13

Conditional estimates

13.1 Estimates for primes

From the explicit formula for $\psi_0(x)$ we see that the contribution to the error term $\psi_0(x) - x$ made by a typical zero $\rho = \beta + i\gamma$ is $-x^\rho/\rho$. This has absolute value $\asymp x^\beta/|\gamma|$, which diminishes as $|\gamma|$ increases, but it depends much more sensitively on the value of β. We recall that if ρ is a zero, then so also is $1 - \rho$. Since at least one of these has real part $\ge 1/2$, we see that the Riemann Hypothesis represents the best of all possible worlds, in the sense that the error term in the Prime Number Theorem is smallest when the Riemann Hypothesis is true. By Theorem 10.13 we find that

$$\sum_{\substack{\rho \\ |\gamma| \le T}} \frac{1}{|\rho|} \ll \sum_{1 \le n \le T} \frac{\log 2n}{n} \ll (\log T)^2. \tag{13.1}$$

Thus by taking $T = x$ in Theorem 12.5, we obtain

Theorem 13.1 *Assume RH. Then for $x \ge 2$,*

$$\psi(x) = x + O\left(x^{1/2}(\log x)^2\right), \tag{13.2}$$

$$\vartheta(x) = x + O\left(x^{1/2}(\log x)^2\right), \tag{13.3}$$

$$\pi(x) = \mathrm{li}(x) + O\left(x^{1/2}\log x\right). \tag{13.4}$$

In Chapter 15 we shall show that these estimates for the error term are within a factor $(\log x)^2$ of being best possible, which is not surprising since each zero individually contributes an amount of the order $x^{1/2}$.

Proof The second assertion follows from the first by Corollary 2.5. By integration by parts we find that

$$\pi(x) = \int_2^x \frac{1}{\log u}\,du + \frac{\vartheta(x) - x}{\log x} + \frac{2}{\log 2} + \int_2^x \frac{\vartheta(u) - u}{u(\log u)^2}\,du, \tag{13.5}$$

and so the third assertion follows from the second. □

https://doi.org/10.1017/CBO9780511618314.015 Published online by Cambridge University Press

The factor $(\log x)^2$ in (13.2) can be avoided if we take smoother weights. For example, put

$$\psi_1(x) = \sum_{n\le x}(x-n)\Lambda(n). \tag{13.6}$$

Then we have the explicit formula

$$\psi_1(x) = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'}{\zeta}(0)x + \frac{\zeta'}{\zeta}(-1) + O\big(x^{-1/2}\big) \tag{13.7}$$

for $x \ge 2$. Assuming RH, it follows easily that

$$\psi_1(x) = \frac{1}{2}x^2 + O\big(x^{3/2}\big). \tag{13.8}$$

Assuming RH, we can also describe more precisely the relationships between the three standard prime-counting functions $\psi(x)$, $\vartheta(x)$, and $\pi(x)$.

Theorem 13.2 *Assume RH. Then*

$$\vartheta(x) = \psi(x) - x^{1/2} + O\big(x^{1/3}\big), \tag{13.9}$$

and

$$\pi(x) - \mathrm{li}(x) = \frac{\vartheta(x)-x}{\log x} + O\left(\frac{x^{1/2}}{(\log x)^2}\right). \tag{13.10}$$

Proof By an easy elaboration on Corollary 2.5, we see that

$$\vartheta(x) = \psi(x) - \psi\big(x^{1/2}\big) + O\big(x^{1/3}\big).$$

Hence (13.9) follows immediately from (13.2). To obtain (13.10), put

$$\vartheta_1(x) = \sum_{p\le x}(x-p)\log p = \int_2^x \vartheta(u)\,du.$$

By (13.8) and (13.9) it follows that $\vartheta_1(x) = x^2/2 + O\big(x^{3/2}\big)$. By integration by parts we see that the final integral in (13.5) is

$$\begin{aligned}\left[\frac{\vartheta_1(u)-u^2/2}{u(\log u)^2}\right]_2^x &+ \int_2^x \frac{\vartheta_1(u)-u^2/2}{(u\log u)^2}(1+2/\log u)\,du \\ &\ll \frac{x^{1/2}}{(\log x)^2} + \int_2^x u^{-1/2}(\log u)^{-2}\,du \\ &\ll \frac{x^{1/2}}{(\log x)^2}.\end{aligned}$$

Thus (13.10) follows from (13.5). □

As for primes in short gaps, we see from (13.4) that

$$\pi(x+h) - \pi(x) = \int_x^{x+h} \frac{1}{\log u}\, du + O\big(x^{1/2}\log x\big).$$

Here the main term on the right is larger than the error term if $h \ge Cx^{1/2}(\log x)^2$. We can do slightly better than this by counting primes between x and $x+h$ with a smoother weight.

Theorem 13.3 (Cramér) *There is a constant $C > 0$ such that if the Riemann Hypothesis is true, then for every $x \ge 2$ the interval $(x, x + Cx^{1/2}\log x)$ contains at least $x^{1/2}$ prime numbers.*

Proof Let h be a parameter to be determined, and put $w(u) = 1 - |u-x|/h$ when $|u-x| \le h$, and $w(u) = 0$ otherwise. Then by three applications of (13.7) we see that

$$\begin{aligned}\sum_n \Lambda(n)w(n) &= \frac{1}{h}(\psi_1(x+h) - 2\psi_1(x) + \psi_1(x-h))\\ &= h - \frac{1}{h}\sum_\rho \frac{(x+h)^{\rho+1} - 2x^{\rho+1} + (x-h)^{\rho+1}}{\rho(\rho+1)} + O\left(\frac{1}{hx}\right).\end{aligned} \tag{13.11}$$

Assuming RH, we note that the summand here is obviously

$$\ll \frac{x^{3/2}}{\gamma^2}. \tag{13.12}$$

Moreover, if $\gamma > x/h$, then the three terms in the numerator may have quite different arguments, in which case the above estimate is the best that we can assert in general. On the other hand, if γ is smaller, then some cancellation must occur in the numerator. To see this, note that the summand may be written

$$\int_{x-h}^{x+h} (h - |x-u|)u^{\rho-1}\, du \ll h^2 x^{-1/2} \tag{13.13}$$

assuming RH. This improves on (13.12) when $|\gamma| < x/h$. We use this estimate for the size of the summand together with Theorem 10.13 to see that the sum in (13.11) is $\ll hx^{1/2}\log x/h$. Hence if $h = Cx^{1/2}\log x$, then

$$\sum_{x-h<n<x+h} \Lambda(n) \ \ge\ \frac{h}{2}.$$

To complete the proof it remains to estimate the contribution made by higher powers of primes on the left-hand side. The number of squares in this interval is $\ll \log x$, so the squares of the primes contribute an amount that is $\ll (\log x)^2$. For each $k > 2$ there is at most one k^{th} power in the interval. Moreover, if p^k is in the interval, then $k \ll \log x$. Hence the higher powers contribute an amount $\ll (\log x)^2$, and the proof is complete. □

Although Cramér's theorem is highly non-trivial, and is significantly stronger than anything that we know how to prove unconditionally, it is nevertheless disappointing that it falls so far short of what we conjecture to be true, namely that for every $\varepsilon > 0$ the interval $[x, x + x^\varepsilon]$ contains a prime, for all $x > x_0(\varepsilon)$. In order to understand the weakness in our approach, write

$$\psi(x+h) - \psi(x) - h = -\sum_\rho \frac{(x+h)^\rho - x^\rho}{\rho} + \cdots. \qquad (13.14)$$

The contribution of zeros with $|\gamma| > x/h$ can be attenuated by employing a smoother weight, but no amount of smoothing will eliminate the smaller zeros. However, if $|\gamma| \le x/h$ then the argument of $(x+h)^\rho$ is near that of x^ρ, so there is some significant cancellation in the numerators above. Indeed,

$$\frac{(x+h)^\rho - x^\rho}{\rho} = \int_x^{x+h} u^{\rho-1}\,du \ll hx^{-1/2}$$

if $0 \le h \le x$ and $\beta = 1/2$. Taking this a step further, we see that the above is

$$= hx^{\rho-1} + O(h^2|\gamma|x^{\beta-2}).$$

Thus the left-hand side of (13.14) bears a passing resemblance to

$$-hx^{-1/2} \sum_{|\gamma| \le x/h} x^{i\gamma}, \qquad (13.15)$$

if we assume RH. Here the sum has $\asymp xh^{-1} \log x/h$ terms, and with sums of independent random variables in mind, we might guess that the above sum is $\ll (x/h)^{1/2+\varepsilon}$, which suggests

Conjecture 13.4 *If* $2 \le h \le x$, *then*

$$\psi(x+h) - \psi(x) = h + O_\varepsilon\big(h^{1/2}x^\varepsilon\big).$$

Although we expect there to be considerable cancellation in (13.15), any such cancellation that might occur among the contributions of the zeros is discarded in the proof of Theorem 13.3. Thus it seems that if we are to argue through zeta zeros to obtain an improvement of Theorem 13.3, then we need not just RH but also some deeper information concerning the distribution of the γ – more precisely that the numbers $\gamma \log x$ are approximately uniformly distributed modulo 2π. Although we cannot demonstrate that the desired cancellation occurs for all x, we can show that there is considerable cancellation in mean square.

Theorem 13.5 *Assume RH. Then for $X \ge 2$,*

$$\int_X^{2X} (\psi(x) - x)^2 \, dx \ll X^2.$$

Note that if we were to use the pointwise bound of Theorem 13.1 to bound the left-hand side above, then we would obtain an estimate that is larger than the above by a factor $(\log X)^4$. From the above we see that $\psi(x) = x + O(x^{1/2})$ on average.

Proof Take $T = X$ in the explicit formula of Theorem 12.5. Then

$$\psi(x) = x - \sum_{|\gamma| \le X} \frac{x^\rho}{\rho} + R(x)$$

where

$$\int_X^{2X} R(x)^2 \, dx \ll X(\log X)^4 + \sum_{X/2 < p^k < 3X} \left(\log p^k\right)^2 \left(1 + \int_1^\infty u^{-2} \, du\right)$$
$$\ll X(\log X)^4.$$

On the other hand, the sum over zeros contributes

$$\int_X^{2X} \Big| \sum_{|\gamma| \le X} \frac{x^\rho}{\rho} \Big|^2 dx = \sum_{\substack{\gamma_1, \gamma_2 \\ |\gamma_i| \le X}} \frac{1}{\rho_1 \overline{\rho_2}} \int_X^{2X} x^{1 + i(\gamma_1 - \gamma_2)} \, dx$$
$$\ll X^2 \sum_{\gamma_1, \gamma_2} \frac{1}{|\rho_1 \rho_2| \, |2 + i(\gamma_1 - \gamma_2)|}.$$

To complete the proof it suffices to show that

$$\sum_{\gamma_1, \gamma_2} \frac{1}{|\gamma_1 \gamma_2|(1 + |\gamma_1 - \gamma_2|)} < \infty. \tag{13.16}$$

In view of the symmetry of zeros about the real axis, we may confine our attention to $\gamma_1 > 0$. For each such zero, we consider γ_2 in various ranges. By Theorem 10.13, the sum over $\gamma_2 < -\gamma_1$ is

$$\sum_{\substack{\gamma_2 \\ \gamma_2 < -\gamma_1}} \frac{1}{|\gamma_2|(1 + |\gamma_1 - \gamma_2|)} \ll \sum_{\substack{\gamma_2 \\ \gamma_2 < -\gamma_1}} \frac{1}{\gamma_2^2} \ll \sum_{n > \gamma_1} \frac{\log n}{n^2} \ll \frac{\log \gamma_1}{\gamma_1}.$$

Similarly, the sum over those γ_2 for which $|\gamma_2| \le \frac{1}{2}\gamma_1$ is

$$\ll \frac{1}{\gamma_1} \sum_{\substack{\gamma_2 \\ 0 < \gamma_2 \le \gamma_1}} \frac{1}{\gamma_2} \ll \frac{1}{\gamma_1} \sum_{1 \le n \le \gamma_1} \frac{\log n}{n} \ll \frac{(\log \gamma_1)^2}{\gamma_1}.$$

The sum over those γ_2 for which $\frac{1}{2}\gamma_1 < \gamma_2 < \frac{3}{2}\gamma_1$ is

$$\ll \frac{1}{\gamma_1} \sum_{\substack{\gamma_2 \\ |\gamma_2-\gamma_1|\le\gamma_1/2}} \frac{1}{1+|\gamma_1-\gamma_2|} \ll \frac{\log\gamma_1}{\gamma_1} \sum_{1\le n\le\gamma_1} \frac{1}{n} \ll \frac{(\log\gamma_1)^2}{\gamma_1},$$

and finally the sum over $\gamma_2 \ge \frac{3}{2}\gamma_1$ is

$$\ll \sum_{\substack{\gamma_2 \\ \gamma_2\ge\frac{3}{2}\gamma_1}} \frac{1}{\gamma_2^2} \ll \sum_{n>\gamma_1} \frac{\log n}{n^2} \ll \frac{\log\gamma_1}{\gamma_1}.$$

We sum these estimates, multiply by $1/\gamma_1$, and sum over γ_1 to see that the expression (13.16) is

$$\ll \sum_{\gamma_1>0} \frac{(\log\gamma_1)^2}{\gamma_1^2} \ll \sum_{n=1}^{\infty} \frac{(\log n)^3}{n^2} < \infty.$$

This completes the proof. □

The oscillations of $x^{i\gamma} = e^{i\gamma\log x}$ become slower as x increases, since $\frac{d}{dx}\log x = 1/x \to 0$ as $x \to \infty$. However, with the change of variable $x = e^u$ we have $x^{i\gamma} = e^{i\gamma u}$, which is a periodic function of u. Put

$$f(u) = \frac{\psi\left(e^u\right) - e^u}{e^{u/2}}. \tag{13.17}$$

Assuming RH, the explicit formula of Theorem 12.5 gives

$$f(u) = -\sum_{\rho} \frac{e^{i\gamma u}}{\rho} + o(1)$$

as $u \to \infty$. This provides a kind of Fourier expansion of $f(u)$. Since

$$\int_U^{U+1} |f(u)|^2\,du = \int_{e^U}^{e^{U+1}} (\psi(x)-x)^2 \frac{dx}{x^2} \asymp e^{-2U} \int_{e^U}^{e^{U+1}} (\psi(x)-x)^2\,dx,$$

Theorem 13.5 is equivalent (assuming RH) to the estimate

$$\int_U^{U+1} |f(u)|^2\,du \ll 1. \tag{13.18}$$

By averaging $|f(u)|^2$ over a longer interval we obtain not just an upper bound, but an asymptotic formula.

Theorem 13.6 *Assume RH, and let $f(u)$ be defined as in* (13.17). *Then*

$$\lim_{U\to\infty} \frac{1}{U} \int_0^U |f(u)|^2\,du = \sum_{\text{distinct }\gamma} \frac{m_\rho^2}{|\rho|^2}$$

where m_ρ denotes the multiplicity of the zero ρ.

https://doi.org/10.1017/CBO9780511618314.015 Published online by Cambridge University Press

Proof Since the explicit formula for $\psi_0(x)$ is uniformly convergent in intervals free of prime powers, and is boundedly convergent in a neighbourhood of a prime power, it follows that

$$\begin{aligned}\frac{1}{U}\int_1^U |f(u)|^2\,du \\ &= \lim_{T\to\infty}\sum_{\substack{\gamma_1,\gamma_2\\ |\gamma_i|\le T}} \frac{1}{\rho_1\overline{\rho_2}U}\int_1^U e^{i(\gamma_1-\gamma_2)u}\,du \;+ o(1)\\ &= \Big(1-\frac{1}{U}\Big)\sum_{\substack{\gamma_1,\gamma_2\\ \gamma_1=\gamma_2}}\frac{1}{|\rho_1|^2} + O\Big(\sum_{\substack{\gamma_1,\gamma_2\\ \gamma_1\ne\gamma_2}}\frac{1}{|\gamma_1\gamma_2|}\min\Big(1,\frac{1}{U|\gamma_1-\gamma_2|}\Big)\Big) + o(1).\end{aligned}$$

Here the sum over $\gamma_1 \ne \gamma_2$ is finite already when $U = 1$, in view of (13.16). Since each term in this sum tends to 0 as $U \to \infty$, it follows that

$$\lim_{U\to\infty}\frac{1}{U}\int_1^U |f(u)|^2\,du = \sum_{\substack{\gamma_1,\gamma_2\\ \gamma_1=\gamma_2}}\frac{1}{|\rho_1|^2}.$$

Suppose that $\rho = 1/2 + i\gamma$ is a zero, and that its multiplicity is m_ρ. Then the equation $\gamma_i = \gamma$ has m_ρ solutions for $i = 1$ and for $i = 2$. Thus there are m_ρ^2 pairs (γ_1, γ_2) such that $\gamma_1 = \gamma_2 = \gamma$, so we have the result. □

We now return to the distribution of primes in arithmetic progressions.

Theorem 13.7 *Let q be given, and suppose that GRH holds for all L-functions modulo q. Then for $x \ge 2$,*

$$\psi(x,\chi) = E_0(\chi)x + O\big(x^{1/2}(\log x)(\log qx)\big), \tag{13.19}$$

$$\vartheta(x,\chi) = E_0(\chi)x + O\big(x^{1/2}(\log x)(\log qx)\big), \tag{13.20}$$

$$\pi(x,\chi) = E_0(\chi)\mathrm{li}(x) + O\big(x^{1/2}\log qx\big) \tag{13.21}$$

where $E_0(\chi) = 1$ or 0 according as $\chi = \chi_0$ or not.

Proof For χ_0 these relations follow from Theorem 1 and (12.14). Suppose that χ is non-principal, and that $\chi^\star$ is a primitive character that induces χ. Thus $\chi^\star$ is a character modulo d for some $d|q$, $1 < d \le q$. By taking $T = x$ in the explicit formula for $\psi(x, \chi^\star)$, and appealing to Theorem 10.17, we see that

$$\psi(x,\chi^\star) \ll x^{1/2}(\log qx)(\log x),$$

and then by (12.14) we have (13.19). By the triangle inequality, $|\psi(x,\chi) - \vartheta(x,\chi)| \le \psi(x) - \vartheta(x)$. From Corollary 2.5 we know that this latter quantity is $\ll x^{1/2}$, so (13.20) follows from (13.19). On inserting (13.20) into the identity

$$\pi(x,\chi) = \frac{\vartheta(x,\chi)}{\log x} + \int_2^x \frac{\vartheta(u,\chi)}{u(\log u)^2}\,du,$$

we obtain (13.21). □

Corollary 13.8 *Let q be given, and assume GRH for all L-functions modulo q. Suppose that $(a, q) = 1$. Then for $x \ge 2$,*

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O\big(x^{1/2}(\log x)^2\big), \tag{13.22}$$

$$\vartheta(x; q, a) = \frac{x}{\varphi(q)} + O\big(x^{1/2}(\log x)^2\big), \tag{13.23}$$

$$\pi(x; q, a) = \frac{\mathrm{li}(x)}{\varphi(q)} + O\big(x^{1/2}\log x\big). \tag{13.24}$$

Note that trivially,

$$0 \le \psi(x; q, a) \le (\log x) \sum_{\substack{0<n\le x \\ n\equiv a\ (q)}} 1 \le (\log x)(1 + x/q).$$

Thus we see that the bound (13.22) is worse than trivial if $q > x^{1/2}$. However, if q is smaller, say $q \le x^{\theta}$ with $\theta < 1/2$, then (13.22) provides a form of the Prime Number Theorem for arithmetic progressions with a much better error term than we were able to prove unconditionally (cf. Corollary 11.17).

Proof In view of the remarks above, we may assume that $q \le x^{1/2}$. By (11.22) we see that

$$\psi(x; q, a) - \frac{x}{\varphi(q)} = \frac{\psi(x, \chi_0) - x}{\varphi(q)} + \frac{1}{\varphi(q)} \sum_{\chi \ne \chi_0} \overline{\chi}(a)\psi(x, \chi). \tag{13.25}$$

Thus by the triangle inequality,

$$|\psi(x; q, a) - \frac{x}{\varphi(q)}| \le \frac{|\psi(x, \chi_0) - x|}{\varphi(q)} + \frac{1}{\varphi(q)} \sum_{\chi \ne \chi_0} |\psi(x, \chi)|, \tag{13.26}$$

and so (13.22) follows from (13.19). The other relations are proved similarly. □

Since $L(s, \chi)$ has $\asymp \log q$ zeros with $\gamma \ll 1$, we expect (assuming GRH) that $\psi(x, \chi)$ is usually about $(x \log q)^{1/2}$ in size. Thus the estimates of Theorem 13.7 are close to what we presume would be best possible. On the right-hand side of (13.25), we have $\varphi(q)$ terms. With sums of independent random variables in mind, we would expect therefore that the right-hand side of (13.25) is usually $\ll (x(\log q)/\varphi(q))^{1/2}$. Since we are unable to prove that there is cancellation in (13.25), we have no recourse but to use the triangle inequality, as in (13.26). However, we conjecture that a lot has been lost at this point.

Conjecture 13.9 If $(a, q) = 1$ and $q \le x$, then

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O_{\varepsilon}\big(x^{1/2+\varepsilon}/q^{1/2}\big).$$

Although we are unable to confirm our speculations concerning cancellation in (13.25) for any individual a, we can show that such cancellation must occur on average.

Corollary 13.10 *Assume GRH for all L-functions modulo q. If $2 \le q \le x$, then*

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} (\psi(x;q,a) - x/\varphi(q))^2 \ll x(\log x)^4.$$

Proof We claim that

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big|\sum_{\chi} c(\chi)\chi(a)\Big|^2 = \varphi(q)\sum_{\chi} |c(\chi)|^2 \tag{13.27}$$

for arbitrary complex numbers $c(\chi)$. To understand why this holds, expand the left-hand side and take the sum over a inside, to see that it is

$$= \sum_{\chi_1}\sum_{\chi_2} c(\chi_1)\overline{c(\chi_2)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \chi_1(a)\overline{\chi_2}(a).$$

By the basic orthogonality property of Dirichlet characters (cf (4.14)), the inner sum here is $\varphi(q)$ if $\chi_1 = \chi_2$, and is 0 otherwise, and this gives (13.27). By taking $c(\chi) = (\psi(x,\overline{\chi}) - E_0(\chi)x)/\varphi(q)$, it follows by (11.22) that

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} (\psi(x;q,a) - x/\varphi(q))^2 = \frac{1}{\varphi(q)}\sum_{\chi} |\psi(x,\chi) - E_0(\chi)x|^2,$$

The stated estimate now follows from (13.19). □

For non-principal χ let $n(\chi)$ denote the least character non-residue of χ, which is to say the least positive integer n such that $\chi(n) \ne 1$ and $\chi(n) \ne 0$. Since

$$\psi(x,\chi_0) = \psi(x) + O((\log q)(\log x)) \asymp x$$

for $x \ge C(\log q)(\log\log q)$, it follows by taking $x = C(\log q)^2(\log\log q)^2$ in (13.19) that $n(\chi) \ll (\log q)^2(\log\log q)^2$. As was the case with Cramér's theorem (Theorem 13.3), we can do slightly better by using a weighted sum of primes.

Theorem 13.11 *Let χ be a non-principal character modulo q, and assume that $L(s,\chi) \ne 0$ for $\sigma > 1/2$. Then $n(\chi) \ll (\log q)^2$.*

Proof By taking $k = 1$ in (5.17)–(5.19), we see that

$$\sum_{n \le x} \chi(n)\Lambda(n)(x-n) = \frac{-1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{L'}{L}(s,\chi)\frac{x^{s+1}}{s(s+1)}\,ds.$$

On pulling the contour to the line $\sigma = 1/4$, we see that the above is

$$-\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - \frac{x^{5/4}}{2\pi}\int_{-\infty}^{\infty} \frac{L'}{L}(1/4+it,\chi)\frac{x^{it}}{(1/4+it)(5/4+it)}\,dt.$$

By Theorem 10.17, the sum over ρ is $\ll x^{3/2}\log q$. By Theorem 10.17 with Lemma 12.7, we see that $\frac{L'}{L}(1/4+it,\chi) \ll \log q\tau$. Hence the second term above is $\ll x^{5/4}\log q$. Thus

$$\sum_{n\le x} \chi(n)\Lambda(n)(x-n) \ll x^{3/2}\log q. \tag{13.28}$$

On the other hand,

$$\sum_{n\le x} \chi_0(n)\Lambda(n)(x-n) = \sum_{n\le x} \Lambda(n)(x-n) + O(x(\log x)(\log q)) \gg x^2 \tag{13.29}$$

if $x \ge C(\log q)(\log\log q)$. If $\chi(n) = \chi_0(n)$ for all prime powers $n \le x$, then the left-hand sides of (13.28) and (13.29) are equal. However, the right-hand sides are inconsistent if we take $x = C(\log q)^2$, so we obtain the stated result. □

Weaker hypotheses concerning the zeros of $L(s,\chi)$ also imply bounds for $n(\chi)$. The argument here depends on a careful selection of the kernel in the inverse Mellin transform.

Theorem 13.12 *Let χ be a non-principal character* (mod q), *and suppose that δ is chosen, $1/\log q \le \delta \le 1/2$, so that $L(s,\chi) \ne 0$ for $1-\delta < \sigma < 1$, $0 < |t| \le \delta^2 \log q$. Then $n(\chi) < (A\delta\log q)^{1/\delta}$. Here A is a suitable absolute constant.*

Proof First we show that if $1/\log q \le R \le 1$, then

$$\sum_{|\rho-1|>R} \frac{1}{|\rho-1|^2} \ll \frac{\log q}{R}. \tag{13.30}$$

To see this, note that

$$\sum_{R<|\rho-1|\le 2R} \frac{1}{|\rho-1|^2} \ll \frac{1}{R^2} n(2R;0,\chi) \ll \frac{\log q}{R}$$

by Theorems 11.5 and 10.17. On replacing R by $2^k R$, and summing, we deduce that

$$\sum_{R<|\rho-1|\le 1} \frac{1}{|\rho-1|^2} \ll \frac{\log q}{R}.$$

As for zeros farther from 1, we note by Theorem 10.17 that

$$\sum_{|\rho-1|>1} \frac{1}{|\rho-1|^2} \ll \sum_{n=1}^{\infty} \frac{\log 2qn}{n^2} \ll \log q,$$

and so we have (13.30) for all $R \ge 1/\log q$.

Let x and y be parameters to be chosen later so that $2 < y \le x^{1/3}$. For $x/y^2 \le u \le xy^2$ set $w(u) = (2\log y - |\log(x/u)|)x/u$, and put $w(u) = 0$ otherwise. Then

$$\sum_n w(n)\chi(n)\Lambda(n) = \frac{-1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{L'}{L}(s,\chi)\left(\frac{y^{s-1}-y^{1-s}}{s-1}\right)^2 x^s\,ds \quad (13.31)$$

for $\sigma_0 > 1$. We move the contour to the abscissa $\sigma_0 = -1/2$, and find that the above is

$$\begin{aligned} &= -\sum_\rho \left(\frac{y^{\rho-1}-y^{1-\rho}}{\rho-1}\right)^2 x^\rho \; - (1-\kappa)(y-1/y)^2 \\ &\quad - \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} \frac{L'}{L}(s,\chi)\left(\frac{y^{s-1}-y^{1-s}}{s-1}\right)^2 x^s\,ds. \end{aligned} \quad (13.32)$$

Here the second term arises because $L(s,\chi)$ has a trivial zero at $s = 0$ if $\chi(-1) = 1$. Suppose that χ is induced by a primitive character $\chi^\star$. Then by (10.20) we see that

$$\frac{L'}{L}(s,\chi) = \frac{L'}{L}(s,\chi^\star) + \sum_{p|q} \frac{\chi^\star(p)\log p}{p^s - \chi^\star(p)}.$$

When $\sigma = -1/2$, the summand above is $\ll \log p$, and so by Lemma 12.9 we see that $\frac{L'}{L}(-1/2+it,\chi) \ll \log q\tau$. Hence the last term in (13.32) is $\ll x^{-1/2}y^3\log q$. If χ is imprimitive, then $L(s,\chi)$ may have infinitely many zeros on the imaginary axis. Such zeros are to be included in the sums in (13.30) and (13.32). If a zero ρ is real, then its contribution in (13.32) is negative. If ρ is a zero for which $\beta \le 1-\delta$, then its contribution to (13.32) is

$$\ll \frac{x^{1-\delta}y^{2\delta}}{|\rho-1|^2}.$$

From (13.30) with $R = \delta$ we see that the total contribution of such zeros is

$$\ll x^{1-\delta}y^{2\delta}(\log q)/\delta.$$

If ρ is a zero for which $\beta > 1-\delta$ and ρ is not real, then by hypothesis we have $|\gamma| \ge \delta^2 \log q$. The summand in (13.32) is $\ll x/|\rho - 1|$, so that from (13.30) with $R = \delta^2 \log q$ we see that such zeros contribute an amount $\ll x/\delta^2$. On combining these estimates we find that there is an absolute constant $c_1 > 0$ such that

$$\Re \sum_n w(n)\chi(n)\Lambda(n) \le c_1\big(x^{1-\delta}y^{2\delta}\delta^{-1}\log q + x\delta^{-2}\big). \tag{13.33}$$

If we replace χ by χ_0 in (13.31) and argue as in the proof of the Prime Number Theorem, we find that

$$\sum_n w(n)\chi_0(n)\Lambda(n) = 4(\log y)^2 x + O\big(x\exp\big(-c\sqrt{\log x}\big)\big) + O(y^2\log q). \tag{13.34}$$

Here the second error term reflects the possible contribution of zeros of $L(s, \chi_0)$ on the imaginary axis. If $\chi(n) = \chi_0(n)$ for all n for which $w(n) \ne 0$, then the left-hand side in (13.33) is identical with that in (13.34). Thus we wish to show that the right-hand sides cannot be equal, with a choice of x and y for which xy^2 is as small as possible. To this end, note that if $x = (C^3\delta\log q)^{1/\delta}$ and $y = C^{1/\delta}$, then the right-hand side of (13.33) is $\asymp (1 + 1/C)x/\delta^2$, while the right-hand side of (13.34) is $\asymp (\log C)^2x/\delta^2$, uniformly for $C \ge 2$. Thus if C is a sufficiently large absolute constant, then the left-hand members of (13.33) and (13.34) cannot be identical, and we have the stated result. □

13.1.1 Exercises

1. Let $\Theta = \sup_\rho \beta$ where ρ runs over all non-trivial zeros of $\zeta(s)$. Show that

$$\psi(x) = x + O(x^\Theta(\log x)^2),$$
$$\vartheta(x) = x + O(x^\Theta(\log x)^2),$$
$$\pi(x) = = \mathrm{li}(x) + O(x^\Theta\log x).$$

2. Let $F(x)$ be as in the proof of Theorem 13.3. Suppose that $2 \le \Delta \le h \le x$, and put $w(u) = 0$ for $u \le x - \Delta$, $w(u) = (u - x + \Delta)/\Delta$ for $x - \Delta \le u \le x$, $w(u) = 1$ for $x \le u \le x + h$, $w(u) = (x + h + \Delta - u)/\Delta$ for $x + h \le u \le x + h + \Delta$, $w(u) = 0$ for $u \ge x + h + \Delta$.
   (a) Show that

$$\sum_n \Lambda(n)w(n) = \frac{1}{\Delta}(F(x+h+\Delta) - F(x+h) - F(x) + F(x-\Delta))$$
$$= h + \Delta - \frac{1}{\Delta}\sum_\rho S(\rho) + O\left(\frac{1}{\Delta x}\right)$$

where

$$S(\rho) = \frac{(x+h+\Delta)^{\rho+1} - (x+h)^{\rho+1} - x^{\rho+1} + (x-\Delta)^{\rho+1}}{\rho(\rho+1)}.$$

(b) Show that if RH holds, then $S(\rho) \ll h\Delta x^{-1/2}$ for $|\gamma| \le x/h$, that $S(\rho) \le \Delta x^{1/2}/|\gamma|$ for $x/h \le |\gamma| \le x/\Delta$, and that $S(\rho) \ll x^{3/2}/\gamma^2$ for $\gamma| \ge x/\Delta$.

(c) Show that if RH holds, then

$$\psi(x+h) - \psi(x) = h + O\left(x^{1/2}(\log x)\log\frac{2h}{x^{1/2}\log x}\right)$$

uniformly for $x^{1/2}\log x \le h \le x$.

3. Assume RH. Show that

$$\int_2^X (\psi(x) - x)^2 \frac{dx}{x^2} \sim (\log X)\sum_\rho \frac{m_\rho^2}{|\rho|^2}$$

as $X \to \infty$.

4. Assume RH. Suppose that T is given, $T \ge 2$, and let $f(u)$ be defined as in (13.17). Show that

$$\lim_{U\to\infty}\frac{1}{U}\int_1^U \Big| f(u) + \sum_{\substack{\rho \\ |\gamma|\le T}} \frac{e^{i\gamma u}}{\rho}\Big|^2 du = \sum_{\substack{\rho \\ |\gamma|>T}} \frac{m_\rho^2}{|\rho|^2}.$$

5. Assume GRH for all L-functions modulo q. (a) Show that

$$\sum_{n\le x}\chi(n)\Lambda(n)(x-n) = E_0(\chi)x^2/2 + O\big(x^{3/2}\log q\big),$$

$$\sum_{p\le x}\chi(p)(\log p)(x-p) = E_0(\chi)x^2/2 + O\big(x^{3/2}\log q\big).$$

(b) Show that if $(a,q) = 1$, then

$$\sum_{\substack{n\le x \\ n\equiv a\,(q)}} \Lambda(n)(x-n) = \frac{x^2}{2\varphi(q)} + O\big(x^{3/2}\log q\big),$$

$$\sum_{\substack{p\le x \\ p\equiv a\,(q)}} (\log p)(x-p) = \frac{x^2}{2\varphi(q)} + O\big(x^{3/2}\log q\big).$$

(c) Deduce that if $(a,q) = 1$, then the least prime $p \equiv a \pmod q$ is $\ll \varphi(q)^2(\log q)^2$.

6. Assume Conjecture 13.9. Show that if $(a,q) = 1$, then there is a prime number $p \equiv a \pmod q$ such that $p \ll_\varepsilon q^{1+\varepsilon}$.

7. Let χ be a non-principal character, and let $n(\chi)$ denote the least positive integer n such that $\chi(n) \ne 1$, $\chi(n) \ne 0$. Show that $n(\chi)$ is a prime number.

8. (Montgomery 1971, p. 121) Let χ be a character modulo q, and let d denote the order of χ.
   (a) Show that
$$\frac{1}{d}\sum_{k=1}^{d}\chi^k(n)e(-ak/d)=\begin{cases}1 & \text{if } \chi(n)=e(a/d),\\ 0 & \text{otherwise.}\end{cases}$$
   (b) Assume that GRH holds for the $d-1$ L-functions $L(s,\chi^k)$ where $0<k<d$. Show that for each d^{th} root of unity $e(a/d)$ there is a prime p such that $\chi(p)=e(a/d)$, with $p \ll d^2(\log q)^2$.
9. (Montgomery 1971, p. 122) Let $\mathcal{P}(y)$ denote the set of those primes p such that $\left(\frac{n}{p}\right)=1$ for all $n\le y$, and let $P(y)$ be the product of all primes not exceeding y. Suppose that $2\le y\le x$.
   (a) Explain why
$$\sum_{\substack{x<p\le 2x\\ p\in\mathcal{P}(y)}}\log p = 2^{-\pi(y)}\sum_{x<p\le 2x}(\log p)\prod_{p_1\le y}\left(1+\left(\frac{p_1}{p}\right)\right).$$
   (b) For each $m|P(y)$, $m>1$, let χ_m be the quadratic character determined by quadratic reciprocity so that $\chi_m(p)=\prod_{p_1|m}\left(\frac{p_1}{p}\right)$. Also, let $\chi_1(n)=1$ for all n. Explain why the above is
$$=2^{-\pi(y)}\sum_{m|P(y)}(\vartheta(2x,\chi_m)-\vartheta(x,\chi_m)).$$
   (c) Assume GRH for all quadratic L-functions. Show that the above is
$$=2^{-\pi(y)}x(1+o(1))+O\big(x^{1/2}(\log x)^2\big).$$
   (d) Show that if $y=\frac{2}{3}(\log x)(\log\log x)$, then the above is positive, for all sufficiently large x.
   (e) Let $n_2(p)$ denote the least quadratic non-residue of p, which is to say the least positive integer n such that $\left(\frac{n}{p}\right)=-1$. Show that if GRH is true for all quadratic L-functions, then there exist infinitely many primes p such that $n_2(p)>\frac{2}{3}(\log p)(\log\log p)$.
10. (Littlewood 1924a; cf. Goldston 1982)
   (a) Show (unconditionally) that
$$\psi(x)\le x-\sum_\rho\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}+O(h)$$
   for $2\le h\le x/2$.
   (b) Show (unconditionally) that
$$\psi(x)\ge x-\sum_\rho\frac{x^{\rho+1}-(x-h)^{\rho+1}}{h\rho(\rho+1)}-O(h)$$
   for $2\le h\le x/2$.

(c) Now, and in the following, assume RH. Show that

$$\sum_{\substack{\rho \\ |\gamma|>x/h}} \frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)} \ll x^{1/2}\log x/h.$$

(d) Show that if $|\gamma| \le x/h$, then

$$\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)} = \frac{x^\rho}{\rho} + O\big(x^{-1/2}h\big).$$

(e) Show that

$$\sum_{\substack{\rho \\ |\gamma|\le x/h}} \frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)} = \sum_{\substack{\rho \\ |\gamma|\le x/h}} \frac{x^\rho}{\rho} + O\big(x^{1/2}\log x/h\big).$$

(f) Show that

$$\psi(x) = x - \sum_{|\gamma|\le\sqrt{x}/\log x} \frac{x^\rho}{\rho} + O\big(x^{1/2}\log x\big).$$

13.2 Estimates for the zeta function

We now show that our estimates of $\zeta(s)$ and of $\frac{\zeta'}{\zeta}(s)$ can be improved if we assume RH. To this end, we begin with a useful explicit formula. For $x \ge 2$, $y \ge 2$, put

$$w(u) = w(x,y;u) = \begin{cases} 1 & \text{if } 1 \le u \le x; \\ 1 - \frac{\log u/x}{\log y} & \text{if } x \le u \le xy; \\ 0 & \text{if } u \ge xy. \end{cases}$$

Then by two applications of (5.20) we find that

$$\sum_{n\le xy} w(n)\frac{\Lambda(n)}{n^s} = \frac{-1}{2\pi i \log y}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{\zeta'}{\zeta}(s+w)\frac{(xy)^w - x^w}{w^2}\,dw,$$

and on pulling the contour to the left we see that this is

$$\begin{aligned} &= -\frac{\zeta'}{\zeta}(s) + \frac{(xy)^{1-s}-x^{1-s}}{(1-s)^2\log y} \\ &\quad - \sum_\rho \frac{(xy)^{\rho-s}-x^{\rho-s}}{(\rho-s)^2\log y} - \sum_{k=1}^\infty \frac{(xy)^{-2k-s}-x^{-2k-s}}{(2k+s)^2\log y} \end{aligned} \tag{13.35}$$

provided that $s \ne 1$ and that $\zeta(s) \ne 0$. This much is true unconditionally, but from now on we assume RH, and show that the sum on the left provides a useful approximation to $-\frac{\zeta'}{\zeta}(s)$ when $\sigma > 1/2$.

https://doi.org/10.1017/CBO9780511618314.015 Published online by Cambridge University Press

Theorem 13.13 *Assume RH. Then*

$$\left|\frac{\zeta'}{\zeta}(s)\right| \le \sum_{n\le(\log\tau)^2}\frac{\Lambda(n)}{n^\sigma} + O((\log\tau)^{2-2\sigma}) \tag{13.36}$$

uniformly for $1/2 + 1/\log\log\tau \le \sigma \le 3/2$, $|t| \ge 1$.

Proof If $\sigma \ge 1/2$, then $|y^{\rho-s} - 1| \le 2$. Hence for $\sigma > 1/2$, the sum over ρ in (13.25) has absolute value not exceeding

$$\frac{2x^{1/2-\sigma}}{\log y}\sum_\rho \frac{1}{|s-\rho|^2}.$$

By (10.29) and (10.30) we see that

$$\begin{aligned}(\sigma - 1/2)\sum_\rho &\frac{1}{(\sigma-1/2)^2+(t-\gamma)^2}\\ &= \Re\frac{\zeta'}{\zeta}(s) + \frac{1}{2}\Re\frac{\Gamma'}{\Gamma}(s/2+1) - \frac{1}{2}\log\pi + \frac{\sigma-1}{(\sigma-1)^2+t^2},\end{aligned}$$

and by Theorem C.1 this is

$$= \Re\frac{\zeta'}{\zeta}(s) + \frac{1}{2}\log\tau + O(1).$$

On inserting this in (13.35), we find that

$$\begin{aligned}\frac{\zeta'}{\zeta}(s) = &-\sum_{n\le xy} w(n)\frac{\Lambda(n)}{n^s} + \frac{\theta 2x^{1/2-\sigma}}{(\sigma-1/2)\log y}\left|\Re\frac{\zeta'}{\zeta}(s)\right|\\ &+ O\left(\frac{x^{1/2-\sigma}\log\tau}{(\sigma-1/2)\log y}\right) + O\left(\frac{(xy)^{1-\sigma}}{\tau^2}\right) + O\left(\frac{y^{1-\sigma}}{\tau^2}\right)\end{aligned} \tag{13.37}$$

where θ is a complex number satisfying $|\theta| \le 1$. Thus

$$\frac{\zeta'}{\zeta}(s) \ll \left|\sum_{n\le xy} w(n)\frac{\Lambda(n)}{n^s}\right| + \frac{x^{1/2-\sigma}\log\tau}{(\sigma-1/2)\log y} + \frac{(xy)^{1-\sigma}}{\tau^2} + \frac{y^{1-\sigma}}{\tau^2} \tag{13.38}$$

provided that

$$\frac{2x^{1/2-\sigma}}{(\sigma-1/2)\log y} \le c < 1. \tag{13.39}$$

We take

$$y = \exp\left(\frac{1}{\sigma-1/2}\right), \qquad x = (\log\tau)^2/y.$$

Then the left-hand side of (13.39) is $2e(\log\tau)^{1-2\sigma}$, and so (13.39) holds with

$c = 2/e$ for $\sigma \ge 1/2 + 1/\log\log\tau$. We observe that

$$\sum_{n\le xy} w(n)\frac{\Lambda(n)}{n^s} \ll \sum_{n\le(\log\tau)^2} \frac{\Lambda(n)}{n^{1/2}} \ll \log\tau$$

uniformly for $\sigma \ge 1/2$. On inserting this in (13.38), we find that

$$\frac{\zeta'}{\zeta}(s) \ll \log\tau$$

uniformly for $\sigma \ge 1/2 + 1/\log\log\tau$, $|t| \ge 1$. We insert this on the right-hand side of (13.37) to obtain the stated estimate. □

Corollary 13.14 *Assume RH. Then*

$$\frac{\zeta'}{\zeta}(s) \ll ((\log\tau)^{2-2\sigma} + 1)\min\left(\frac{1}{|\sigma - 1|}, \log\log\tau\right)$$

uniformly for $1/2 + 1/\log\log\tau \le \sigma \le 3/2$, $|t| \ge 1$.

Proof By Chebyshev's estimate (Theorem 2.4) we know that

$$\sum_{U\le n<eU} \frac{\Lambda(n)}{n^\sigma} \ll U^{1-\sigma}.$$

On summing this over $U = e^k$ for $0 \le k \le 2\log\log\tau$, we obtain the stated bound from Theorem 13.13. □

Corollary 13.15 *Assume RH. Then*

$$|\log\zeta(s)| \le \sum_{n\le(\log\tau)^2} \frac{\Lambda(n)}{n^\sigma \log n} + O\left(\frac{(\log\tau)^{2-2\sigma}}{\log\log\tau}\right) \tag{13.40}$$

uniformly for $1/2 + 1/\log\log\tau \le \sigma \le 3/2$, $|t| \ge 1$.

Proof Since

$$\log\zeta(\sigma + it) = \log\zeta(3/2 + it) - \int_\sigma^{3/2} \frac{\zeta'}{\zeta}(\alpha + it)\,d\alpha,$$

it follows by the triangle inequality that

$$|\log\zeta(\sigma + it)| \le |\log\zeta(3/2 + it)| + \int_\sigma^{3/2} \left|\frac{\zeta'}{\zeta}(\alpha + it)\right| d\alpha,$$

which by Corollary 13.13 is

$$\le |\log\zeta(3/2 + it)| + \sum_{n\le(\log\tau)^2} \frac{\Lambda(n)}{\log n}\left(n^{-\sigma} - n^{-3/2}\right) + O\left(\frac{(\log\tau)^{2-2\sigma}}{\log\log\tau}\right).$$

But

$$|\log\zeta(3/2+it)| = \Big|\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}n^{-3/2-it}\Big| \le \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}n^{-3/2},$$

so it follows that

$$|\log\zeta(\sigma+it)| \le \sum_{n\le(\log\tau)^2}\frac{\Lambda(n)}{\log n}n^{-\sigma} + \sum_{n>(\log\tau)^2}\frac{\Lambda(n)}{\log n}n^{-3/2} + O\left(\frac{(\log\tau)^{2-2\sigma}}{\log\log\tau}\right). \tag{13.41}$$

By the Chebyshev estimate $\psi(x) \ll x$ we see that

$$\sum_{U<n\le 2U}\frac{\Lambda(n)}{\log n}n^{-3/2} \ll U^{-1/2}(\log U)^{-1}.$$

By taking $U = (\log\tau)^2 2^k$, and summing over $k \ge 0$, we deduce that

$$\sum_{n>(\log\tau)^2}\frac{\Lambda(n)}{\log n}n^{-3/2} \ll (\log\tau)^{-1}(\log\log\tau)^{-1}.$$

Since this is majorized by the error term in (13.41), we have (13.40). □

Corollary 13.16 *Assume RH. If* $|t| \ge 1$, *then*

$$|\log\zeta(s)| \le \log\frac{1}{\sigma-1} + O(\sigma-1) \tag{13.42}$$

for $1 + 1/\log\log\tau \le \sigma \le 3/2$,

$$|\log\zeta(s)| \le \log\log\log\tau + O(1) \tag{13.43}$$

for $1 - 1/\log\log\tau \le \sigma \le 1 + 1/\log\log\tau$, *and*

$$|\log\zeta(s)| \le \log\frac{1}{1-\sigma} + O\left(\frac{(\log\tau)^{2-2\sigma}}{(1-\sigma)\log\log\tau}\right) \tag{13.44}$$

for $1/2 + 1/\log\log\tau \le \sigma \le 1 - 1/\log\log\tau$.

Proof To establish (13.42), we note that if $1 < \sigma \le 3/2$, then

$$|\log\zeta(s)| = \Big|\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}n^{-s}\Big| \le \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}n^{-\sigma} = \log\zeta(\sigma)$$
$$= \log\big(1/(\sigma-1)+O(1)\big) = \log\frac{1}{\sigma-1} + O(\sigma-1).$$

As for (13.43), we note first that

$$\sum_{n\le z}\frac{\Lambda(n)}{n\log n} = \log\log z + O(1)$$

https://doi.org/10.1017/CBO9780511618314.015 Published online by Cambridge University Press

by Mertens' estimates (Theorem 2.7). Also, if $\sigma = 1 + O(1/\log z)$, then

$$n^{-\sigma} - n^{-1} = \int_1^\sigma n^{-\alpha}\, d\alpha \log n \ll |\sigma - 1| n^{-1} \log n$$

for $1 \le n \le z$, so that

$$\sum_{n\le z} \frac{\Lambda(n)}{\log n}(n^{-\sigma} - n^{-1}) \ll |\sigma - 1| \sum_{n\le z} \frac{\Lambda(n)}{n} \ll |\sigma - 1| \log z \ll 1.$$

On combining these estimates with $z = (\log \tau)^2$, we see that the sum in (13.40) is $\le \log\log\log \tau + O(1)$, which gives the desired estimate.

Concerning (13.44), we note that

$$\begin{aligned}\sum_{n\le z} \frac{\Lambda(n)}{\log n} n^{-\sigma} &= \int_{2^-}^z \frac{1}{u^\sigma \log u}\, d\psi(u) \\ &= \int_2^z \frac{1}{u^\sigma \log u}\, du + \frac{\psi(z) - z}{z^\sigma \log z} + 2^{1-\sigma}/\log 2 \\ &\quad + \int_2^z \frac{\psi(u) - u}{u^{\sigma+1} \log u}\left(\sigma + \frac{1}{\log u}\right) du. \end{aligned} \tag{13.45}$$

By the change of variable $v = u^{1-\sigma}$, the first integral immediately above is $\mathrm{li}(z^{1-\sigma}) - \mathrm{li}(2^{1-\sigma})$. But

$$\mathrm{li}(z^{1-\sigma}) \ll \frac{z^{1-\sigma}}{(1-\sigma)\log z}$$

for $\sigma \le 1 - 1/\log z$, and

$$\begin{aligned} -\mathrm{li}\big(2^{1-\sigma}\big) &= \int_{2^{1-\sigma}}^2 \frac{dv}{\log v} = \int_{2^{1-\sigma}}^2 \left(\frac{1}{v-1} + O(1)\right) dv \\ &= -\log(2^{1-\sigma} - 1) + O(1) = \log \frac{1}{\sigma - 1} + O(1). \end{aligned}$$

By Theorem 13.1, the second term in (13.45) is $\ll z^{1/2-\sigma} \log z$, and the final integral in (13.45) is

$$\ll \int_2^\infty u^{-\sigma-1/2} \log u\, du \ll (\sigma - 1/2)^{-2}.$$

On combining these estimates, we find that

$$\sum_{n\le z} \frac{\Lambda(n)}{n^\sigma \log n} = \log \frac{1}{1-\sigma} + O\left(\frac{z^{1-\sigma}}{(1-\sigma)\log z}\right),$$

uniformly for $1/2 < \sigma \le 1 - 1/\log z$. On taking $z = (\log \tau)^2$, the desired estimate now follows from (13.40). □

From Corollary 13.16 we see that if RH holds, then

$$\frac{1}{\log\log\tau} \ll |\zeta(1+it)| \ll \log\log\tau$$

for $|t| \geq 1$. We can make this more precise by taking a little more care.

Corollary 13.17 *Assume RH. Then* $|\zeta(1+it)| \leq 2e^{C_0}\log\log\tau + O(1)$.

Proof We observe that

$$\sum_{n\leq z}\frac{\Lambda(n)}{n\log n} = \sum_{p^k\leq z}\frac{\Lambda(n)}{n\log n} \leq \sum_{p\leq z}\sum_{k=1}^{\infty}\frac{1}{kp^k} = \log\prod_{p\leq z}\left(1-\frac{1}{p}\right)^{-1}$$
$$= C_0 + \log\log z + O(1/\log z)$$

by Mertens' estimate (Theorem 2.7). We take $z = (\log\tau)^2$, insert this in Corollary 13.15, and exponentiate to obtain the stated bound. □

To complete the picture, we estimate $|\zeta(s)|$ and $\arg\zeta(s)$ when σ is near 1/2. Of these estimates, the upper bound for $|\zeta(s)|$ is the most immediate.

Theorem 13.18 *Assume RH. There is an absolute constant* $C > 0$ *such that*

$$|\zeta(s)| < \exp\left(\frac{C\log\tau}{\log\log\tau}\right)$$

uniformly for $\sigma \geq 1/2$, $|t| \geq 1$.

Note that this is a quantitative form of the Lindelöf Hypothesis (LH).

Proof Put $\sigma_1 = 1/2 + 1/\log\log\tau$. For $\sigma \geq \sigma_1$, the above is contained in Corollary 13.14. Suppose that $1/2 \leq \sigma \leq \sigma_1$. Since $\Re 1/(s-\rho) \geq 0$ for all zeros ρ, from Lemma 12.1 it follows that there is an absolute constant $A > 0$ such that

$$\Re\frac{\zeta'}{\zeta}(s) \geq -A\log\tau$$

uniformly for $1/2 \leq \sigma \leq 2$, $|t| \geq 1$. Hence

$$\log|\zeta(s)| = \log|\zeta(\sigma_1+it)| - \int_{\sigma}^{\sigma_1}\Re\frac{\zeta'}{\zeta}(\alpha+it)\,d\alpha$$
$$\leq \log|\zeta(\sigma_1+it)| + A(\sigma_1-\sigma)\log\tau.$$

Here the first member on the right-hand side is bounded by Corollary 13.15, and $0 \leq \sigma_1 - \sigma \leq 1/\log\log\tau$, so we have the stated bound. □

To obtain the remaining estimates, we first establish two lemmas, which are of interest in their own right.

Lemma 13.19 *Assume RH. Then for $T \ge 4$,*

$$N(T + 1/\log\log T) - N(T) \ll \frac{\log T}{\log\log T}.$$

Proof Take $s = 1/2 + 1/\log\log T + iT$. Then $\frac{\zeta'}{\zeta}(s) \ll \log T$ by Corollary 13.14. Hence by Lemma 12.1 it follows that

$$\sum_{\substack{\rho \\ |\gamma - T| \le 1}} \frac{1}{s - \rho} \ll \log T.$$

Here each summand has positive real part, and for $T \le \gamma \le T + 1/\log\log T$ the real part is $\ge \frac{1}{2}\log\log T$, so we obtain the stated bound. □

By mimicking the proof of Lemma 12.1, we obtain

Lemma 13.20 *Assume RH. If $|\sigma - 1/2| \le 1/\log\log\tau$, then*

$$\frac{\zeta'}{\zeta}(s) = \sum_{\substack{\rho \\ |\gamma - t| \le 1/\log\log\tau}} \frac{1}{s - \rho} + O(\log\tau).$$

In applying the above, one is free to replace the condition $|\gamma - t| \le 1/\log\log\tau$ by a different condition, say $|\gamma - t| \le \delta$, provided that $\delta \asymp 1/\log\log\tau$. To see why this is so, note that a summand in one sum that is missing in the other has absolute value $\asymp \log\log\tau$, and that by Lemma 13.19 there are $\ll (\log\tau)/\log\log\tau$ such summands. Hence the total contribution made by terms in one sum but not the other is $\ll \log\tau$, and a discrepancy of this size may be absorbed in the error term.

Proof Put $\sigma_1 = 1/2 + 1/\log\log\tau$, and set $s_1 = \sigma_1 + it$. We apply Lemma 12.1 at s_1 and at s, and difference, to see that

$$\frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(s_1) + \sum_{|\gamma - t| \le 1} \left(\frac{1}{s - \rho} - \frac{1}{s_1 - \rho} \right) + O(\log\tau).$$

Here the first term on the right-hand side is $\ll \log\tau$, by Corollary 13.14. Let k be a positive integer, and consider zeros for which $k/\log\log\tau \le |\gamma - t| \le (k+1)/\log\log\tau$. By the preceding lemma, there are $\ll (\log\tau)/\log\log\tau$ such zeros, each one of which contributes an amount $\ll (\log\log\tau)/k^2$ to the above sum. On summing over k we see that the contribution of zeros for which $|\gamma - t| > 1/\log\log\tau$ is $\ll \log\tau$. Finally, for the zeros with $|\gamma - t| \le 1$, we observe that $|1/(s_1 - \rho)| \le \log\log\tau$, and there are $\ll (\log\tau)/\log\log\tau$ such zeros, so we have the stated result. □

If t is not the ordinate of a zero of the zeta function, then we define $\arg\zeta(s)$ by continuous variation along the ray $\alpha + it$ where α runs from σ to $+\infty$, and $\arg(+\infty + it) = 0$. If t is the ordinate of a zero, then we put $\arg \zeta(s) = (\arg \zeta(\sigma + it^+) + \arg \zeta(\sigma + it^-))/2$.

Theorem 13.21 *Assume RH. Then*

$$\arg \zeta(s) \ll \frac{\log \tau}{\log \log \tau}$$

uniformly for $\sigma \geq 1/2$, $|t| \geq 1$.

Proof We may assume that t is not the ordinate of a zero. Let σ_1 and s_1 be defined as in the preceding proof. If $\sigma \geq \sigma_1$, then the above follows from Corollary 13.16. Suppose now that $1/2 \leq \sigma \leq \sigma_1$. Then

$$\arg \zeta(s) = \arg \zeta(s_1) - \int_\sigma^{\sigma_1} \Im \frac{\zeta'}{\zeta}(\alpha + it)\, d\alpha.$$

Since $0 \leq \sigma_1 - \sigma \leq 1/\log\log\tau$, by Lemma 13.20 the right-hand side above is

$$= - \sum_{|\gamma - t| \leq 1/\log\log\tau} \int_\sigma^{\sigma_1} \Im \frac{1}{\alpha + it - \rho}\, d\alpha \; + O\left(\frac{\log \tau}{\log\log\tau}\right).$$

Here the summand is

$$\arctan \frac{\sigma - 1/2}{\gamma - t} - \arctan \frac{\sigma_1 - 1/2}{\gamma - t}.$$

If $\gamma > t$, then the above lies between 0 and $\pi/2$, while if $\gamma < t$, then it lies between $-\pi/2$ and 0. In either case, the contribution is bounded, and there are $\ll (\log \tau)/\log\log\tau$ summands by Lemma 13.19, so we have the result. □

Although a lower bound for $|\zeta(s)|$ at all heights is out of the question, we can show, assuming RH, that there are heights for which a lower bound can be established.

Theorem 13.22 *Assume RH. There is an absolute constant C such that for every $T \geq 4$ there is a t, $T \leq t \leq T + 1$, such that*

$$|\zeta(s)| \;\geq\; \exp\left(\frac{-C \log T}{\log\log T}\right)$$

uniformly for $-1 \leq \sigma \leq 2$.

Proof By Corollary 10.5 we see that if $-1 \leq \sigma \leq 1/2$, then $|\zeta(s)| \gg |\zeta(1 - \sigma + it)|$. Thus we may restrict our attention to $1/2 \leq \sigma \leq 2$. Put $\sigma_1 = 1/2 + 1/\log\log T$. From Corollary 13.16 we have the desired lower bound for all heights, for $\sigma_1 \leq \sigma \leq 2$. For the remaining interval, $I = [1/2, \sigma_1]$, we show that

$$\int_T^{T+1} \log \frac{1}{\min_{\sigma\in I} |\zeta(s)|}\, dt \ll \frac{\log T}{\log\log T}. \tag{13.46}$$

Put $s_1 = \sigma_1 + it$. Then

$$\log|\zeta(s)| = \log|\zeta(s_1)| - \int_\sigma^{\sigma_1} \Re \frac{\zeta'}{\zeta}(\alpha + it)\, d\alpha.$$

By Corollary 13.16 and Lemma 13.20, this is

$$= -\int_\sigma^{\sigma_1} \sum_{\substack{\rho \\ |\gamma - t|\le\delta}} \Re \frac{1}{\alpha + it - \rho}\, d\alpha + O\left(\frac{\log T}{\log\log T}\right)$$

where $\delta = 1/\log\log T$. The summands are non-negative, so the above is

$$\ge -\int_{1/2}^{\sigma_1} \sum_{\substack{\rho \\ |\gamma - t|\le\delta}} \Re \frac{1}{\alpha + it - \rho}\, d\alpha + O\left(\frac{\log T}{\log\log T}\right).$$

Since this lower bound applies for all $\sigma \in I$, the above provides a lower bound for $\log \min_{\sigma\in I} |\zeta(s)|$. We note that

$$\begin{aligned}\int_{1/2}^{\sigma_1}\int_{\gamma-\delta}^{\gamma+\delta} \Re \frac{1}{\alpha + it - \rho}\, dt\, d\alpha &= \int_0^\delta \int_{-\delta}^{\delta} \frac{x}{x^2+y^2}\, dy\, dx \\ &\le \int_{-\pi/2}^{\pi/2}\int_0^{2\delta} \frac{r\cos\theta}{r^2}\, r dr\, d\theta = 4\delta.\end{aligned}$$

Hence

$$\int_T^{T+1}\int_{1/2}^{\sigma_1} \sum_{\substack{\rho \\ |\gamma - t|\le\delta}} \Re \frac{1}{\alpha + it - \rho}\, d\alpha\, dt \ll \sum_{\substack{\rho \\ T-1\le\gamma\le T+2}} \delta \ll \frac{\log T}{\log\log T},$$

so we have (13. 46), and the proof is complete. □

By Theorem 5.2 and Corollary 5.3 with $\sigma_0 = 1 + 1/\log x$ and $1 \le T \le x$, we see that

$$M(x) = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{x^s}{\zeta(s)s}\, ds \ + O\left(\frac{x\log x}{T}\right). \tag{13.47}$$

By Corollary 13.16 we see (assuming RH) that $|\zeta(1/2+\varepsilon+it)| \gg_\varepsilon \tau^{-\varepsilon}$. Hence, by moving the contour to the abscissa $1/2 + \varepsilon$, we deduce that $M(x) \ll_\varepsilon x^{1/2+\varepsilon}$. This can be made more precise, by determining ε as a function of x, but in order to do so we need a lower bound for $|\zeta(s)|$ when $1/2 < \sigma \le 1/2 + 1/\log\log\tau$.

Theorem 13.23 *Assume RH. There is a constant $C > 0$ such that if $|t| \ge 1$, then*

$$\left|\frac{1}{\zeta(s)}\right| \le \begin{cases} \exp\left(\frac{C\log\tau}{\log\log\tau}\right) & \text{for } \sigma \ge 1/2 + 1/\log\log\tau, \\ \exp\left(\frac{C\log\tau}{\log\log\tau}\log\frac{e}{(\sigma-1/2)\log\log\tau}\right) & \text{for } 1/2 < \sigma \le 1/2 + 1/\log\log\tau. \end{cases}$$

Proof The first part follows from Corollary 13.14. Let σ_1 and s_1 be defined as in the proof of Lemma 13.20, and suppose that $1/2 < \sigma \le \sigma_1$. Then

$$\log\zeta(s) = \log\zeta(s_1) - \int_\sigma^{\sigma_1}\frac{\zeta'}{\zeta}(\alpha + it)\,d\alpha.$$

Here the first term on the right is $\ll (\log\tau)/\log\log\tau$, by Corollary 13.16. By Lemma 13.19 we know that the sum in Lemma 13.20 has $\ll (\log\tau)/\log\log\tau$ terms. Since each term has absolute value $\le 1/(\sigma - 1/2)$, it follows that

$$\frac{\zeta'}{\zeta}(\alpha + it) \ll \frac{\log\tau}{(\alpha - 1/2)\log\log\tau}$$

for $1/2 < \alpha \le \sigma_1$. Hence

$$\log\zeta(s) \ll \left(1 + \log\frac{\sigma_1 - 1/2}{\sigma - 1/2}\right)\frac{\log\tau}{\log\log\tau},$$

which gives the stated bound. □

Theorem 13.24 *Assume RH. Then there is an absolute constant $C > 0$ such that*

$$M(x) \ll x^{1/2}\exp\left(\frac{C\log x}{\log\log x}\right)$$

for $x \ge 4$.

Proof Put $\sigma_1 = 1/2 + 1/\log\log x$, and let $\mathcal{C}$ denote the contour that passes by straight line segments from $\sigma_0 - ix$ to $\sigma_1 - ix$ to $\sigma_1 + ix$ to $\sigma_0 + ix$. Then

$$\int_{\sigma_0 - ix}^{\sigma_0 + ix}\frac{x^s}{\zeta(s)s}\,ds = \int_{\mathcal{C}}\frac{x^s}{\zeta(s)s}\,ds,$$

since the integrand is analytic in the rectangle enclosed by these contours. By the first case of Theorem 13.22 we see that

$$\int_{\sigma_1 + ix}^{\sigma_0 + ix}\frac{x^s}{\zeta(s)s}\,ds \ll \exp\left(\frac{C\log x}{\log\log x}\right)\int_{\sigma_1}\sigma_0 x^{\sigma - 1}\,d\sigma \ll \exp\left(\frac{C\log x}{\log\log x}\right),$$

and the same estimate applies to the integral from $\sigma_1 - ix$ to $\sigma_0 - ix$. Similarly, by the second part of Theorem 13.22 we see that

$$\int_{\sigma_1 - ix}^{\sigma_1 + ix}\frac{x^s}{\zeta(s)s}\,ds \ll x^{\sigma_1}\int_0^x \exp\left(\frac{C\log\tau}{\log\log\tau}\log\frac{e\log\log x}{\log\log\tau}\right)\frac{dt}{\tau}.$$

https://doi.org/10.1017/CBO9780511618314.015 Published online by Cambridge University Press

By logarithmic differentiation we may confirm that the argument of the exponential is an increasing function of t for $0 \le t \le x$. Thus we obtain the stated bound by taking $T = x$ in (13.47). □

13.2.1 Exercises

1. (a) Show (unconditionally) that
$$\Re \frac{\xi'}{\xi}(s) = \sum_\rho \Re \frac{1}{s-\rho}$$
whenever $\xi(s) \ne 0$.

(b) Show (unconditionally) that
$$\Re \frac{\xi'}{\xi}(1/2+it) = 0$$
for all t such that $\xi(1/2+it) \ne 0$.

(c) Assume RH. Show that
$$\Re \frac{\xi'}{\xi}(s) \begin{cases} > 0 & \text{if } \sigma > 1/2, \\ = 0 & \text{if } \sigma = 1/2 \text{ and } \xi(s) \ne 0, \\ < 0 & \text{if } \sigma < 1/2. \end{cases}$$

(d) Assume RH. Show that if $\xi'(s) = 0$, then $\Re s = 1/2$.

(e) Assume RH, and let t be any fixed real number. Show that $|\xi(\sigma + it)|$ is a strictly increasing function of σ for $1/2 \le \sigma < \infty$, and that $|\xi(\sigma + it)|$ is a strictly decreasing function of σ for $-\infty < \sigma \le 1/2$.

(f) Assume RH, and suppose that t is a fixed real number. Show that $(\sigma - 1/2)\Re\frac{\xi'}{\xi}(\sigma+it)$ is an increasing function of σ for $1/2 \le \sigma < \infty$.

(g) Assume RH. Show that if $1/2 < \sigma_2 \le \sigma_1$, then
$$|\xi(\sigma_2+it)| \ge |\xi(\sigma_1+it)| \cdot \left(\frac{\sigma_2 - 1/2}{\sigma_1 - 1/2}\right)^{(\sigma_1 - 1/2)\Re\frac{\xi'}{\xi}(\sigma_1+it)}.$$

2. (a) Show (unconditionally) that if $\xi(s) \ne 0$, then
$$\frac{\xi''}{\xi}(s) - \left(\frac{\xi'}{\xi}(s)\right)^2 = -\sum_\rho \frac{1}{(s-\rho)^2}.$$

(b) Show (unconditionally) that if t is real, then $\xi'(1/2+it) \in i\mathbb{R}$.

(c) Show (unconditionally) that if t is real, then $\xi''(1/2+it) \in \mathbb{R}$.

(d) Show (unconditionally) that if t is real, then
$$\sum_\rho \frac{1}{(1/2+it-\rho)^2}$$
is real.

(e) Assume RH. Show that if $\xi(1/2+it) \ne 0$, then

$$\frac{\xi''}{\xi}(1/2+it) > \left(\frac{\xi'}{\xi}\right)^2 (1/2+it).$$

(f) Assume RH. Show that if $\xi(1/2+it) \ne 0$ and $\xi'(1/2+it) = 0$, then $\operatorname{sgn} \xi''(1/2+it) = \operatorname{sgn} \xi(1/2+it)$.

(g) Assume RH. Show that if $\xi(1/2+it) \ne 0$ and $\xi'(1/2+it) = 0$, then

$$\operatorname{sgn} \frac{\partial^2}{\partial t^2}\xi(1/2+it) = -\operatorname{sgn} \xi(1/2+it).$$

(h) Assume RH. Suppose that $\xi(1/2+i\gamma) = \xi(1/2+i\gamma') = 0$, and that $\xi(1/2+it) \ne 0$ for $\gamma < t < \gamma'$. Show that $\xi'(1/2+it)$ has exactly one zero with $\gamma < t < \gamma'$, and that this zero is necessarily simple.

(i) Assume RH. In the above notation, show that the number of zeros of $\xi'(1/2+it)$ in the interval $[\gamma, \gamma')$, counting multiplicity, is the same as the number of zeros of $\xi(1/2+it)$ in the same interval.

(j) Assume RH. Let $N_1(T)$ denote the number of zeros of $\xi'(s)$ with imaginary part in the interval $[0, T]$. Show that $N_1(T) = N(T) + O(1)$.

3. Let χ be a primitive character modulo q, $q > 1$, and suppose that $L(s,\chi) \ne 0$ for $\sigma > 1/2$. Show that

$$\left|\frac{L'}{L}(s,\chi)\right| \le \sum_{n \le (\log q\tau)^2} \frac{\Lambda(n)}{n^\sigma} + O\left(\frac{(\log q\tau)^{2-2\sigma}}{\log\log\tau}\right)$$

uniformly for $1/2 + 1/\log\log q\tau \le \sigma \le 3/2$.

4. Let χ be a primitive character modulo q, $q > 1$, and suppose that $L(s,\chi) \ne 0$ for $\sigma > 1/2$. Show that

$$\frac{L'}{L}(s,\chi) \ll ((\log q\tau)^{2-2\sigma} + 1)\min\left(\frac{1}{|\sigma-1|}, \log\log q\tau\right)$$

uniformly for $1/2 + 1/\log\log q\tau \le \sigma \le 3/2$.

5. Let χ be a primitive character modulo q, $q > 1$, and suppose that $L(s,\chi) \ne 0$ for $\sigma > 1/2$. Show that

$$|\log L(s,\chi)| \le \sum_{n \le (\log q\tau)^2} \frac{\Lambda(n)}{n^\sigma \log n} + O\left(\frac{(\log q\tau)^{2-2\sigma}}{\log\log q\tau}\right)$$

uniformly for $1/2 + 1/\log\log q\tau \le \sigma \le 3/2$.

6. Let χ be a primitive character modulo q, $q > 1$, and suppose that $L(s,\chi) \ne 0$ for $\sigma > 1/2$.

(a) Show that

$$|L(s,\chi)| \le \log\frac{1}{\sigma-1} + O(\sigma-1)$$

uniformly for $1 + 1/\log\log q\tau \le \sigma \le 3/2$.

(b) Show that

$$|L(s,\chi)| \le \log\log q\tau + O(1)$$

uniformly for $1 - 1/\log\log q\tau \le \sigma \le 1 + 1/\log\log q\tau$.

(c) Show that

$$|L(s,\chi)| \le \log\frac{1}{1-\sigma} + O\left(\frac{(\log q\tau)^{2-2\sigma}}{(1-\sigma)\log\log q\tau}\right)$$

uniformly for $1/2 + 1/\log\log q\tau \le \sigma \le 1 - 1/\log\log q\tau$.

7. Let χ be a primitive character modulo q, $q > 1$, and suppose that $L(s,\chi) \ne 0$ for $\sigma > 1/2$. Show that $|L(1+it,\chi)| \le 2e^{C_0}\log\log q\tau$.

8. Let χ be a primitive Dirichlet character modulo q with $q > 1$, and suppose that $L(s,\chi) \ne 0$ for $\sigma > 1/2$. Show that there is an absolute constant $C > 0$ such that

$$|L(s,\chi)| \le \exp\left(\frac{C\log q\tau}{\log\log q\tau}\right)$$

uniformly for $1/2 \le \sigma \le 3/2$.

9. Let χ be a primitive character modulo q, $q > 1$, and suppose that $L(s,\chi) \ne 0$ for $\sigma > 1/2$. Show that the number of zeros $\rho = 1/2 + i\gamma$ of $L(s,\chi)$ with $T \le \gamma \le T + 1/\log\log q\tau$ is $\ll (\log q\tau)/(\log\log q\tau)$ uniformly in T.

10. Let χ be a primitive character modulo q, $q > 1$, and suppose that $L(s,\chi) \ne 0$ for $\sigma > 1/2$. Show that if $|\sigma - 1/2| \le 1/\log\log q\tau$, then

$$\frac{L'}{L}(s,\chi) = \sum_{|\gamma - t| \le 1/\log\log q\tau} \frac{1}{s-\rho} + O(\log q\tau).$$

11. (Selberg 1946b, Section 5) Let χ be a primitive character modulo q, $q > 1$, and suppose that $L(s,\chi) \ne 0$ for $\sigma > 1/2$. Show that

$$\arg L(s,\chi) \ll \frac{\log q\tau}{\log\log q\tau}$$

uniformly for $\sigma \ge 1/2$.

12. Let χ be a character modulo q, and suppose that χ is induced by a primitive character $\chi^\star$ where $\chi^\star$ is a character modulo d for some $d|q$. Show that

$$\frac{L'}{L}(s,\chi) - \frac{L'}{L}(s,\chi^\star) \ll \big((\log q)^{1-\sigma} + 1\big)\min\left(\frac{1}{|\sigma - 1|}, \log\log q\right).$$

13. (Vorhauer 2006) Let χ be a primitive character modulo q, $q > 1$, and suppose that $L(s,\chi) \ne 0$ for $\sigma > 1/2$. Show that

$$\lim_{T\to\infty} \sum_{|r| \le T} \frac{1}{\rho} = \frac{1}{2}\log q + O(\log\log q).$$

14. (Axer 1911) Assume RH.
   (a) Show that if $c = 1/4 + \varepsilon$, then
$$\int_{c-iT}^{c+iT} \left|\frac{\zeta(s)x^s}{\zeta(2s)s}\right| |ds| \ll x^{1/4+\varepsilon}T^{1/4+\varepsilon}.$$
   (b) Let $Q(x)$ denote the number of square-free integers not exceeding x. Show that if RH is true, then
$$Q(x) = \frac{6}{\pi^2}x + O\big(x^{2/5+\varepsilon}\big).$$
   (A better estimate is obtained in Exercise 16 below.)
15. Assume RH.
   (a) Show that if $c = 1/2 + \varepsilon$, then
$$\int_{c-iT}^{c+iT} \left|\frac{\zeta(s)x^s}{\zeta(2s)s(s+1)}\right| |ds| \ll x^{1/4+\varepsilon}T^{\varepsilon}.$$
   (b) Show that if RH is true, then
$$\sum_{n\le x} \mu(n)^2(1 - n/x) = \frac{3}{\pi^2}x + O\big(x^{1/4+\varepsilon}\big).$$
16. (Montgomery & Vaughan 1981)
   (a) Show that
$$Q(x) = \sum_{\substack{d,m\\ d^2m\le x}} \mu(d).$$
   Let Σ_1 denote the sum of the above terms for which $d \le y$, and let Σ_2 denote the sum of the above terms for which $d > y$. Here y is a parameter to be determined later, $1 \le y \le x^{1/2}$.
   (b) Put
$$S(x, y) = \sum_{d\le y} \mu(d)B_1(x/d^2)$$
   where $B_1(u) = u - 1/2$ is the first Bernoulli polynomial. Show that
$$\Sigma_1 = x\sum_{d\le y}\frac{\mu(d)^2}{d} - \frac{1}{2}M(y) - S(x, y).$$
   (c) Assume RH. Show that if $\sigma \ge 1/2 + 2\varepsilon$, then
$$\sum_{d\le y}\frac{\mu(d)}{d^s} = \frac{1}{2\pi i}\int_{\mathcal{C}_0}\frac{y^{w-s}}{\zeta(w)(w-s)}\,dw$$

where $\mathcal{C}_0$ is a contour running from $\sigma_0 - i\infty$ to $\sigma_0 - iy$ to $1/2 + \varepsilon - iy$ to $1/2 + \varepsilon + iy$ to $\sigma_0 + iy$ to $\sigma_0 + i\infty$ and $\sigma_0 = 1 + 1/\log y$. Deduce that

$$\sum_{d \le y} \frac{\mu(d)}{d^s} = \frac{1}{\zeta(s)} + O\big(y^{1/2-\sigma+\varepsilon}\tau^{\varepsilon}\big).$$

(d) Put $f_y(s) = 1/\zeta(s) - \sum_{d \le y} \mu(d)/d^s$. Show that

$$\Sigma_2 = \frac{1}{2\pi i}\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \zeta(s) f_y(2s) \frac{x^s}{s}\, ds$$

where $\sigma_1 = 1 + 1/\log x$.

(e) Show (unconditionally) that

$$\Sigma_2 = f_y(2) + \frac{1}{2\pi i}\int_{\mathcal{C}_1} \zeta(s) f_y(2s) \frac{x^s}{s}\, ds$$

where $\mathcal{C}_1$ is a contour running from $\sigma_1 - i\infty$ to $\sigma_1 - ix$ to $1/2 - ix$ to $1/2 + ix$ to $\sigma_1 + ix$ to $\sigma_1 + i\infty$.

(f) Assume RH. Show that $\Sigma_2 \ll x^{1/2+\varepsilon} y^{-1/2}$.

(g) Note that the estimate $S(x, y) \ll y$ is trivial.

(h) Show that if RH is true, then

$$Q(x) = \frac{6}{\pi^2}x + O\big(x^{1/3+\varepsilon}\big).$$

13.3 Notes

Section 13.1. Theorem 13.1 is due to von Koch (1901). Theorems 13.3 and 13.5 are due to Cramér (1921). The order of magnitude of the estimate in Theorem 13.5 is optimal, in view of Theorem 13.6, which is from Cramér (1922). Wintner (1941) showed (assuming RH) that the function $f(u)$ defined in (13.17) has a limiting distribution. That is, there is a weakly monotonic function $F(x)$ with $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to+\infty} F(x) = 1$, such that

$$\lim_{U\to\infty} \frac{1}{U}\operatorname{meas}\{u \in [0, U] : f(u) \le x\} = F(x)$$

whenever x is a point of continuity of F. The result of Exercise 13.1.4 is useful in this connection. If in addition to RH, the ordinates $\gamma > 0$ are linearly independent over the field $\mathbb{Q}$ of rational numbers, then this distribution function is the same as the distribution function of the random variable

$$X = 2\sum_{\gamma > 0} \frac{\cos 2\pi X_\gamma}{\rho}$$

https://doi.org/10.1017/CBO9780511618314.015 Published online by Cambridge University Press

where the X_γ are independent random variables, each one uniformly distributed on [0, 1]. It can be shown (unconditionally) that the distribution function F_X of X satisfies the inequalities

$$\exp\left(-c_1\sqrt{x}e^{\sqrt{2\pi x}}\right) < 1 - F_X(x) < \exp\left(-c_2\sqrt{x}e^{\sqrt{2\pi x}}\right) \quad (13.48)$$

for $x \ge 2$ where c_1 and c_2 are positive absolute constants.

Concerning the mean square distribution of primes in short intervals, Selberg (1943) showed (assuming RH) that

$$\int_0^X (\psi((1+\delta)x) - \psi(x) - \delta x)^2 \, \frac{dx}{x^2} \ll \delta(\log X)^2$$

uniformly for $1/X \le \delta \le 1/\log X$. Theorem 13.7 and Corollary 13.8 are due to Titchmarsh (1930). Corollary 13.10 is due to Turán (1937). Theorem 13.11, in the case of the Legendre symbol, is due to Ankeny (1952), who used deeper estimates of Selberg (1946b) found in Exercise 13.1.11. Our simpler proof, and the extension to general non-principal characters, is from Montgomery (1971, p. 120). Theorem 13.12 is from Montgomery (1994, p. 164). See also Lagarias, Montgomery & Odlyzko (1979).

Section 13.2. All results here from Theorem 13.13 through Theorem 13.21 are due to Littlewood (1922, 1924b, 1926, 1928), although our proofs are much simpler than in the original ones. Indeed, referring to Theorem 13.21, Littlewood commented that, 'The proof of this theorem is long and difficult, and depends on a singularly varied set of ideas.' Precursors to Theorem 13.21 were established by Bohr, Landau & Littlewood (1913), Cramér (1918), and Landau (1920). See Titchmarsh (1927) for an alternative proof. Our simpler approach is that of Selberg (1944). Littlewood (1928) not only established Corollary 13.17, but also showed (assuming RH) that

$$|\zeta(1+it)| \ge \frac{\pi^2}{12e^{C_0}\log\log\tau} + O((\log\log\tau)^{-2}).$$

In the opposite direction, Titchmarsh (1928) showed (unconditionally) that

$$\limsup_{t\to+\infty} \frac{|\zeta(1+it)|}{\log\log t} \ge e^{C_0}.$$

Also, Titchmarsh (1933) showed (unconditionally) that

$$\liminf_{t\to+\infty} |\zeta(1+it)| \log\log t \ge \frac{\pi^2}{6e^{C_0}}.$$

Here we see a factor of 2 between the two sets of bounds. The same factor of 2 arises when we consider what is known concerning large values of the zeta function in the critical strip. Let $\alpha(\sigma)$ denote the least number such that

$$\zeta(\sigma + it) \ll \exp\left((\log \tau)^{\alpha(\sigma)+\varepsilon}\right)$$

as $t \to \infty$. From Corollary 13.16 we see that $\alpha(\sigma) \le 2 - 2\alpha$, assuming RH. In the opposite direction, Titchmarsh (1928) showed (unconditionally) that $\alpha(\sigma) \ge 1 - \alpha$. More precisely, it is known that if $1/2 \le \sigma < 1$, then there is a $c(\sigma) > 0$ such that

$$|\zeta(\sigma + it)| = \Omega\left(\exp\left(\frac{c(\sigma)(\log \tau)^{1-\sigma}}{(\log\log \tau)^{\sigma}}\right)\right).$$

For $1/2 < \sigma < 1$ this is due to Montgomery (1977); the case $\sigma = 1/2$ is due to Balasubramanian & Ramachandra (1977). Opinions as to where the truth lies between these bounds vary widely among experts. For more on the value distribution of the zeta function and L-functions, see Titchmarsh (1986), Joyner (1986), and Laurinčikas (1996).

That the estimate $M(x) \ll x^{1/2+\varepsilon}$ is equivalent to RH was proved by Littlewood (1912). Theorems 13.22 through 13.24 are due to Titchmarsh (1927). Theorem 13.24 has been improved upon by Maier & Montgomery (2006), who showed (assuming RH) that

$$M(x) \ll x^{1/2} \exp\left((\log x)^{39/61}\right).$$

13.4 References

Ankeny, N. C. (1952). The least quadratic non residue, *Ann. of Math.* **55**, 65–72.

Axer, A. (1911). Über einige Grenzwertsätze, *S.-B. Wiss. Wien* IIa **120**, 1253–1298.

Balasubramanian, R. & Ramachandra, K. (1977). On the frequency of Titchmarsh's phenomenon for $\zeta(s)$, III, *Proc. Indian Acad. Sci. Sect.* A **86**, 341–351.

Bohr, H., Landau, E., & Littlewood, J. E. (1913). Sur la fonction $\zeta(s)$ dans le voisinage de la droite $\sigma = 1/2$, *Acad. Roy. Belgique Bull. Cl. Sci.*, 1144–1175; *Bohr's Collected Works*, Vol. 1. København: Dansk Mat. Forening, 1952, B.2; *Landau's Collected Works*, Vol. 6. Essen: Thales Verlag, 1986, pp. 61–93; *Littlewood's Collected Papers*, Vol. 2. Oxford: Oxford University Press, 1982, pp. 797–828.

Cramér, H. (1918). Über die Nullstellen der Zetafunktion, *Math. Z.* **2**, 237–241; *Collected Works*, Vol. 1. Berlin: Springer-Verlag, 1994, 92–96.

(1921). Some theorems concerning prime numbers, *Arkiv för Mat. Astr. Fys.* **15**, no. 5, 33 pp.; *Collected Works*, Vol. 1. Berlin: Springer-Verlag, 1994, pp. 138–170.

(1922). Ein Mittelwertsatz der Primzahltheorie, *Math. Z.* **12**, 147–153; *Collected Works*, Vol. 1. Berlin: Springer-Verlag, 1994, pp. 229–235.

Goldston, D. A. (1982). On a result of Littlewood concerning prime numbers, *Acta Arith.* **40**, 263–271.

Joyner, D. (1986). *Distribution Theorems of L-functions*, Pitman Research Notes in Math. 142. Harlow: Longman.

von Koch, H. (1901). Sur la distribution des nombres premiers, *Acta Math.* **24**, 159–182.

Lagarias, J. C., Montgomery, H. L., & Odlyzko, A. M. (1979). A bound for the least prime ideal in the Chebotarev density theorem, *Invent. Math.* **54**, 271–296.

Landau, E. (1920). Über die Nullstellen der Zetafunktion, *Math. Z.* **6**, 151–154; *Collected Works*, Vol. 7. Essen: Thales Verlag, 1986, pp. 226–229.

Laurinčikas, A. (1996). *Limit Theorems for the Riemann Zeta-function*, Mathematics and its Applications 352. Dordrecht: Kluwer.

Littlewood, J. E. (1912). Quelques conséquences de l'hypothèse que la fonction $\zeta(s)$ de Riemann n'a pas de zéros dans le demi-plan $\mathrm{R}(s) > \frac{1}{2}$, *Comptes Rendus Acad. Sci. Paris* **154**, 263–266; *Collected Papers*, Vol. 2. Oxford: Oxford University Press, 1882, pp. 793–796.

(1922). Researches in the theory of the Riemann ζ-function, *Proc. London Math. Soc.* (2) **20**, xxii–xxviii; *Collected Papers*, Vol. 2. Oxford: Oxford University Press, 1982, pp. 844–850.

(1924a). Two notes on the Riemann Zeta-function, *Proc. Cambridge Philos. Soc.* **22**, 234–242; *Collected Papers*, Vol. 2. Oxford: Oxford University Press, 1982, pp. 851–859.

(1924b). On the zeros of the Riemann zeta-function, *Proc. Cambridge Philos. Soc.* **22**, 295–318; *Collected Papers*, Vol. 2. Oxford: Oxford University Press, 1982, pp. 860–883.

(1926). On the Riemann zeta function, *Proc. London Math. Soc.* (2) **24**, 175–201; *Collected Papers*, Vol. 2. Oxford: Oxford University Press, 1982, pp. 844–910.

(1928). Mathematical Notes (5): On the function $1/\zeta(1+ti)$, *Proc. London Math. Soc.* (2) **27**, 349–357; *Collected Papers*, Vol. 2, Oxford: Oxford University Press, 1982, pp. 911–919.

Maier, H. & Montgomery, H. L. (2006). *On the sum of the Möbius function*, to appear, 16 pp.

Montgomery, H. L. (1971). *Topics in Multiplicative Number Theory*, Lecture Notes in Math. 227. Berlin: Springer-Verlag.

(1977). Extreme values of the Riemann zeta-function, *Comment. Math. Helv.* **52**, 511–518.

(1994). *Ten lectures on the interface between analytic number theory and harmonic analysis*, CMBS 84. Providence: Amer. Math. Soc.

Montgomery, H. L. & Vaughan, R. C. (1981). The distribution of square-free numbers, *Recent Progress in Analytic Number Theory* (Durham, 1979), Vol. 1. London: Academic Press, pp. 247–256.

Selberg, A. (1943). On the normal density of primes in small intervals, *Arch. Math. Natur-vid.* **47**, 87–105; *Collected Papers*, Vol. 1, New York: Springer Verlag, 1989, pp. 160–178.

(1944). On the Remainder in the Formula for $N(T)$, the Number of Zeros of $\zeta(s)$ in the Strip $0 < t < T$. *Avhandl. Norske Vid.-Akad. Oslo I. Mat.-Naturv. Kl.*, no. 1; *Collected Papers*, Vol. 1, New York: Springer Verlag, 1989, pp. 179–203.

(1946a). Contributions to the Theory of the Riemann zeta-function, *Arch. Math. Naturvid.* **48**, 89–155; *Collected Papers*, Vol. 1, New York: Springer Verlag, 1989, pp. 214–280.

(1946b). Contributions to the Theory of Dirichlet's L-functions, *Skrifter Norske Vid.-Akad. Oslo I. Mat.-Naturvid. Kl.*, no. 3; *Collected Papers*, Vol. 1, New York: Springer Verlag, 1989, pp. 281–340.

Titchmarsh, E. C. (1927). A consequence of the Riemann hypothesis, *J. London Math. Soc.* **2**, 247–254.

(1928). On an inequality satisfied by the zeta-function of Riemann, *Proc. London Math. Soc.* (2) **28**, 70–80.

(1930). A divisor problem, *Rend. Circ. Mat. Palermo* **54**, 414–429.

(1933). On the function $1/\zeta(1+it)$, *Quart. J. Math.* Oxford **4**, 64–70.

(1986). *The Theory of the Riemann Zeta-function*, Second edition. Oxford: Oxford University Press.

Turán, P., (1937). Über die Primzahlen der Arithmetischen Progression, I, *Acta Sci. Szeged* **8**, 226–235; *Collected Papers*, Vol. 1. Budapest: Akadémiai Kiadó, 1990, pp. 64–73.

Vorhauer, U. M. A. (2006). *The Hadamard product formula for Dirichlet L-functions*, to appear.

Wintner, A. (1941). On the distribution function of the remainder term of the Prime Number Theorem, *Amer. J. Math.* **63**, 233–248.