14

Zeros

14.1 General distribution of the zeros

If $T > 0$ is not the ordinate of a zero of the zeta function, then we let $N(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle $0 < \beta < 1, 0 < \gamma < T$. If T is the ordinate of a zero, then we set $N(T) = (N(T^+) + N(T^-))/2$. By the argument principle we obtain

Theorem 14.1 *For any real t, put*

$$S(t) = \frac{1}{\pi} \arg \zeta(1/2 + it). \tag{14.1}$$

If $T > 0$, then

$$N(T) = \frac{1}{\pi} \arg \Gamma(1/4 + iT/2) - \frac{T}{2\pi} \log \pi + S(T) + 1. \tag{14.2}$$

Proof Since

$$N(T) = \frac{1}{2}(N(T^+) + N(T^-)), \qquad S(T) = \frac{1}{2}(S(T^+) + S(T^-)),$$

it suffices to prove (14.2) when T is not the ordinate of a zero. Let $\mathcal{C}$ denote the contour that proceeds by straight lines from 2 to $2 + iT$ to $-1 + iT$ to -1 to 2. Then by the argument principle,

$$N(T) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\xi'}{\xi}(s)\, ds.$$

Now let $\mathcal{C}_1$ denote the contour that proceeds by line segments from $1/2$ to 2 to $2 + iT$ to $1/2 + iT$, and let $\mathcal{C}_2$ be the contour that proceeds from $1/2 + iT$ to $-1 + iT$ to -1 to $1/2$. Thus $\int_{\mathcal{C}} = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2}$. For $s \in \mathcal{C}_2$ we use the identity

$$\frac{\xi'}{\xi}(s) = -\frac{\xi'}{\xi}(1 - s),$$

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and thus we see that

$$\int_{\mathcal{C}_2} \frac{\xi'}{\xi}(s)\,ds = -\int_{\mathcal{C}_2} \frac{\xi'}{\xi}(1-s)\,ds = \int_{\mathcal{C}_3} \frac{\xi'}{\xi}(s)\,ds$$

where $\mathcal{C}_3$ proceeds from $1/2 - iT$ to $2 - iT$ to 2 to 1/2. On adding this to the integral over $\mathcal{C}_1$, we see that the contribution of the interval [1/2, 2] cancels, and hence

$$N(T) = \frac{1}{2\pi i} \int_{\mathcal{C}_4} \frac{\xi'}{\xi}(s)\,ds$$

where $\mathcal{C}_4$ runs from $1/2 - iT$ to $2 - iT$ to $2 + iT$ to $1/2 + iT$. By (10.25) we see that the above is

$$= \frac{1}{2\pi i}\Big[\log s + \log(s-1) + \log\zeta(s) + \log\Gamma(s/2) - \frac{s}{2}\log\pi\Big|_{1/2-iT}^{1/2+iT}.$$

By the Schwarz reflection principle, the real parts cancel and the imaginary parts reinforce. Thus the above is

$$= \frac{1}{\pi}\bigg(\arg(1/2+iT) + \arg(-1/2+iT) \ + \arg\zeta(1/2+iT)$$
$$+ \arg\Gamma(1/4 + iT/2) - \frac{T}{2}\log\pi\bigg).$$

Here $\arg(1/2+iT) + \arg(-1/2+iT) = \pi$, so we have the stated result. □

By Stirling's formula (Theorem C.1) we know that

$$\log\Gamma(s) = (s - 1/2)\log s - s + \frac{1}{2}\log 2\pi + O(1/|s|). \qquad (14.3)$$

By using this, we obtain

Corollary 14.2 *For $T \ge 2$,*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(1/T).$$

Proof Clearly

$$\Im\big((-1/4 + iT/2)\log(1/4 + iT/2) - (1/4 + iT/2)\big)$$
$$= -\frac{1}{4}\arg\left(\tfrac{1}{4} + i\tfrac{T}{2}\right) + \frac{T}{4}\log\left(\tfrac{1}{16} + \tfrac{T^2}{4}\right) - \frac{T}{2}.$$

But $\arg(1/4 + iT/2) = \pi/2 + O(1/T)$, and $\log(1/16 + T^2/4) = 2\log T/2 + O(1/T^2)$, so we obtain the stated result. □

By combining the above with Lemma 12.3 or Theorem 13.20, we obtain

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Corollary 14.3 *For $T \ge 4$,*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Corollary 14.4 *If the Riemann Hypothesis is true, then*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O\left(\frac{\log T}{\log\log T}\right).$$

Note that these estimates imply the estimates of Theorem 10.13 and Lemma 13.18, respectively. In addition, from the first estimate above we see that there is an absolute constant $C > 0$ such that

$$N(T+h) - N(T) \asymp h\log T \tag{14.4}$$

uniformly for $C \le h \le T$. Similarly, there is an absolute constant $C > 0$ such that if RH is true, then (14.4) holds for $C/\log\log T \le h \le T$, $T \ge 4$. By modifying our method we obtain corresponding estimates for the number of zeros of a Dirichlet L-function.

Theorem 14.5 *Let χ be a primitive character modulo q, with $q > 1$. For $T > 0$, let $N(T,\chi)$ denote the number of zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $0 < \beta < 1$ and $0 \le \gamma \le T$. Any zeros with $\gamma = 0$ or $\gamma = T$ should be counted with weight $1/2$. Also, for any real number T, put*

$$S(T,\chi) = \frac{1}{\pi}\arg L(1/2 + iT, \chi). \tag{14.5}$$

Then

$$N(T,\chi) = \frac{1}{\pi}\arg\Gamma(1/4 + \kappa/2 + iT/2) + \frac{T}{2\pi}\log\frac{q}{\pi} + S(T,\chi) - S(0,\chi)$$

where $\kappa = 0$ or 1 according as $\chi(-1) = 1$ or -1.

There is no need to establish a separate result pertaining to zeros with $\gamma < 0$, since the number of zeros of $L(s,\chi)$ with $-T \le \gamma \le 0$ is $N(T,\overline{\chi})$.

Proof We may assume that T is not the ordinate of a zero, for if it were, then we have only to replace T by $T^{\pm}$, and average. However, we must take some precautions against the possibility that $L(s,\chi)$ has a zero on the real axis in the interval $(0, 1)$. Let $\mathcal{C}^{\pm}$ be the contour from $2 \pm i\varepsilon$ to $2 + iT$ to $-1 + iT$ to $-1 \pm i\varepsilon$ to $2 \pm i\varepsilon$, let $\mathcal{C}_1^{\pm}$ be the contour from $1/2 \pm i\varepsilon$ to $2 \pm i\varepsilon$ to $2 + iT$ to $1/2 + iT$, let $\mathcal{C}_2^{\pm}$ be the path from $1/2 + iT$ to $-1 + iT$ to $-1 \pm i\varepsilon$ to $1/2 \pm i\varepsilon$, and let $\mathcal{C}_3^{\pm}$ be the path from $1/2 - iT$ to $2 - iT$ to $2 \mp i\varepsilon$ to $1/2 \mp i\varepsilon$. By the argument principle, the number of zeros with $0 < \gamma \le T$ is

$$\frac{1}{2\pi i}\int_{\mathcal{C}^+}\frac{\xi'}{\xi}(s,\chi)\,ds = \frac{1}{2\pi i}\int_{\mathcal{C}_1^+}\frac{\xi'}{\xi}(s,\chi)\,ds + \frac{1}{2\pi i}\int_{\mathcal{C}_2^+}\frac{\xi'}{\xi}(s,\chi)\,ds.$$

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For $s \in \mathcal{C}_2^+$ we write

$$\frac{\xi'}{\xi}(s, \chi) = -\frac{\xi'}{\xi}(1 - s, \overline{\chi}),$$

and thus we find that

$$\int_{\mathcal{C}_2^+} \frac{\xi'}{\xi}(s, \chi)\, ds = -\int_{\mathcal{C}_2^+} \frac{\xi'}{\xi}(1 - s, \overline{\chi})\, ds = \int_{\mathcal{C}_3^+} \frac{\xi'}{\xi}(s, \overline{\chi})\, ds.$$

By (10.33), it follows that

$$\begin{aligned}\int_{\mathcal{C}_1^+} \frac{\xi'}{\xi}(s, \chi)\, ds &= \Big[\log L(s, \chi) + \log \Gamma((s + \kappa)/2) + \frac{s}{2} \log q/\pi \Big|_{1/2+i\varepsilon}^{1/2+iT} \\ &= \log L(1/2 + iT, \chi) - \log L(1/2 + i\varepsilon, \chi) \\ &\quad + \log \Gamma(1/4 + \kappa/2 + iT/2) - \log \Gamma(1/4 + \kappa/2 + i\varepsilon/2) \\ &\quad + i\frac{T - \varepsilon}{2} \log \frac{q}{\pi},\end{aligned}$$

and that

$$\begin{aligned}\int_{\mathcal{C}_3^+} \frac{\xi'}{\xi}(s, \overline{\chi})\, ds &= \Big[\log L(s, \overline{\chi}) + \log \Gamma((s + \kappa)/2) + \frac{s}{2} \log q/\pi \Big|_{1/2-iT}^{1/2-i\varepsilon} \\ &= \log L(1/2 - i\varepsilon, \overline{\chi}) - \log L(1/2 - iT, \overline{\chi}) \\ &\quad + \log \Gamma(1/4 + \kappa/2 - i\varepsilon/2) - \log \Gamma(1/4 + \kappa/2 - iT/2) \\ &\quad + i\frac{T - \varepsilon}{2} \log \frac{q}{\pi}.\end{aligned}$$

When these quantities are added, the real parts cancel and the imaginary parts are doubled, so after dividing by $2\pi i$ we find that the number of zeros with $0 < \gamma \le T$ is

$$\frac{1}{\pi} \arg \Gamma(1/4 + \kappa/2 + iT/2) + S(T, \chi) - S(0^+, \chi) + \frac{T}{2\pi} \log \frac{q}{\pi}.$$

By proceeding similarly with the opposite sign, we find that the number of zeros with $0 \le \gamma \le T$ is

$$\frac{1}{\pi} \arg \Gamma(1/4 + \kappa/2 + iT/2) + S(T, \chi) - S(0^-, \chi) + \frac{T}{2\pi} \log \frac{q}{\pi}.$$

We form the average of these two identities to obtain the stated result. □

Corollary 14.6 *Let χ be a primitive character modulo q, with $q > 1$. Then for $T > 0$,*

$$N(T, \chi) = \frac{T}{2\pi} \log \frac{qT}{2\pi} - \frac{T}{2\pi} + S(T, \chi) - S(0, \chi) - \chi(-1)/8 + O(1/(T + 1)).$$

Proof If $0 < T \le 2$, then $\arg \Gamma(1/4 + \kappa/2 + iT/2) \ll 1$ and $T \log T/2 - T \ll 1$, so the estimate is immediate in this case. Suppose that $T \ge 2$.

Clearly

$$\begin{aligned}&\Im((-1/4+\kappa/2+iT/2)\log(1/4+\kappa/2+iT/2)-(1/4+\kappa/2+iT/2))\\&=(-1/4+\kappa/2)\arg(1/4+\kappa/2+iT/2)+\frac{T}{4}\log((1/4+\kappa/2)^2+T^2/4)-\frac{T}{2}.\end{aligned}$$

Here $\arg(1/4+\kappa/2+iT/2)=\pi/2+O(1/T)$, $\log((1/4+\kappa/2)^2+T^2/4)=2\log T/2+O(1/T^2)$, and $2\kappa-1=-\chi(-1)$, so the result follows by Stirling's formula in the form (14.3). □

By combining the above with Lemma 12.8 we obtain

Corollary 14.7 *Let χ be a primitive character modulo q, $q>1$. Then for $T\ge 4$,*

$$N(T,\chi)=\frac{T}{2\pi}\log\frac{qT}{2\pi}-\frac{T}{2\pi}+O(\log qT).$$

14.1.1 Exercise

1. Let χ be a primitive character modulo q with $q>1$. Show that if $L(s,\chi)\ne 0$ for $\sigma>1/2$, then

$$N(T,\chi)=\frac{T}{2\pi}\log\frac{qT}{2\pi}-\frac{T}{2\pi}+O\left(\frac{\log qT}{\log\log qT}\right)$$

for $T\ge 2$.

14.2 Zeros on the critical line

At present we are unable to prove the Riemann Hypothesis, which asserts that all non-trivial zeros of the zeta function lie on the critical line $\sigma=1/2$. However, we are able to show that infinitely many zeros lie on this line.

Theorem 14.8 (Hardy) *There exist infinitely many real numbers γ such that $\zeta(1/2+i\gamma)=0$.*

For real t, let

$$Z(t)=\zeta(1/2+it)\frac{\Gamma(1/4+it/2)\pi^{-1/4-it/2}}{|\Gamma(1/4+it/2)\pi^{-1/4-it/2}|}. \tag{14.6}$$

Thus, as depicted in Figure 14.1, $Z(t)$ is real-valued, $|Z(t)|=|\zeta(1/2+it)|$, and $Z(t)$ changes sign at γ if and only if $\zeta(s)$ has a zero at $1/2+i\gamma$ of odd

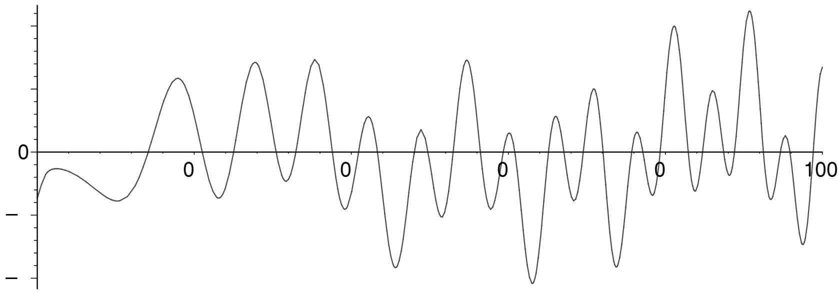

Figure 14.1 Graph of $Z(t)$ for $0 \le t \le 100$.

multiplicity. If $T > 0$ is a real number such that

$$\left| \int_T^{2T} Z(t)\,dt \right| < \int_T^{2T} |Z(t)|\,dt, \tag{14.7}$$

then $Z(t)$ is not of constant sign in the interval $(T, 2T)$, which is to say that $\zeta(s)$ has at least one zero $1/2 + i\gamma$ of odd multiplicity, with $T < \gamma < 2T$. Although it is possible to show that (14.7) holds for all large T, the requisite arguments involve technical tools that we have not yet developed. Fortunately, there is a family of weights $W(t)$ such that the integral $\int W(t)Z(t)\,dt$ can be evaluated by interpreting it as an inverse Mellin transform with a familiar kernel. Thus we are able to establish a weighted variant of (14.7), which suffices for our purpose. In preparation for the main argument, we establish two preliminary results.

Lemma 14.9 *If* $\Re z > 0$ *and* $\sigma_0 > 1$*, then*

$$\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \zeta(s)\Gamma(s/2)(\pi z)^{-s/2}\,ds = 2\sum_{n=1}^{\infty} e^{-\pi n^2 z}.$$

This is the inverse of the Mellin transform relationship (10.7) that Riemann used to establish the functional equation.

Proof By Theorem C.4 we see that if $\Re w > 0$ and $\sigma_0 > 0$, then

$$\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \Gamma(s/2) w^{-s/2}\,ds = 2e^{-w}.$$

We take $w = \pi n^2 z$, and sum over n, to obtain the desired identity. Here the exchange of summation and integration is permissible since the Dirichlet series for $\zeta(s)$ is uniformly convergent on the abscissa σ_0, and since

$$\int_{-\infty}^{\infty} \left|\Gamma((\sigma_0 + it)/2)(\pi z)^{-s/2}\right|\,dt < \infty.$$

□

Lemma 14.10 *We have*

$$\int_1^T \zeta(1/2+it)\,dt = T + O\big(T^{1/2}\big)$$

uniformly for $T \ge 2$.

Proof Let $\mathcal{C}$ denote the rectangular contour with vertices $1/2+i$, $2+i$, $2+iT$, $1/2+iT$. Since $\zeta(s)$ is analytic in this rectangle, we have

$$\int_{\mathcal{C}} \zeta(s)\,ds = 0$$

by Cauchy's theorem. The integral from $1/2+i$ to $2+i$ is an absolute constant, and by Corollary 1.17 the integral from $1/2+iT$ to $2+iT$ is

$$\ll \int_{1/2}^2 \big(1+T^{1-\sigma}\big)(\log T)\,d\sigma \ll T^{1/2}.$$

Thus

$$\int_1^T \zeta(1/2+it)\,dt = \int_1^T \zeta(2+it)\,dt + O\big(T^{1/2}\big).$$

This latter integral is

$$= \sum_{n=1}^\infty n^{-2}\int_1^T n^{-it}\,dt = T-1+\sum_{n=2}^\infty \frac{n^{-i}-n^{-iT}}{in^2\log n} = T+O(1),$$

so we have the stated result. □

Proof of Theorem 14.8 The integrand in Lemma 14.9 has a pole at $s=1$ with residue $z^{-1/2}$, but is otherwise analytic for $\sigma > 0$. We move the path of integration to the line $\sigma = 1/2$, and multiply both sides by $z^{1/4}$ to see that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} &\zeta(1/2+it)\Gamma(1/4+it/2)\pi^{-1/4-it/2}z^{-it/2}\,dt\\ &= -z^{-1/4} + 2z^{1/4}\sum_{n=1}^\infty e^{-\pi n^2 z}.\end{aligned} \tag{14.8}$$

Here the left-hand side is of the form $\int_{-\infty}^{\infty} W(t)Z(t)\,dt$ with

$$W(t) = \frac{|\Gamma(1/4+it/2)|}{2\pi^{5/4}z^{it/2}}.$$

Write z in polar coordinates, $z = re^{i\theta}$. Then $z^{-it/2} = r^{-it/2}e^{\theta t/2}$. For our approach to work, $W(t)$ must have constant argument. Accordingly, we take $r=1$, and set $\theta = \pi/2-\delta$ where δ is small and positive. By (C.19) we see that

$$|\Gamma(s/2)| \asymp \tau^{(\sigma-1)/2}e^{-\pi\tau/4}.$$

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Hence

$$W(t) \asymp \tau^{-1/4} e^{\pi(t-\tau)/4} e^{-\delta t/2} \asymp \begin{cases} \tau^{-1/4} e^{-(\pi-\delta)\tau/2} & \text{if } t \geq 0, \\ \tau^{-1/4} e^{-(1-\delta)\pi\tau/2} & \text{if } t \leq 0. \end{cases}$$

Thus $W(t)$ tends to 0 very rapidly as $t \to -\infty$, but relatively slowly as $t \to +\infty$. In particular,

$$W(t) \asymp \tau^{-1/4}$$

uniformly for $0 \leq t \leq 1/\delta$.

By the above and Lemma 14.10 we see that

$$\int_{-\infty}^{\infty} W(t)|Z(t)|\,dt \gg \delta^{1/4} \int_{1/(2\delta)}^{1/\delta} |Z(t)|\,dt = \delta^{1/4} \int_{1/(2\delta)}^{1/\delta} |\zeta(1/2+it)|\,dt \gg \delta^{-3/4}.$$

In order to exhibit a disparity, we must show that the right-hand side of (14.8) is $o(\delta^{-3/4})$. To this end it suffices to argue fairly crudely. Since $z = ie^{-i\delta} = \sin\delta + i\cos\delta$, by the triangle inequality the right-hand side of (14.8) is

$$\ll \sum_{n=1}^{\infty} e^{-\pi n^2 \sin\delta}.$$

By the integral test this is

$$\leq \int_0^{\infty} e^{-\pi u^2 \sin\delta}\,du = (\sin\delta)^{-1/2} \int_0^{\infty} e^{-\pi v^2}\,dv \ll \delta^{-1/2}.$$

If $\zeta(s)$ had only finitely many zeros on the critical line, then we would have

$$\left| \int_{-\infty}^{\infty} W(t)Z(t)\,dt \right| = \int_{-\infty}^{\infty} W(t)|Z(t)|\,dt + O(1)$$

uniformly as $\delta \to 0^+$. On the contrary, we have shown that

$$\int_{-\infty}^{\infty} W(t)Z(t)\,dt \ll \delta^{-1/2}, \qquad \int_{-\infty}^{\infty} W(t)|Z(t)|\,dt \gg \delta^{-3/4},$$

so the theorem is proved. □

14.2.1 Exercise

1. (a) Show that the right-hand side of (14.8) is

$$= -z^{-1/4} - z^{1/4} + z^{1/4}\vartheta(z),$$

in the notation of (10.8).

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(b) Show that if $z = ie^{-i\delta} = \sin\delta + i\cos\delta$, then

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} (-1)^n (1 + O(n^2\delta^2)) e^{-\pi n^2 \sin\delta}.$$

(c) Show that

$$\sum_{n=-\infty}^{\infty} n^2 e^{-\pi n^2 \sin\delta} \asymp \delta^{-3/2}$$

for $0 < \delta \le 1$.

(d) By taking $\alpha = 1/2$ in Theorem 10.1, or otherwise, show that

$$\sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi n^2 x} \asymp x^{-1/2} e^{-\pi/(4x)}$$

uniformly for $0 < x \le 1$.

(e) Show that if z is taken as in (b), then $\vartheta(z) \ll \delta^{1/2}$.

(f) Conclude that the right-hand side of (14.8) is $= -2\cos\pi/8 + O(\delta^{1/2})$.

14.3 Notes

Section 14.1. Theorem 14.1 and Corollary 14.2 are due to Backlund (1914, 1918), and this gave a shorter proof of Corollary 14.3 which had been obtained by von Mangoldt (1905). Earlier von Mangoldt (1895) had the error term $O((\log T)^2)$. Riemann (1859) proposed Corollary 14.3 but with no indication of a proof. It is remarkable that Corollary 14.3 is perhaps the only theorem on the Riemann zeta function that has not seen some significant improvement in the last 100 years.

Although the maximum order of $S(t)$ is unclear, even assuming the Riemann Hypothesis, we have considerable (unconditional) knowledge of its moments and distribution. Selberg (1944) showed that if k is a fixed non-negative even integer, then

$$\int_0^T S(t)^k\,dt = \frac{k!}{(k/2)!(2\pi)^k} T(\log\log T)^{k/2} + O(T(\log\log T)^{k/2-1}).$$

Although Selberg did not mention it, his techniques can also be used to show that

$$\int_0^T S(t)^k\,dt = o(T(\log\log T)^{k/2})$$

when k is odd. From these estimates it follows that the distribution of $S(t)$ is asymptotically normal, in the sense that

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\{t \in [0, T] : 2\pi S(t) \le c \log\log T\} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{c} e^{-t^2/2}\, dt$$

for any given real number c. Similar results apply to the distribution of the real part of $\log \zeta(1/2 + it)$, and indeed Selberg (unpublished) showed that the real and imaginary parts can be treated simultaneously. Specifically,

$$\int_0^T (\log \zeta(1/2+it))^h (\log \zeta(1/2-it))^k\, dt = \delta_{h,k} k! T (\log\log T)^k + O_{h,k}\left(T(\log\log T)^{(h+k-1)/2}\right)$$

where

$$\delta_{h,k} = \begin{cases} 1 & \text{if } h = k, \\ 0 & \text{otherwise.} \end{cases}$$

From this it follows that $\log \zeta(1/2 + it)$ is asymptotically normally distributed in the complex plane, in the sense that if Ω is a set in the complex plane with Jordan content, then

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{t \in [4, T] : \frac{\log \zeta(1/2+it)}{\sqrt{\log\log t}} \in \Omega\right\} = \frac{1}{\pi} \iint_{\Omega} e^{-|z|^2}\, dx\, dy.$$

Section 14.2. Theorem 14.8 was announced and a proof sketched in Hardy (1914). Further details are given in Hardy & Littlewood (1917). Let $N_0(T)$ denote the number of zeros of the form $1/2 + i\gamma$ with $0 < \gamma \le T$. Hardy & Littlewood (1921) showed that $N_0(T) \gg T$. Later Selberg improved this, first (1942a) to $N_0(T) \gg T \log\log T$ and then (1942b) to $N_0(T) \gg T \log T$, so that a positive proportion of the zeros are on the $\frac{1}{2}$-line. Levinson (1974) introduced an alternative method that enabled him to show that at least one-third of the non-trivial zeros are on the $\frac{1}{2}$-line. Selberg's method detects only zeros of odd multiplicity. This should not be a handicap, since presumably all zeros are simple. Heath-Brown (1979) has observed that Levinson's method detects only simple zeros. Conrey (1989) used Levinson's method to show that $N_0(T) \gtrsim \frac{2}{5} N(T)$.

The proof we have given of Hardy's Theorem 14.8 is but one of several described by Titchmarsh (1986, Chapter 10).

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