| \le \delta$. We show that

$$|S(x_n, \xi_n) - S(x'_n, \xi'_n)| \le 2\varepsilon \operatorname{Var}_{[a,b]}(g) \tag{A.2}$$

provided that $\operatorname{mesh}\{x_n\} \le \delta$ and that $\operatorname{mesh}\{x'_n\} \le \delta$. This clearly suffices.

Suppose first that the partition $\{x_n\}$ is a subsequence of a second partitioning $\{x''_n\}$. Let $\mathcal{M}(n) = \{m : x_{n-1} < x''_m \le x_n\}$. The sets $\mathcal{M}(n)$ partition the set $\{1, 2, \ldots, M\}$, so we may write

$$\begin{aligned} &S(x_n, \xi_n) - S(x''_m, \xi''_m) \\ &\quad = \sum_{n=1}^{N} \left(f(\xi_n)(g(x_n) - g(x_{n-1})) - \sum_{m \in \mathcal{M}(n)} f(\xi''_m)(g(x''_m) - g(x''_{m-1})) \right). \end{aligned}$$

Since the sequence $\{x_n\}$ is an increasing subsequence of the increasing sequence $\{x''_m\}$, it follows that

$$g(x_n) - g(x_{n-1}) = \sum_{m \in \mathcal{M}(n)} g(x''_m) - g(x''_{m-1}).$$

On inserting this in the former expression, we find that it is

$$\sum_{n=1}^{N} \sum_{m \in \mathcal{M}(n)} (f(\xi_n) - f(\xi''_m))(g(x''_m) - g(x''_{m-1})).$$

Since $|\xi_n - \xi''_m| \le \delta$, it follows that

$$\begin{aligned} |S(x_n, \xi_n) - S(x''_m, \xi''_m)| &\le \varepsilon \sum_{n} \sum_{m \in \mathcal{M}(n)} |g(x''_m) - g(x''_{m-1})| \\ &= \varepsilon \sum_{m=1}^{M} |g(x''_m) - g(x''_{m-1})| \\ &\le \varepsilon \operatorname{Var}_{[a,b]} g. \end{aligned} \tag{A.3}$$

We now take $\{x''_m\}$ to be the union of $\{x_n\}$ and $\{x'_n\}$, so that both $\{x_n\}$ and $\{x'_n\}$ are subsequences of $\{x''_m\}$. Since

$$\begin{aligned} |S(x_n, \xi_n) - S(x'_n, \xi'_n)| &= |S(x_n, \xi_n) - S(x''_m, \xi''_m) + S(x''_m, \xi''_m) - S(x'_n, \xi'_n)| \\ &\le |S(x_n, \xi_n) - S(x''_m, \xi''_m)| + |S(x''_m, \xi''_m) - S(x'_n, \xi'_n)| \end{aligned}$$

by the triangle inequality, the desired bound (A.2) follows by applying (A.3) twice. □

The main negative feature of the Riemann–Stieltjes integral is that $\int_a^b f\,dg$ does not exist if f and g have a common discontinuity in (a, b). However, if f is continuous, the Riemann–Stieltjes integral enables us to express the sum $\sum_{n=1}^{N} a_n f(n)$ in terms of the unweighted partial sums $A(x) = \sum_{1\le n\le x} a_n$. Indeed,

$$\sum_{n=1}^{N} a_n f(n) = \int_0^N f(x)\,dA(x). \tag{A.4}$$

There is some freedom in the interval of integration, since the left endpoint can be any number in [0, 1), and the right endpoint can be any number in $[N,\ N+1)$ without affecting the value of the integral. Frequently it is useful to integrate from 1^- to N, i.e. to consider $\lim_{\varepsilon\to 0^+} \int_{1-\varepsilon}^{N}$. Some care must be exercised in choosing the endpoints of integration, since for example

$$\int_1^N f(x)\,dA(x) = \sum_{n=2}^{N} a_n f(n).$$

Theorem A.2 *If $\int_a^b f\,dg$ exists, then $\int_a^b g\,df$ also exists, and*

$$\int_a^b g\,df = f(b)g(b) - f(a)g(a) - \int_a^b f\,dg.$$

As we see in the above, we lose no information by writing $\int_a^b f\,dg$ instead of the longer $\int_a^b f(x)\,dg(x)$. On combining Theorems A.1 and A.2 we see that $\int_a^b f\,dg$ exists if f is of bounded variation on $[a, b]$ and g is continuous on $[a, b]$.

Proof Put $\xi_0 = a$ and $\xi_{N+1} = b$. Then

$$\sum_{n=1}^{N} g(\xi_n)(f(x_n) - f(x_{n-1}))$$
$$= f(b)g(b) - f(a)g(a) - \sum_{n=1}^{N+1} f(x_{n-1})(g(\xi_n) - g(\xi_{n-1})).$$

Here the sum on the right-hand side is a Riemann–Stieltjes sum $S(\xi_n, x_{n-1})$ approximating to $\int_a^b f\,dg$, since $x_{n-1} \in [\xi_{n-1},\ \xi_n]$. Moreover, $\operatorname{mesh}\{\xi_n\} \le 2\operatorname{mesh}\{x_n\}$, so that the sum on the right tends to $\int_a^b f\,dg$ as $\operatorname{mesh}\{x_n\}$ tends to 0. □

This proof displays the close relation between partial summation and integration by parts. Rather than sum the series $\sum a_n f(n)$ by parts, we can integrate by parts in (A.4) to see that

$$\sum_{n=1}^{N} a_n f(n) = A(N)f(N) - \int_0^N A(x)\,df(x). \tag{A.5}$$

It is to be expected that if g is differentiable, then $\int_a^b f\,dg$ should resemble $\int_a^b fg'\,dx$. In this direction we establish

Theorem A.3 *If g' is continuous on $[a, b]$, then*

$$Var_{[a,b]}g = \int_a^b |g'(x)|\,dx.$$

If in addition f is Riemann integrable, then

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)g'(x)\,dx.$$

Proof By the mean value theorem there is a $\zeta_n \in [x_{n-1},\ x_n]$ such that

$$g(x_n) - g(x_{n-1}) = g'(\zeta_n)(x_n - x_{n-1}).$$

Hence

$$\sum_{n=1}^{N} |g(x_n) - g(x_{n-1})| = \sum_{n=1}^{N} |g'(\zeta_n)|(x_n - x_{n-1}),$$

which tends to $\int_a^b |g'|\,dx$ as mesh$\{x_n\}$ tends to 0. Since $g'(x)$ is uniformly continuous on $[a,\ b]$, there is a $\delta > 0$ such that $|g'(\xi) - g'(\zeta)| < \varepsilon$ whenever $|\xi - \zeta| < \delta$. Clearly

$$\begin{aligned}\sum_{n=1}^{N} f(\xi_n)(g(x_n) - g(x_{n-1})) &= \sum_{n=1}^{N} f(\xi_n)g'(\zeta_n)(x_n - x_{n-1})\\ &= \sum_{n=1}^{N} f(\xi_n)g'(\xi_n)(x_n - x_{n-1})\\ &\quad + \sum_{n=1}^{N} f(\xi_n)(g'(\zeta_n) - g'(\xi_n))(x_n - x_{n-1})\\ &= \Sigma_1 + \Sigma_2,\end{aligned}$$

say. The function fg' is Riemann integrable, and hence Σ_1 tends to $\int_a^b fg'\,dx$ as mesh$\{x_n\}$ tends to 0. Suppose that M is chosen so that $|f(x)| \le M$ for all $x \in [a,\ b]$. If mesh$\{x_n\} < \delta$, then $|\Sigma_2| \le M\varepsilon(b - a)$. Hence $\int_a^b f\,dg$ exists and has the value $\int_a^b fg'\,dx$. □

Continuing from (A.4), we see that if f' is continuous, then

$$\sum_{n=1}^{N} a_n f(n) = A(N)f(N) - \int_0^N A(x)f'(x)\,dx. \qquad \text{(A.6)}$$

This useful identity can be verified without mention of Riemann–Stieltjes integration, but its formulation and derivation is most natural through (A.4) and (A.5).

Suppose that f is Riemann integrable. A version of the triangle inequality asserts that $|\int_a^b f| \le \int_a^b |f|$. We now derive an analogue of this for the Riemann–Stieltjes integral.

Theorem A.4 *Suppose that g has bounded variation, and put $g^*(x) = \mathrm{Var}_{[a,x]} g$. Then*

$$\left| \int_a^b f(x)\, dg(x) \right| \le \int_a^b |f(x)|\, dg^*(x).$$

provided that both integrals exist.

Proof Clearly

$$\begin{aligned} |S(x_n, \xi_n)| &\le \sum_{n=1}^{N} |f(\xi_n)||g(x_n) - g(x_{n-1})| \\ &\le \sum_{n=1}^{N} |f(\xi_n)|(g^*(x_n) - g^*(x_{n-1})), \end{aligned}$$

which gives the result. □

The differential dg^* is sometimes abbreviated $|dg|$. From Theorem A.4 we see that if $|f(x)| \le M$ for $a \le x \le b$ and g is of bounded variation, then

$$\left| \int_a^b f(x)\, dg(x) \right| \le M \mathrm{Var}_{[a,b]} g \tag{A.7}$$

provided that the integral exists. As with Riemann integrals, we set $\int_a^a f\, dg = 0$. If $a > b$ we set $\int_a^b f\, dg = -\int_b^a f\, dg$, so that $\int_a^c + \int_c^b = \int_a^b$ for any real numbers a, b, c. Finally, improper Riemann–Stieltjes integrals are defined as limits of proper integrals, e.g.

$$\int_a^\infty f(x)\, dg(x) = \lim_{b\to\infty} \int_a^b f(x)\, dg(x).$$

Exercises

1. Suppose that $\varphi(t)$ is continuous and strictly increasing for $\alpha \le t \le \beta$, and that $\varphi(\alpha) = a$, $\varphi(\beta) = b$. Put $F(t) = f(\varphi(t))$, $G(t) = g(\varphi(t))$. Show that

   $$\int_a^b f(x)\, dg(x) = \int_\alpha^\beta F(t)\, dG(t)$$

   provided that either integral exists.

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2. Let f and g be continuous, and h have bounded variation. Put $I(x) = \int_a^x g\,dh$. Show that
$$\int_a^b f(x)g(x)\,dh(x) = \int_a^b f(x)\,dI(x).$$
3. The proof of Theorem A.2 depends on summation by parts. We now show that, conversely, summation by parts can be recovered from Theorem A.2. Suppose that the numbers $a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ are given. Put $A_n = a_1 + \cdots + a_n$ for $1 \le n \le N$. For $1 \le x < N+1$ put $A(x) = A_{[x]}$; set $A(x) = 0$ for $x < 1$. For $1/2 \le x \le N + 1/2$ let $B(x) = b_{[x+1/2]}$. (The discontinuities of $B(x)$ are displaced in order to ensure that $A(x)$ and $B(x)$ do not have a common discontinuity.)
   (a) Show that
$$\sum_{n=1}^N a_n b_n = \int_{1^-}^N B(x)\,dA(x).$$
   (b) Show that
$$\sum_{n=1}^{N-1} A_n(b_n - b_{n+1}) = -\int_{1^-}^N A(x)\,dB(x).$$
   (c) Use Theorem 2 to derive Abel's lemma:
$$\sum_{n=1}^N a_n b_n = A_N b_N + \sum_{n=1}^{N-1} A_n(b_n - b_{n+1}).$$
4. Show that
$$\left|\int_a^b fg\,dh\right|^2 \le \left(\int_a^b |f|^2\,|dh|\right)\left(\int_a^b |g|^2\,|dh|\right)$$
provided that these integrals exist.
5. Suppose that f is non-negative and decreasing, that $g(a) = h(a)$, and that $g(x) \le h(x)$ for $a \le x \le b$. Show that
$$\int_a^b f\,dg \le \int_a^b f\,dh$$
provided that these integrals exist.
6. (First mean value theorem) Suppose that f and g are real-valued functions with f continuous on $[a, b]$, and g weakly increasing on this interval. Put $m = \min_{x\in[a,b]} f(x)$, $M = \max_{x\in[a,b]} f(x)$.
   (a) Show that
$$m(g(b) - g(a)) \le \int_a^b f\,dg \le M(g(b) - g(a))\,.$$

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(b) Show that there is an $x_0 \in [a, b]$ such that

$$\int_a^b f\,dg = f(x_0)(g(b) - g(a)).$$

7. (Second mean value theorem) Suppose that f and g are real-valued functions with f weakly increasing on $[a, b]$, and g continuous on this interval. Show that there is an $x_0 \in [a, b]$ such that

$$\int_a^b f\,dg = f(a)(g(x_0) - g(a)) + f(b)(g(b) - g(x_0)).$$

8. (Darst & Pollard 1970) Suppose that f and g are real-valued functions with f of bounded variation on $[a, b]$, and g continuous on this interval. (a) Show that if $\xi \in [a, b]$ and $f(\xi) = 0$, then

$$\int_\xi^b f\,dg \le \mathrm{Var}_{[\xi,b]}(f) \max_{\xi \le x \le b}(g(b) - g(x)),$$

$$\int_a^\xi f\,dg \le \mathrm{Var}_{[a,\xi]}(f) \max_{\xi \le x \le b}(g(x) - g(a)).$$

(b) Show that if $\inf_{a \le x \le b} f(x) = 0$, then

$$\int_a^b f\,dg \le \mathrm{Var}_{[a,b]}(f) \max_{a \le \alpha \le \beta \le b}(g(\beta) - g(\alpha)).$$

(c) Show that in general,

$$\int_a^b f\,dg \le (g(b) - g(a)) \inf_{a \le x \le b} f(x) + \mathrm{Var}_{[a,b]}(f) \max_{a \le \alpha \le \beta \le b}(g(\beta) - g(\alpha)).$$

9. Suppose that

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \le 1, \\ 0 & \text{otherwise;} \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int_{-1}^0 f\,dg$ and $\int_0^1 f\,dg$ both exist, but that $\int_{-1}^1 f\,dg$ does not exist.

A.1 Notes

Our treatment follows that of Ingham in his lectures at Cambridge University. Several variants of the Riemann–Stieltjes (R-S) integral have been proposed. The integral as we have defined it is known as the *uniform* Riemann–Stieltjes integral. A slightly more powerful variant is the *refinement* Riemann–Stieltjes integral, in which $\int_a^b f\,dg$ is said to have the value I if for every $\varepsilon > 0$ there is a partition $\{x_n\}$ such that if $\{x'_m\}$ is a refinement of $\{x_n\}$, then $|S(x'_m, \xi'_m) - I| < \varepsilon$

for all choices of $\xi'_m \in [x'_{m-1}, x'_m]$. The refinement Riemann–Stieltjes integral is developed in considerable detail by Apostol (1974, Chapter 9) and Bartle (1964, Section 22), and is used by Bateman & Diamond (2004). If $\int_a^b f\,dg$ exists in the sense of uniform R–S integration, then it also exists in the refinement R–S sense, and has the same value. The refinement integral has the attractive property that if $a < b < c$, and if $\int_a^b f\,dg$, $\int_b^c f\,dg$ both exist, then $\int_a^c f\,dg$ exists and

$$\int_a^c f\,dg = \int_a^b f\,dg + \int_b^c f\,dg\,.$$

This is not true for the uniform R–S integral, as we see by the example in Exercise A.9.

We mention without proof two more advanced properties of the Riemann–Stieltjes integral: If f is continuous on $[a, b]$, and if g is absolutely continuous on the same interval, then

$$\int_a^b f\,dg = \int_a^b fg'$$

where the integral on the right is a Lebesgue integral. Secondly, the *Riesz representation theorem*, which is fundamental to functional analysis, asserts that if G is a positive bounded linear functional on the space $C[a, b]$ of continuous functions on $[a, b]$, then there exists a weakly increasing function g on $[a, b]$ such that

$$G(f) = \int_a^b f\,dg$$

for all $f \in C[a, b]$. An account of this is given in Kestelman (1960, pp. 265–269).

For more extensive accounts of Riemann–Stieltjes integration, see Apostol (1974, Chapter 9), Hildebrandt (1938), Kestelman (1960, Chapter 11), Rankin (1963, Section 29), or Widder (1946, Chapter 1).

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