15

Oscillations of error terms

15.1 Applications of Landau's theorem

In this section we make repeated use of the following simple analogue of Landau's theorem (Theorem 1.7) concerning Dirichlet series with non-negative coefficients.

Lemma 15.1 *Suppose that $A(x)$ is a bounded Riemann-integrable function in any finite interval $1 \le x \le X$, and that $A(x) \ge 0$ for all $x > X_0$. Let σ_c denote the infimum of those σ for which $\int_{X_0}^{\infty} A(x)x^{-\sigma}\,dx < \infty$. Then the function*

$$F(s) = \int_1^{\infty} A(x)x^{-s}\,dx$$

is analytic in the half-plane $\sigma > \sigma_c$, but not at the point $s = \sigma_c$.

Proof Write

$$F(s) = \int_1^{X_0} A(x)x^{-s}\,dx + \int_{X_0}^{\infty} A(x)x^{-s}\,dx = F_1(s) + F_2(s),$$

say. Then the function $F_1(s)$ is entire, and the proof of Theorem 1.7 can be adapted to $F_2(s)$ to give the stated result. □

In Exercise 13.1.1 we saw that if Θ denotes the supremum of the real parts of the zeros of the zeta function, then $\psi(x) = x + O(x^{\Theta}(\log x)^2)$. Conversely, if $\psi(x) = x + O(x^{\alpha+\varepsilon})$, then by Theorem 1.3 the Dirichlet series $\sum_{n=1}^{\infty}(\Lambda(n) - 1)n^{-s}$ converges for $\sigma > \alpha$, and hence $\zeta(s) \ne 0$ in this half-plane. That is, $\psi(x) - x = \Omega(x^{\Theta-\varepsilon})$. We now sharpen this, by showing that $\psi(x) - x$ must be large in both signs.

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Theorem 15.2 *Let Θ denote the supremum of the real parts of the zeros of the zeta function. Then for every $\varepsilon > 0$,*

$$\psi(x) - x = \Omega_\pm(x^{\Theta-\varepsilon}) \tag{15.1}$$

and

$$\pi(x) - \mathrm{li}(x) = \Omega_\pm(x^{\Theta-\varepsilon}) \tag{15.2}$$

as $x \to \infty$.

Proof By Theorem 1.3 we have

$$-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \psi(x)x^{-s-1}\,dx$$

for $\sigma > 1$. Hence

$$-\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} = \int_1^\infty (\psi(x) - x)x^{-s-1}\,dx$$

for $\sigma > 1$. Suppose that

$$\psi(x) - x < x^{\Theta-\varepsilon} \text{ for all } x > X_0(\varepsilon). \tag{15.3}$$

Then we apply Lemma 15.1 to the function

$$\frac{1}{s-\Theta+\varepsilon} + \frac{\zeta'(s)}{s\zeta(s)} + \frac{1}{s-1} = \int_1^\infty (x^{\Theta-\varepsilon} - \psi(x) + x)x^{-s-1}\,dx.$$

Here the left-hand side has a pole at $\Theta - \varepsilon$, but is analytic for real $s > \Theta - \varepsilon$, in view of Corollary 1.14. Hence the above identity holds for $\sigma > \Theta - \varepsilon$, and both sides are analytic in this half-plane. But by the definition of Θ, the function ζ'/ζ has poles with real part $> \Theta - \varepsilon$. From this contradiction we deduce that the assertion (15.3) is false. That is, $\psi(x) - x = \Omega_+(x^{\Theta-\varepsilon})$. To obtain the corresponding Ω_- estimate we argue similarly using the identity

$$\frac{1}{s-\Theta+\varepsilon} - \frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} = \int_1^\infty (x^{\Theta-\varepsilon} + \psi(x) - x)x^{-s-1}\,dx.$$

In contrast to the situation of Corollary 2.5 or Theorem 13.2, it does not seem possible to derive (15.2) from (15.1) by integrating by parts. Instead, we pursue an argument modelled on the one just given. First we examine the Mellin transform of $\mathrm{li}(x)$. By integrating by parts we see that

$$s\int_2^\infty \mathrm{li}(x)x^{-s-1}\,dx = \int_2^\infty \frac{dx}{x^s\log x} = \int_{(s-1)\log 2}^\infty e^{-u}\,\frac{du}{u}.$$

Clearly this is

$$= \int_1^\infty e^{-u}\,\frac{du}{u} + \int_{(s-1)\log 2}^1 \frac{e^{-u}-1}{u}\,du - \log(s-1) - \log\log 2.$$

By (7.31) we see that this is

$$= -\int_0^{(s-1)\log 2} \frac{e^{-u}-1}{u}\,du - C_0 - \log(s-1) - \log\log 2.$$

Thus we find that

$$s\int_2^\infty \mathrm{li}(x)x^{-s-1}\,dx = -\log(s-1) + r(s)$$

where $r(s)$ is an entire function. Put

$$\Pi(x) = \sum_{n\le x} \frac{\Lambda(n)}{\log n}\,.$$

By Theorem 1.3 we know that

$$s\int_2^\infty \Pi(x)x^{-s-1}\,dx = \log\zeta(s)$$

for $\sigma > 1$. Hence

$$\frac{1}{s-\Theta+\varepsilon} - \frac{1}{s}\log(\zeta(s)(s-1)) + \frac{r(x)}{s}$$
$$= \int_2^\infty (x^{\Theta-\varepsilon} - \Pi(x) + \mathrm{li}(x))x^{-s-1}\,dx$$

for $\sigma > 1$. We observe that this function is analytic on the real axis for $s > \Theta - \varepsilon$. Thus by Lemma 1, if $\Pi(x) - \mathrm{li}(x) < x^{\Theta-\varepsilon}$ for all sufficiently large x, then the identity above holds in the half-plane $\sigma > \Theta - \varepsilon$. However, we are assuming that the zeta function has a zero $\rho = \beta + i\gamma$ with $\beta > \Theta - \varepsilon$, and the left-hand side above has a logarithmic singularity at $s = \rho$. Thus we have a contradiction, and so $\Pi(x) - \mathrm{li}(x) = \Omega_+(x^{\Theta-\varepsilon})$. Since $\pi(x) = \Pi(x) + O(x^{1/2}/\log x)$, and since $\Theta \ge 1/2$, it follows that $\pi(x) - \mathrm{li}(x) = \Omega_+(x^{\Theta-\varepsilon})$. For the corresponding Ω_- estimate, we argue similarly from the identity

$$\frac{1}{s-\Theta+\varepsilon} + \frac{1}{s}\log(\zeta(s)(s-1)) - \frac{r(x)}{s}$$
$$= \int_2^\infty (x^{\Theta-\varepsilon} + \Pi(x) - \mathrm{li}(x))x^{-s-1}\,dx.$$

□

Next we show that if there is a zero of $\zeta(s)$ on the line $\sigma = \Theta$, then we may draw a stronger conclusion.

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Theorem 15.3 *Suppose that Θ is the supremum of the real parts of the zeros of $\zeta(s)$, and that there is a zero ρ with $\Re\rho = \Theta$, say $\rho = \Theta + i\gamma$. Then*

$$\limsup_{x\to\infty} \frac{\psi(x) - x}{x^\Theta} \ge \frac{1}{|\rho|}, \tag{15.4}$$

and

$$\liminf_{x\to\infty} \frac{\psi(x) - x}{x^\Theta} \le -\frac{1}{|\rho|}. \tag{15.5}$$

Proof Suppose that $\psi(x) \le x + cx^\Theta$ for all $x \ge X_0$. Then by Lemma 15.1,

$$\frac{c}{s - \Theta} + \frac{\zeta'(s)}{s\zeta(s)} + \frac{1}{s-1} = \int_1^\infty (cx^\Theta - \psi(x) + x)x^{-s-1}\,dx \tag{15.6}$$

for $\sigma > \Theta$. Call this function $F(s)$. Then

$$\begin{aligned} F(s) &+ \frac{1}{2}e^{i\phi}F(s + i\gamma) + \frac{1}{2}e^{-i\phi}F(s - i\gamma) \\ &= \int_1^\infty (cx^\Theta - \psi(x) + x)(1 + \cos(\phi - \gamma\log x))x^{-s-1}\,dx \end{aligned}$$

for $\sigma > \Theta$. We now consider the behaviour of these two expressions as s tends to Θ from above through real values. On the right-hand side, the integral from 1 to X_0 is uniformly bounded, while the integral from X_0 to ∞ is non-negative. Thus the lim inf of the right-hand side is $> -\infty$ as $s \to \Theta^+$. On the other hand, the left-hand side is a meromorphic function that has a pole at $s = \Theta$ with residue

$$c + \frac{me^{i\phi}}{2\rho} + \frac{me^{-i\phi}}{2\overline{\rho}}$$

where $m \ge 1$ denotes the multiplicity of the zero ρ. We choose ϕ so that $e^{i\phi}/\rho = -1/|\rho|$. Then the above is $c - m/|\rho|$. This quantity must be non-negative, for if it were negative, then the left-hand side would tend to $-\infty$ as $s \to \Theta^+$. Hence $c \ge 1/|\rho|$, and we have (15.4). The proof of (15.5) is similar. □

Corollary 15.4 *As x tends to $+\infty$,*

$$\psi(x) - x = \Omega_\pm\big(x^{1/2}\big), \tag{15.7}$$
$$\vartheta(x) - x = \Omega_-\big(x^{1/2}\big), \tag{15.8}$$

and

$$\pi(x) - \mathrm{li}(x) = \Omega_-\big(x^{1/2}(\log x)^{-1}\big). \tag{15.9}$$

The problem of proving Ω_+ companions of (15.8) and (15.9) is more difficult, and is dealt with in the next section.

Proof We first prove (15.7). If RH is false, then $\Theta > 1/2$, and we have a stronger result by Theorem 15.2. If RH holds, then we have (15.7) by Theorem 15.3, and the remaining assertions follow by Theorem 13.2. □

Many similar results can be proved using the above ideas. For example, for $M(x) = \sum_{n\le x} \mu(n)$ we find, in the manner of Theorem 15.2, that

$$M(x) = \Omega_\pm(x^{\Theta-\varepsilon}). \tag{15.10}$$

In analogy to (15.6) we put

$$G(s) = \frac{1}{s\zeta(s)} - \frac{c}{s-\Theta} = \int_1^\infty (M(x) - cx^\Theta)x^{-s-1}\,dx.$$

Then in the manner of the proof of Theorem 15.3, we find that if $\Theta + i\gamma$ is a zero of $\zeta(s)$, then

$$\limsup_{x\to\infty} \frac{M(x)}{x^\Theta} \ge \frac{1}{|\rho\zeta'(\rho)|}, \tag{15.11}$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{x^\Theta} \le -\frac{1}{|\rho\zeta'(\rho)|}. \tag{15.12}$$

Here we are assuming that $\zeta'(\rho) \ne 0$. In the contrary case ρ would be a multiple zero of $\zeta(s)$, and our method would allow us to replace the right-hand side of (15.11) by $+\infty$ and that of (15.12) by $-\infty$. In fact we can prove still more, by considering the function

$$H(s) = \frac{1}{s\zeta(s)} - \frac{c(m-1)!}{(s-\Theta)^m} = \int_1^\infty (M(x) - cx^\Theta(\log x)^{m-1})x^{-s-1}\,dx.$$

Then our method allows us to deduce that if $\Theta + i\gamma$ is a zero of multiplicity $m \ge 1$, then

$$M(x) = \Omega_\pm(x^\Theta(\log x)^{m-1}).$$

Then in the manner of Corollary 15.4 we find that in any case

$$M(x) = \Omega_\pm\big(x^{1/2}\big), \tag{15.13}$$

and that if $\zeta(s)$ has a multiple zero, then

$$M(x) = \Omega_\pm\big(x^{1/2}\log x\big). \tag{15.14}$$

In the explicit formula for $\psi(x) - x$, or for $M(x)$, the arguments of the terms in the sum over the zeros are governed by the quantities $x^{i\gamma}$. If the ordinates $\gamma > 0$ are linearly independent over $\mathbb{Q}$, then these arguments will tend to be statistically independent as x runs over a long range. Numerical experiments have failed to disclose any linear dependences, and in the absence of any indication to the contrary, we presume that the ordinates $\gamma > 0$ are linearly independent. Under this assumption, we can improve on the estimate (15.13).

Theorem 15.5 *Let $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_K$ and γ be ordinates of zeros of $\zeta(s)$. For $1 \le k \le K$ let ε_k take one of the values -1, 0, 1. Suppose that*

$$\sum_{k=1}^{K} \varepsilon_k \gamma_k = 0 \tag{15.15}$$

for such ε_k only when $\varepsilon_k = 0$ for all k. Suppose also that the equation

$$\sum_{k=1}^{K} \varepsilon_k \gamma_k = \gamma \tag{15.16}$$

has a solution only if γ is one of the γ_k, say $\gamma = \gamma_{k_0}$ and that in this case the only solution is obtained by taking $\varepsilon_{k_0} = 1$, $\varepsilon_k = 0$ for $k \ne k_0$. Then

$$\limsup_{x\to\infty} \frac{M(x)}{x^{1/2}} \ge \sum_{k=1}^{K} \frac{1}{|\rho_k \zeta'(\rho_k)|} \tag{15.17}$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{x^{1/2}} \le -\sum_{k=1}^{K} \frac{1}{|\rho_k \zeta'(\rho_k)|}. \tag{15.18}$$

Proof In view of (15.10) and (15.14), we may assume that RH holds and that all zeros of the zeta function are simple. We suppose that $M(x) \le cx^{1/2}$ for all large x and consider the integral

$$I(s) = \int_1^\infty \frac{M(x) - cx^{1/2}}{x^{s+1}} \prod_{k=1}^{K} (1 + \cos(\phi_k - \gamma_k \log x))\, dx.$$

With $G(s)$ defined as above (with $\Theta = 1/2$), we multiply out the product to see that this integral is a linear combination of G at various arguments. More precisely, we see that

$$I(s) = G(s) + \frac{1}{2}\sum_{k=1}^{K} (e^{i\phi_k} G(s + i\gamma_k) + e^{-i\phi_k} G(s - i\gamma_k)) + J(s)$$

where $J(s)$ is a linear combination of G at arguments of the form

$$s + i\sum_{k=1}^{K} \varepsilon_k \gamma_k$$

with more than one of the ε_k non-zero. The function $G(s)$ is analytic in the half-plane $\sigma > 0$, except for poles at $s = 1/2$ and at the non-trivial zeros ρ.

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Hence by Landau's theorem we see that $I(s)$ converges for $\sigma > 1/2$, and our hypotheses (15.15), (15.16) imply that $J(s)$ is analytic at the point $s = 1/2$. Thus the integral $I(s)$ has a pole at $s = 1/2$ with residue

$$-c + \Re \sum_{k=1}^{K} \frac{e^{i\phi_k}}{\rho_k \zeta'(\rho_k)}.$$

We choose the ϕ_k so that the summands here are positive real. Since $I(s)$ is bounded above uniformly for $s > 1/2$, by letting s tend to $1/2$ from above we deduce that

$$c \ge \sum_{k=1}^{K} \frac{1}{|\rho_k \zeta'(\rho_k)|}.$$

This gives (15.17), and the proof of (15.18) is similar. □

It is not known whether it is possible to choose zeros ρ in such a way that the hypotheses (15.15), (15.16) hold, and for which the sum in (15.17) and (15.18) is large, but at least we are able to establish

Theorem 15.6 *Suppose that the Riemann Hypothesis is true and that the zeros of the zeta function are simple. Then*

$$\sum_{0<\gamma\le T} \frac{1}{|\zeta'(\rho)|} \gg T$$

as $T \to \infty$.

From this it follows by partial summation that

$$\sum_{0<\gamma\le T} \frac{1}{|\rho\zeta'(\rho)|} \gg \log T$$

as $T \to \infty$. Thus by combining Theorems 15.5 and 15.6 we have

Corollary 15.7 *If the ordinates* $\gamma > 0$ *of the Riemann zeta function are linearly independent over* $\mathbb{Q}$, *then*

$$\limsup_{x\to\infty} \frac{M(x)}{x^{1/2}} = +\infty$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{x^{1/2}} = -\infty.$$

Proof of Theorem 15.6 It is enough to prove the inequality with T restricted to the special sequence of values T_ν of Theorem 13.21, for which $|\zeta(s)| \gg \tau^{-\varepsilon}$

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uniformly for $-1 \le \sigma \le 2$. By the calculus of residues we see that

$$\sum_{0<\gamma\le T_\nu} \frac{1}{\zeta'(\rho)} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{\zeta(s)}\,ds$$

where $\mathcal{C}$ is the rectangular contour with vertices $2+i$, $2+iT_\nu$, $-1+iT_\nu$, $-1+i$. The top of this rectangle contributes an amount $\ll T_\nu^\varepsilon$. For s on the left side of this contour, $|\zeta(s)| \asymp \tau^{3/2}$ by Corollary 10.5, so that the integral along the left-hand side is $\ll 1$. The integral along the bottom of the rectangle is clearly $\ll 1$ as well. To estimate the integral along the right-hand side, we expand $1/\zeta(s)$ in its Dirichlet series, and integrate term by term. The integral of 1 contributes $T_\nu - 1$, while for $n > 1$ the integral of n^{-2-it} is $\ll n^{-2}/\log n$. On summing over n we find that the integral of $1/\zeta(s)$ over the right-hand side of the rectangle is $T_\nu + O(1)$. On combining these estimates we see that the sum above is $T_\nu + O(T_\nu^\varepsilon)$, and this gives the stated result. □

15.1.1 Exercises

1. (a) Suppose that ε is small and positive, and let $\mathrm{Li}(x)$ be defined as in Exercise 6.2.22. Explain why

$$s\int_{1+\varepsilon}^\infty \mathrm{Li}(x)x^{-s-1}\,dx = \mathrm{Li}(1+\varepsilon)(1+\varepsilon)^{-s} + \int_{1+\varepsilon}^\infty \frac{dx}{x^s \log x} = T_1 + T_2.$$

(b) Show that $\mathrm{Li}(1-\varepsilon) = \mathrm{Li}(1+\varepsilon) + O(\varepsilon)$.

(c) Show that

$$\mathrm{Li}(1-\varepsilon) = -\int_\varepsilon^\infty e^{-v}\,\frac{dv}{v}\,.$$

(d) Show that $\mathrm{Li}(1+\varepsilon) \ll \log 1/\varepsilon$.

(e) Deduce that

$$T_1 = -\int_\varepsilon^\infty e^{-v}\,\frac{dv}{v} + O\left(\varepsilon \log\frac{1}{\varepsilon}\right).$$

(f) Show that

$$T_2 = \int_{(s-1)\log(1+\varepsilon)}^\infty e^{-v}\,\frac{dv}{v}\,.$$

(g) Show that

$$T_2 = \int_{(s-1)\varepsilon}^\infty e^{-v}\,\frac{dv}{v} + O(\varepsilon)\,.$$

(h) Show that

$$T_1 + T_2 = -\log(s-1) - \int_\varepsilon^{(s-1)\varepsilon} (e^{-v} - 1)\,\frac{dv}{v} + O(\varepsilon \log 1/\varepsilon).$$

(i) Conclude that

$$s\int_1^\infty \mathrm{Li}(x)x^{-s-1}\,dx = -\log(s-1)$$

for $\sigma > 1$.

2. Let $\psi_1(x) = \sum_{n\le x}\Lambda(n)(x-n)$. Show that $\psi_1(x) - \frac{1}{2}x^2 = \Omega_\pm(x^{3/2})$.
3. Show that $\psi(2x) - 2\psi(x) = \Omega_\pm(x^{1/2})$.
4. (a) Show that as $x \to \infty$,

$$\sum_{n\le x}(1 - n/x)\mu(n) = \Omega_\pm\big(x^{1/2}\big).$$

(b) Show that as $x \to \infty$,

$$\sum_{n\le x}\mu(n)/n = \Omega_\pm\big(x^{-1/2}\big).$$

(c) Show that as $x \to \infty$,

$$\sum_{n=1}^{\infty}\mu(n)e^{-n/x} = \Omega_\pm\big(x^{1/2}\big).$$

5. Let $Q(x)$ denote the number of square-free numbers not exceeding x.

(a) Show that

$$Q(x) - \frac{6}{\pi^2}x = \Omega_\pm\big(x^{1/4}\big).$$

(b) Show that

$$Q(2x) - 2Q(x) = \Omega_\pm\big(x^{1/4}\big).$$

6. (a) Suppose that $\zeta(1/2 + i\gamma) = 0$ and that $\zeta(1/2 + 2i\gamma) \ne 0$. Show that

$$\limsup_{x\to\infty}\frac{\psi(x) - x}{x^{1/2}} \ge \frac{4}{3|\rho|}$$

and that

$$\liminf_{x\to\infty}\frac{\psi(x) - x}{x^{1/2}} \le -\frac{4}{3|\rho|}.$$

(b) Show that if $\zeta(1/2 + i\gamma_1) = \zeta(1/2 + i\gamma_2) = 0$ but $\zeta(1/2 + i(\gamma_1 + \gamma_2)) \ne 0$ and $\zeta(1/2 + i(\gamma_1 - \gamma_2)) \ne 0$, then

$$\limsup_{x\to\infty}\frac{\psi(x) - x}{x^{1/2}} \ge \frac{1}{|1/2 + i\gamma_1|} + \frac{1}{|1/2 + i\gamma_2|}$$

and that

$$\liminf_{x\to\infty} \frac{\psi(x)-x}{x^{1/2}} \le -\frac{1}{|1/2+i\gamma_1|} - \frac{1}{|1/2+i\gamma_2|}.$$

7. Show that $\sum_{n\le x}(-1)^{\omega(n)} \ll x^{1/2+\varepsilon}$ if and only if $(3^s-2)/\zeta(s)$ is analytic for $\sigma > 1/2$.
8. (Ingham 1942; cf. Haselgrove 1958) Let $L(x) = \sum_{n\le x}\lambda(n)$.
   (a) Show that if $\Theta > 1/2$, then for every $\varepsilon > 0$, $L(x) = \Omega_\pm(x^{\Theta-\varepsilon})$ as $x \to \infty$.
   (b) Show that $\liminf_{x\to\infty} L(x)/x^{1/2} \le 1/\zeta(1/2)$ $(= -0.685\ldots)$.
   (c) Show that if $\zeta(s)$ has a multiple zero, then $L(x) = \Omega_\pm\big(x^{1/2}\log x\big)$.
   (d) Show that if RH holds and σ is fixed, $1/4 < \sigma < 1/2$, then $|\zeta(2s)/\zeta(s)| = \tau^{\sigma-1/2+o(1)}$.
   (e) Show that if RH holds, then there is a sequence of $T_\nu \to \infty$ in such a way that $T_{\nu+1} \le T_\nu + 2$, and

$$\sum_{0<\gamma\le T_\nu} \frac{\zeta(2\rho)}{\zeta'(\rho)} = T_\nu + O\big(T_\nu^{3/4+\varepsilon}\big).$$

   (f) Show that if RH holds and the ordinates $\gamma > 0$ of the zeros of the zeta function are linearly independent over $\mathbb{Q}$, then

$$\limsup_{x\to\infty} \frac{L(x)}{x^{1/2}} = +\infty$$

   and

$$\liminf_{x\to\infty} \frac{L(x)}{x^{1/2}} = -\infty.$$

9. (Turán 1948; cf. Haselgrove 1958)
   (a) Show that if $\sum_{n\le x}\lambda(n)/n \ge 0$ for all $x \ge 1$, then the Riemann Hypothesis is true.
   (b) Show that

$$\sum_{n\le x}\lambda(n)/n = \Omega_+\big(x^{-1/2}\big)$$

   as $x \to \infty$.
10. Let the positive integer q be fixed. Suppose that if χ is a character (mod q), then $L(\sigma,\chi) \ne 0$ for $0 < \sigma < 1$. Suppose also that a and b are integers such that $(ab, q) = 1$ and $a \not\equiv b \pmod q$.
   (a) Let $\Theta = \Theta(q; a, b)$ denote the supremum of the real parts of the poles of the function

$$\sum_\chi (\overline{\chi}(a) - \overline{\chi}(b))\frac{L'}{L}(s,\chi).$$

Show that

$$\psi(x;q,a)-\psi(x;q,b)=\Omega_\pm(x^{\Theta-\varepsilon})$$

for any $\varepsilon>0$.

(b) Let $r(a)$ denote the number of solutions of the congruence $x^2\equiv a \pmod q$. Show that

$$\vartheta(x;q,a)=\psi(x;q,a)-\frac{r(a)}{\varphi(q)}x^{1/2}+o\big(x^{1/2}\big).$$

(c) Show that if $\Theta(q;a,b)>1/2$, then

$$\vartheta(x;q,a)-\vartheta(x;q,b)=\Omega_\pm(x^{\Theta-\varepsilon}),$$
$$\pi(x;q,a)-\pi(x;q,b)=\Omega_\pm(x^{\Theta-\varepsilon})$$

for any $\varepsilon>0$.

(d) Show that $\Theta(q;a,b)\ge 1/2$.

(e) Show that

$$\psi(x;q,a)-\psi(x;q,b)=\Omega_\pm\big(x^{1/2}\big).$$

(f) Show that if $r(a)\ge r(b)$, then

$$\vartheta(x;q,a)-\vartheta(x;q,b)=\Omega_-\big(x^{1/2}\big),$$
$$\pi(x;q,a)-\pi(x;q,b)=\Omega_-\big(x^{1/2}/\log x\big).$$

(g) Show that if $r(a)\le r(b)$, then

$$\vartheta(x;q,a)-\vartheta(x;q,b)=\Omega_+\big(x^{1/2}\big),$$
$$\pi(x;q,a)-\pi(x;q,b)=\Omega_+\big(x^{1/2}/\log x\big).$$

(h) Show that

$$\pi(x;4,1)-\pi(x;4,3)=\Omega_-\big(x^{1/2}/\log x\big).$$

11. (Hardy & Littlewood 1918; Landau 1918a, b) Let $\chi_{-4}(n)=(\frac{-4}{n})$ denote the non-principal character modulo 4, and let

$$T_1(x)=\sum_{n\le x}\Lambda(n)\chi_{-4}(n)(x-n).$$

(a) Show that

$$T_1(x)=-\sum_\rho\frac{x^{\rho+1}}{\rho(\rho+1)}+O(x)$$

where ρ runs over the non-trivial zeros of $L(s,\chi_{-4})$. In parts (b)–(l) below, assume that all these zeros lie on the line $\sigma=1/2$.

(b) Show that

$$\sum_\rho \frac{1}{|\rho|^2} = 2\log 2 - \log \pi - C_0 + 2\frac{L'}{L}(1, \chi_{-4}).$$

(c) Show that $L(1, \chi_{-4}) = \pi/4$.

(d) Show that

$$L'(1, \chi_{-4}) = \frac{\log 3}{6} + \sum_{k=2}^{\infty} \frac{(-1)^k}{2}\Big(\frac{\log 2k-1}{2k-1} - \frac{\log 2k+1}{2k+1}\Big),$$

and apply the alternating series test to show that $0.19 < L'(1, \chi_{-4}) < 0.196$.

(e) Deduce that

$$0.148 < \sum_\rho \frac{1}{|\rho|^2} < 0.164.$$

(f) Show that $|T_1(x)| < (0.165)x^{3/2}$ for all large x.

(g) Show that

$$\sum_{p \le x^{1/2}} (\log p)(x - p^2) = \frac{2}{3}x^{3/2} + o\big(x^{3/2}\big).$$

(h) Let $T_2(x) = \sum_{2<p\le x} (\log p)(-1)^{(p-1)/2}(x - p)$. Show that

$$-\frac{5}{6}x^{3/2} < T_2(x) < -\frac{1}{2}x^{3/2}$$

for all large x.

(i) Let $T_3(x) = \sum_{2<p\le x}(-1)^{(p-1)/2}(x - p)$. Show that

$$\begin{aligned} T_3(x) &= \frac{T_2(x)}{\log x} + \int_3^x \frac{T_2(u)}{u^2(\log u)^2}\Big(x + \frac{2(x-u)}{\log u}\Big)\, du \\ &= \frac{T_2(x)}{\log x} + O\Big(\frac{x^{3/2}}{(\log x)^2}\Big). \end{aligned}$$

(j) Let $P(x) = \sum_{p>2}(-1)^{(p-1)/2}e^{-p/x}$. Show that

$$P(x) = \frac{1}{x^2}\int_0^\infty T_3(u)e^{-u/x}\, du.$$

(k) Show that

$$\int_2^\infty u^{3/2}(\log u)^{-1}e^{-u/x}\, du = \frac{3}{4}\sqrt{\pi}x^{5/2}(\log x)^{-1} + O\big(x^{5/2}(\log x)^{-2}\big).$$

(l) Deduce that

$$P(x) < -\frac{3}{5}\frac{x^{1/2}}{\log x}$$

for all large x.

(m) Chebyshev (1853) proposed that $P(x) < 0$ for all sufficiently large x. Conclude that Chebyshev's conjecture is equivalent to the assertion that $L(s, \chi_{-4}) \neq 0$ for $\sigma > 1/2$.

15.2 The error term in the Prime Number Theorem

We have seen that $\psi(x) - x$ changes sign infinitely often. We now show that these sign changes can be localized if there is a zero on the abscissa Θ.

Theorem 15.8 *Let Θ denote the supremum of the real parts of the zeros of $\zeta(s)$. If $\zeta(s)$ has a zero with real part Θ, then there exists a constant $C > 0$ such that $\psi(x) - x$ changes sign in every interval $[x, Cx]$ for which $x \geq 2$.*

Proof For each integer $k \geq 0$, put

$$R_k(y) = \frac{1}{k!}\sum_{n \leq e^y}(y - \log n)^k \Lambda(n) - e^y.$$

We see easily that $R_k(y)$ is differentiable for $k > 1$, and that $R_k'(y) = R_{k-1}(y)$. By the method used to prove explicit formulæ we see also that

$$R_k(y) = -\sum_\rho \frac{e^{\rho y}}{\rho^{k+1}} + O(y^{k+1}).$$

Suppose that the numbers γ_j are determined, $0 < \gamma_1 < \gamma_2 < \ldots$ so that the numbers $\Theta \pm i\gamma_j$ constitute all the zeros of $\zeta(s)$ on the line $\sigma = \Theta$, and let m_j denote the multiplicity of the zero $\rho_j = \Theta + i\gamma_j$. Since $\sum_\rho |\rho|^{-\alpha} < \infty$ for $\alpha > 1$, we see that if $k \geq 1$, then

$$R_k(y) = -2e^{\Theta y}\Re \sum_j \frac{m_j e^{i\gamma_j y}}{\rho_j^{k+1}} + o(e^{\Theta y}) \tag{15.19}$$

as $y \to \infty$. Let K be the least number for which

$$\frac{m_1}{|\rho_1|^K} > \sum_{j>1} \frac{m_j}{|\rho_j|^K}.$$

Choose ϕ so that $e^{i\gamma_1\phi}/\rho_1^K > 0$. By taking $k = K$ in (15.19) and using the above inequality, we see that for all large numbers n, $R_K(\phi + \pi n/\gamma_1)$ is positive or negative according as n is odd or even. Take $C = \exp(\pi(K+2)/\gamma_1)$. Then any interval $[y_0, y_0 + \log C]$ contains at least $K+2$ points of the form $\phi + \pi n/\gamma_1$. Thus if y_0 is large, then such an interval contains $K+2$ points at which $R_K(y)$ alternates in sign. By the mean value theorem for derivatives we know that if f is differentiable on an interval $[\alpha, \beta]$ and $f(\alpha) < 0$, $f(\beta) > 0$, then there must be a number $\xi, \alpha < \xi < \beta$, such that $f'(\xi) > 0$. Thus we can choose $K+1$ points in the interval $[y_0, y_0 + \log C]$ at which $R_{K-1}(y)$ alternates in sign. Continuing in this manner, we conclude that we can find three points in this interval at which $R_1(y)$ alternates in sign. Now $R_1(y)$ is continuous, and $R_1'(y) = R_0(y)$ in intervals containing no prime power, so that $R_1(y)$ is an indefinite integral of $R_0(y)$. Thus, although $R_0(y)$ is not everywhere differentiable, it is nevertheless true that R_1 will be monotonic in any interval in which R_0 is of constant sign. Since R_1 is not monotonic in the interval in question, we deduce that R_0 changes sign. □

The method used to prove Corollary 15.7 could be applied to $\psi(x) - x$, but for this function we have a different approach that succeeds without any unproved hypothesis. In view of Theorem 15.2 we may assume that the Riemann Hypothesis is true. By substituting e^y for x in the explicit formula for $\psi(x)$, we see that

$$\frac{\psi(e^y) - e^y}{e^{y/2}} = -\sum_\rho e^{i\gamma y}/\rho + O\big(e^{-y/2}\big)$$

uniformly for $y \ge 1$. Since $1/\rho = 1/(i\gamma) + O(1/\gamma^2)$ and $\sum 1/\gamma^2 < \infty$, the above is

$$-2\sum_{\gamma>0} \frac{\sin\gamma y}{\gamma} + O(1).$$

Here each term in the sum is periodic, and if γ is large, then both the period and the amplitude of the term are small. The sum is not absolutely convergent, but by suitably averaging this with respect to y we may arrange that the γ beyond a chosen point make a small contribution. Suppose, for simplicity, that by such an averaging we could truncate the sum, which would leave us to consider the partial sum

$$-2\sum_{0<\gamma\le T} \frac{\sin\gamma y}{\gamma}. \tag{15.20}$$

Here the sum of the absolute values of the coefficients is $\asymp (\log T)^2$, and the sum will be of this order of magnitude if we can find a y for which the fractional parts $\{\gamma y/(2\pi)\}$ are approximately $1/4$ for all the above γ. This, however, is an inhomogeneous problem of Diophantine approximation, and in general such a problem has a solution only if the coefficients γ are linearly independent over $\mathbb{Q}$. Moreover, in order to obtain a quantitative result it would be necessary to have quantitative lower bounds for the absolute values of linear forms in the γ. Since we have no such information, we are confined to homogeneous approximation. Dirichlet's theorem assures us that there exist large y for which each of the numbers $\gamma y/(2\pi)$ is near an integer. That is, $\|\gamma y/(2\pi)\|$ is small for $0 < \gamma \le T$, where $\|\theta\|$ denotes the distance from θ to the nearest integer, $\|\theta\| = \min_{n\in\mathbb{Z}} |\theta - n|$. However, the sum (15.20) vanishes when $y = 0$, and will therefore be small when the numbers $\|\gamma y/(2\pi)\|$ are small. On the other hand, if we take $y = \pi/T$ in (15.20), then $\sin\gamma y \asymp \gamma/T$, and the sum is $\asymp N(T)/T \asymp \log T$. While this is smaller than the $(\log T)^2$ that we might have hoped for, it is definitely large. This y is small, but by Dirichlet's theorem there exists a large number y_0 for which the numbers $\|\gamma y_0/(2\pi)\|$ are small, and then we may take $y = y_0 \pm \pi/T$ to make the sum (15.20) large in either sign.

The truth of the matter is that the sum (15.20) is not an average of the error term in the Prime Number Theorem, but we can form a weighted sum that resembles (15.20).

Lemma 15.9 *If the Riemann Hypothesis is true, then*

$$\frac{1}{(e^\delta - e^{-\delta})x}\int_{e^{-\delta}x}^{e^\delta x}(\psi(u) - u)\,du = -2x^{1/2}\sum_{\gamma>0}\frac{\sin\gamma\delta}{\gamma\delta}\cdot\frac{\sin(\gamma\log x)}{\gamma} + O\left(x^{1/2}\right)$$

uniformly for $x \ge 4$, $1/(2x) \le \delta \le 1/2$.

The first factor in the sum is near 1 if γ is small compared to $1/\delta$, and then becomes small for larger γ. Thus, despite its more complicated appearance, the above sum behaves like the partial sum (15.20) with $T \asymp 1/\delta$.

Proof We recall that

$$\int_0^x (\psi(u) - u)\,du = -\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'}{\zeta}(0)x + O(1)$$

for $x \ge 2$. We replace x by $e^{\pm\delta}x$ and difference to see that the left-hand side in the lemma is

$$-\frac{\delta}{\sinh\delta}\sum_\rho \frac{(e^{\delta(\rho+1)} - e^{-\delta(\rho+1)})x^\rho}{2\delta\rho(\rho+1)} + O(1). \tag{15.21}$$

We appeal to RH, and observe that $e^{\pm\delta(\rho+1)} = e^{\pm i\gamma\delta}(1 + O(\delta)) = e^{\pm i\gamma\delta} + O(\delta)$. Since $N(T+1) - N(T) \ll \log T$, we see easily that $\sum_\gamma \gamma^{-2} \ll 1$. Thus when we replace $e^{\pm\delta(\rho+1)}$ by $e^{\pm i\gamma\delta}$ in (15.21), we introduce an error term that is $\ll x^{1/2}$. Hence the expression (15.21) is

$$-ix^{1/2}\Big(\frac{\delta}{\sinh\delta}\Big)\sum_{\rho}\frac{\sin\gamma\delta}{\delta}\cdot\frac{x^{i\gamma}}{\rho(\rho+1)}+O\big(x^{1/2}\big).$$

The factor in parentheses is $1+O(\delta^2)$, and the sum over ρ is

$$\ll \sum_{0<\gamma\le 1/\delta}\frac{1}{\gamma}+\frac{1}{\delta}\sum_{\gamma>1/\delta}\frac{1}{\gamma^2}\ll(\log 1/\delta)^2,$$

so our expression is

$$-ix^{1/2}\sum_{\rho}\frac{\sin\gamma\delta}{\delta}\cdot\frac{x^{i\gamma}}{\rho(\rho+1)}+O\big(x^{1/2}\big).$$

Now $1/\rho=1/(i\gamma)+O(1/\gamma^2)$, and the first factor in the above sum is $\ll|\gamma|$, so that if we replace $1/\rho$ by $1/(i\gamma)$, then we introduce an error term that is $\ll x^{1/2}\sum_\gamma 1/\gamma^2\ll x^{1/2}$. Similarly we may replace $1/(\rho+1)$ by $1/(i\gamma)$. Thus we see that the above sum is

$$-x^{1/2}\sum_{\rho}\frac{\sin\gamma\delta}{\gamma\delta}\cdot\frac{x^{i\gamma}}{i\gamma}+O\big(x^{1/2}\big).$$

We now obtain the stated result by combining the contributions of γ and $-\gamma$. □

We now formulate a simple form of Dirichlet's theorem that is suitable for our use.

Lemma 15.10 (Dirichlet) *If $\theta_1,\ldots,\theta_K$ are real numbers, and N is a positive integer, then there is a positive integer $n\le N^K$ such that $\|\theta_k n\|<1/N$ for $1\le k\le K$.*

Proof The point $\boldsymbol{p}(n)=(\{\theta_1 n\},\ldots,\{\theta_K n\})$ lies in the hypercube $[0,1)^K$. We partition this hypercube into N^K hypercubes of side length $1/N$. We allow n to take the values $0,1,\ldots,N^K$, which gives us N^K+1 points. Hence by the pigeon-hole principle there are two values of n, say $0\le n_1<n_2\le N^K$, for which the points $\boldsymbol{p}(n_1)$, $\boldsymbol{p}(n_2)$ lie in the same hypercube. Thus

$$\|\theta_k n_1-\theta_k n_2\|\le|\{\theta_k n_1\}-\{\theta_k n_2\}|<1/N$$

for $1\le k\le K$. We take $n=n_2-n_1$ to obtain the desired result. □

Theorem 15.11 (Littlewood) *As $x\to\infty$,*

$$\psi(x)-x=\Omega_\pm\big(x^{1/2}\log\log\log x\big),\tag{15.22}$$

and

$$\pi(x) - \operatorname{li}(x) = \Omega_\pm\left(x^{1/2}(\log x)^{-1}\log\log\log x\right). \qquad (15.23)$$

Proof We consider (15.22). If RH is false, then Theorem 15.2 is stronger. Thus it remains to prove (15.22) if RH holds. Let N be a large integer. We apply Lemma 15.10 to those numbers $\gamma(\log N)/(2\pi)$ for which $0 < \gamma \le T = N \log N$. Thus in Lemma 15.10 we have $K = N(T) \asymp T \log T$, and there exists an integer n, $1 \le n \le N^K$ such that

$$\left\| \frac{\gamma n}{2\pi} \log N \right\| < \frac{1}{N}$$

for $0 < \gamma \le T$. We take $x = N^n e^{\pm 1/N}$, $\delta = 1/N$ in Lemma 15.9. From the general inequality $|\sin 2\pi\alpha - \sin 2\pi\beta| \le 2\pi \|\alpha - \beta\|$ we see that

$$|\sin(\gamma \log x) \mp \sin \gamma/N| \le 2\pi/N.$$

Since

$$\sum_\gamma \left| \frac{\sin\gamma/N}{\gamma/N} \cdot \frac{1}{\gamma} \right| \ll (\log N)^2$$

and $\sum_{\gamma > T} 1/\gamma^2 \ll T^{-1} \log T \ll 1/N$, we deduce that the right-hand side in Lemma 15.9 is

$$\mp 2x^{1/2} N^{-1} \sum_{\gamma > 0} \left(\frac{\sin\gamma/N}{\gamma/N} \right)^2 + O\left(x^{1/2}\right).$$

The sum over γ is $\asymp N \log N$. But $x \le N^{N^K} e^{1/N}$ and $K = N(T) \asymp T \log T \asymp N(\log N)^2$, so that

$$\log\log x \ll N(\log N)^3,$$

and hence $\log N \ge (1 + o(1)) \log\log\log x$. The left-hand side in Lemma 15.9 is simply the average of $\psi(u) - u$ over a neighbourhood of x. Since $x \gg N$ and N is arbitrarily large, we have (15.22).

As for (15.23), we note that if RH holds, then (15.22) and (15.23) are equivalent, in view of Theorem 13.2. If RH is false, then Theorem 15.2 gives a stronger result. □

15.2.1 Exercises

1. Show that

$$\pi(x; 4, 1) - \pi(x; 4, 3) = \Omega_\pm\left(x^{1/2}(\log x)^{-1}\log\log\log x\right)$$

as $x \to \infty$.

2. (a) Show that if $f^{(k-1)}(x)$ is continuous in $[a, a+kh]$ and if $f^{(k)}(x)$ exists throughout $(a, a+kh)$, then there exists a $\xi \in (a, a+kh)$ such that
$$h^k f^{(k)}(\xi) = \sum_{j=0}^{k}(-1)^k \binom{k}{j} f(a+jh).$$
(b) Show that there exist constants $C > 0$, $c > 0$ such that if RH holds, then for all $x \ge 2$,
$$\sup_{x \le u \le Cx} (\psi(u) - u) \ge cx^{1/2}$$
and
$$\inf_{x \le u \le Cx} (\psi(u) - u) \le -cx^{1/2}.$$
3. Show that for every $C > 1$ there is a $\delta = \delta(C) > 0$ such that if RH holds, then
$$\sup_{x \le u \le Cx} |\psi(u) - u| \ge \delta x^{1/2}$$
for all $x \ge 2$.
4. (Ingham 1936)
(a) Let N be a positive integer, Y a positive real number, and let $\theta_1, \ldots, \theta_K$ be arbitrary real numbers. By using Dirichlet's theorem, or otherwise, show that there is a real number y, $Y \le y \le YN^K$ such that $\|\theta_k y\| < 1/N$ for $1 \le k \le K$.
(b) Let N be an integer > 1, Y a positive real number. Show that there exist real numbers $\theta_1, \ldots, \theta_K$ such that $\max_k \|\theta_k y\| \ge 1/N$ uniformly for all real y in the interval $Y \le y \le Y(N-1)^K$.
(c) Suppose that RH holds. Show that there exists an absolute constant $c > 0$ such that for any real numbers $X \ge 2$ and $Z \ge 16$ there exists an x, $X \le x \le XZ$, for which
$$\pi(x) - \mathrm{li}(x) > cx^{1/2}(\log x)^{-1} \log\log\log Z,$$
and an x' in the same interval for which
$$\pi(x) - \mathrm{li}(x) < -cx^{1/2}(\log x)^{-1} \log\log\log Z.$$
(d) Deduce that there is an absolute constant $C > 0$ such that if RH holds, then $\pi(x) - \mathrm{li}(x)$ changes sign in every interval $[X, CX]$ for $X \ge 2$.

5. Show that the implicit constant in Littlewood's theorem can be taken to be 1/2. That is,

$$\limsup_{x\to\infty} \frac{\psi(x) - x}{x^{1/2} \log\log\log x} \ge 1/2,$$

with similar inequalities for the lim inf and for $\pi(x) - \mathrm{li}(x)$.

6. Suppose that q is an integer such that $\prod_\chi L(\sigma, \chi) \ne 0$ for $\sigma > 1/2$. Show that if $(b, q) = 1$, $b \not\equiv 1 \pmod q$, then

$$\pi(x; q, 1) - \pi(x; q, b) = \Omega_\pm\left(x^{1/2}(\log x)^{-1} \log\log\log x\right).$$

7. Suppose that $\sum_n |c_n| < \infty$, and put $g(y) = \sum_n c_n e^{i\lambda_n y}$ where the λ_n are real. Show that for any y_0 and any $\varepsilon > 0$, there exist arbitrarily large numbers y such that $|g(y) - g(y_0)| < \varepsilon$.
8. Suppose that $g(y) = \sum_n c_n e^{i\lambda_n y}$ is uniformly convergent for y in a neighbourhood of y_0, and put

$$M_\delta = \frac{1}{\delta}\int_{-\delta}^{\delta} \left(1 - \frac{|y|}{\delta}\right) g(y_0 + y)\, dy.$$

   (a) Show that

$$M_\delta = \sum_n c_n \left(\frac{\sin \lambda_n\delta/2}{\lambda_n\delta/2}\right)^2 e^{i\lambda_n y_0}$$

   for all small positive δ.

   (b) Show that $M_\delta \to g(y_0)$ as $\delta \to 0^+$.
9. (Jurkat 1973, Anderson 1991) Suppose that there is a constant K such that $M(x) \le Kx^{1/2}$ for all $x \ge 1$, or that there is a constant K such that $-Kx^{1/2} \le M(x)$ for all $x \ge 1$.
   (a) Show that the Riemann Hypothesis is true, that the zeros of $\zeta(s)$ are simple, and that $|\zeta'(\rho)| \gg 1/|\rho|$.
   (b) Show that there is a sequence of T_ν tending to infinity such that

$$M(x) = \lim_{\nu\to\infty} \sum_{|\gamma|\le T_\nu} \frac{x^\rho}{\rho\zeta'(\rho)} - 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\pi/x)^{2n}}{(2n)!n\zeta(2n+1)}$$

   for $x > 0$, and that the convergence is uniform in intervals that do not contain a square-free number.

   (c) Let

$$g(y) = \lim_{\nu\to\infty} \sum_{|\gamma|\le T_\nu} \frac{e^{i\gamma y}}{\rho\zeta'(\rho)}.$$

Show that if $g(y)$ is continuous at y_0, then for any $\varepsilon > 0$ there exist arbitrarily large y such that $|g(y) - g(y_0)| < \varepsilon$.

(d) Show that $g(0^+) - g(0^-) = 1$.

(e) Deduce that $\limsup_{x\to\infty} |M(x)|/x^{1/2} \ge 1/2$.

10. (a) Let $h(x) = (M(2x) - M(x))/x^{1/2}$. Show that $h(1^+) = -1$ and that $h(1^-) = 1$.

(b) Show that

$$\limsup_{x\to\infty} \Big| \sum_{x<n\le 2x} \mu(n) \Big| x^{-1/2} \ge 1.$$

15.3 Notes

Theorems 15.2 and 15.3, and Corollary 15.4, are due in substance to E. Schmidt (1903). Mertens (1897) conjectured that $|M(x)| \le x^{1/2}$ for all $x \ge 1$. This 'Mertens Hypothesis' was disproved by Odlyzko and te Riele (1984), who showed that

$$\limsup_{x\to\infty} \frac{M(x)}{x^{1/2}} \ge 1.06$$

and that

$$\liminf_{x\to\infty} \frac{M(x)}{x^{1/2}} \le -1.009.$$

One would expect that here the lim sup is $+\infty$ and the lim inf is $-\infty$, but neither of these assertions has been proved. Ingham (1942) proved Theorem 15.5 under the stronger hypothesis that the ordinates $\gamma > 0$ are joined by at most a finite number of linear relations. That one may restrict the coefficients of the linear relations, and thus in principle verify the hypothesis for the first several zeros, was shown by Bateman *et al.* (1971). The product used in the proof of Theorem 15.5 is very similar to the Riesz products used in the study of lacunary Fourier series (see Zygmund 1959, pp. 208–212).

The method used to prove Theorem 15.8 was introduced by Littlewood (1927) for the purpose of providing a simple proof of Theorem 15.3.

Theorem 15.11 was announced by Littlewood (1914), who sketched the proof. Full details were given later by Hardy and Littlewood (1918). The initial proofs depended on an appeal to the Phragmén–Lindelöf principle. Ingham (1936) found that this could be dispensed with. Ingham considered a more complicated weighted average of $\psi(u) - u$ which led to the simpler weighted partial sum

$$\sum_{0<\gamma\le T}(1-\gamma/T)\frac{\sin\gamma y}{\gamma}$$

of the sum (15.20). The present exposition was inspired by Ingham's editorial remark in Hardy's Collected Works (1967, p. 99).

The proof given of Theorem 15.11 is non-effective in the sense that it does not permit one to determine an explicit constant c about which one can assert that $\pi(x) > \mathrm{li}(x)$ for some $x < c$. Skewes (1933, 1955) formulated a slightly different division into cases (RH 'nearly true' vs. RH 'significantly false'), which permitted him to show that one can take

$$c = \exp(\exp(\exp(\exp(7.705)))).$$

One of the problems here is to construct a function $f(x)$ about which one can assert that in any interval $[x_0, f(x_0)]$ there exist x for which the sum over the non-trivial zeros is not highly cancelling. That is, the conclusion of Theorem 15.2 must be put in a more quantitative, localized form. In this connection, Littlewood (1937) was led to consider a question concerning a sum of cosines. Turán (1946) discovered that the theorem formulated by Littlewood is false – the argument provided establishes a weaker result than claimed. Turán undertook a detailed study of such power sums. His 'power sum method' has many important applications to the oscillatory error terms that arise in analytic number theory (see Turán 1984). In particular, Knapowski (1961) used Turán's method to show, without need of extensive numerical calculations, that an effective upper bound for the constant c can be determined. Subsequently, Lehman (1966) used extensive numerical information concerning the zeros ρ to show that one can take $c = 1.65\times 10^{1165}$. Using the same method te Riele (1989) shows that $\pi(x) > \mathrm{li}\,x$ for at least 10^{180} consecutive integers in the interval $[6.627\ldots\times 10^{370}, 6.687\ldots\times 10^{370}]$. More recently Bays & Hudson (2000) have given some new regions where $\pi(x) > \mathrm{li}(x)$, the first of these being around 1.39×10^{316}. An extension of Littlewood's theorem to Beurling primes has been given by Kahane (1999).

Monach & Montgomery (cf. Monach 1980) have conjectured that for every $\varepsilon > 0$ and every $K > 0$ there is a $T_0(\varepsilon, K)$ such that

$$\Big|\sum_{0<\gamma\le T} k_\gamma\gamma\Big| > \exp(-T^{1+\varepsilon}) \tag{15.24}$$

whenever $T \ge T_0$ and the k_γ are integers, not all 0, for which $|k_\gamma| \le K$. From this they have shown that

$$\limsup_{x\to\infty} \frac{\psi(x) - x}{x^{1/2}(\log\log\log x)^2} \geq \frac{1}{2\pi}, \tag{15.25}$$

and that

$$\liminf_{x\to\infty} \frac{\psi(x) - x}{x^{1/2}(\log\log\log x)^2} \leq \frac{-1}{2\pi}. \tag{15.26}$$

In view of (13.48), it is plausible that equality holds in (15.25) and (15.26).

Let $L(x) = \sum_{n\le x} \lambda(n)$. It was conjectured by Pólya (1919) that $L(x) \le 0$ for all $x \ge 2$, and it has been verified that this inequality holds for $2 \le x \le 10^6$. Pólya's conjecture was disproved by Haselgrove (1958), whose extensive computer calculations led to the conclusion that

$$\limsup_{x\to\infty} \frac{L(x)}{x^{1/2}} > 0.$$

Subsequently Lehman (1960) found that $L(906,180,359) = 1$.

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