12
Explicit formulæ

12.1 Classical formulæ

When we proved the Prime Number Theorem, we confined the contour of integration to the zero-free region. If we pull the contour further to the left, then we encounter a number of poles that leave residues, and thus we can express the error term in the Prime Number Theorem as a sum over the zeros of $\zeta(s)$. Let $\psi_0(x) = (\psi(x^+) + \psi(x^-))/2$. By applying Perron's formula (Theorem 5.1) to the Dirichlet series $-\frac{\zeta'}{\zeta}(s) = \sum_n \Lambda(n)n^{-s}$, we see that

$$\psi_0(x) = \lim_{T\to\infty} \frac{-1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s}\, ds.$$

Here the integrand has a pole at $s = 1$, at zeros ρ, at $s = 0$, and at the trivial zeros $-2k$. Since x^s decays very rapidly as $\sigma \to -\infty$, it is reasonable to expect that we can pull the contour to the left, and thus show that the above is

$$= x - \lim_{T\to\infty} \sum_{\substack{\rho \\ |\gamma|\le T}} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) + \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k}. \tag{12.1}$$

Here $\frac{\zeta'}{\zeta}(0) = \log 2\pi$ by (10.11) and (10.14), and the sum over the trivial zeros is

$$-\frac{1}{2}\log(1 - 1/x^2)\,,$$

which is continuous and tends to 0 as $x \to \infty$. In order to give a rigorous proof of the above, we first establish estimates for $\frac{\zeta'}{\zeta}(s)$.

Lemma 12.1 *We have*

$$\frac{\zeta'}{\zeta}(s) = \frac{-1}{s-1} + \sum_{\substack{\rho \\ |\gamma - t|\le 1}} \frac{1}{s-\rho} + O(\log\tau) \tag{12.2}$$

uniformly for $-1 \le \sigma \le 2$.

Here the first term on the right is significant only for $|t| \le 1$. We could prove the above by the same method that we used to prove Lemma 6.4, but we find it instructive to argue instead from Corollary 10.14.

Proof By combining (10.29) and Theorem C.1, it is immediate that

$$\frac{\zeta'}{\zeta}(s) = \frac{-1}{s-1} + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{1}{2}\log\tau + O(1).$$

On applying this at $\sigma + it$ and at $2 + it$, and differencing, it follows that

$$\frac{\zeta'}{\zeta}(s) = \frac{-1}{s-1} + \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) + O(1).$$

By Theorem 10.13 it is clear that

$$\sum_{\substack{\rho \\ |\gamma - t| \le 1}} \frac{1}{2+it-\rho} \ll \sum_{\substack{\rho \\ |\gamma - t| \le 1}} 1 \ll \log\tau.$$

Now suppose that n is a positive integer, and consider those zeros ρ for which $n \le |\gamma - t| \le n+1$. Since

$$\frac{1}{s-\rho} - \frac{1}{2+it-\rho} = \frac{2-\sigma}{(s-\rho)(2+it-\rho)} \ll \frac{1}{n^2},$$

it follows that such zeros contribute an amount

$$\ll \frac{N(t+n+1) - N(t+n) + N(t-n) - N(t-n-1)}{n^2} \ll \frac{\log(\tau+n)}{n^2}.$$

On summing over n we obtain the stated estimate. □

Lemma 12.2 *For each real number $T \ge 2$ there is a T_1, $T \le T_1 \le T+1$, such that*

$$\frac{\zeta'}{\zeta}(\sigma + iT_1) \ll (\log T)^2$$

uniformly for $-1 \le \sigma \le 2$.

Proof By Theorem 10.13, there is a $T_1 \in [T, T+1]$ such that $|T_1 - \gamma| \gg 1/\log T$ for all zeros ρ. Since each summand in (12.2) is $\ll \log T$, and there are $\ll \log T$ summands, the estimate is immediate. □

The next lemma is useful in Chapter 14, but we establish it here since it is a also an immediate corollary of Lemma 12.1.

Lemma 12.3 *For any real number t,*

$$\arg\zeta(\sigma + it) \ll \log\tau$$

uniformly for $-1 \le \sigma \le 2$.

The function $\log\zeta(s)$ has a branch point at $s = 1$, and also at zeros ρ of the zeta function. To obtain a single branch of the logarithm, we remove from the complex plane the interval $(-\infty, 1]$, and also intervals of the form $(-\infty + i\gamma, \beta + i\gamma]$. What remains is simply connected, and in this region we take that branch of $\log\zeta(s)$ for which $\log\zeta(s) \to 0$ as $\sigma \to \infty$. This is the branch of the logarithm that we have expanded as a Dirichlet series, for $\sigma > 1$ (cf. Corollary 1.11). Thus, if t is not the ordinate of a zero, we define $\arg\zeta(s) = \Im\log\zeta(s)$ by continuous variation from $\infty + it$ to $\sigma + it$, which is to say that

$$\arg\zeta(s) = -\int_\sigma^\infty \Im\frac{\zeta'}{\zeta}(\alpha + it)\,d\alpha.$$

If t is the ordinate of a zero then we set $\arg\zeta(s) = (\arg\zeta(\sigma + it^+) + \arg\zeta(\sigma + it^-))/2$.

Proof Suppose that $-1 \le \sigma \le 2$, and that t is not the ordinate of a zero. Then

$$\arg\zeta(\sigma + it) = \arg\zeta(2 + it) - \int_\sigma^2 \Im\frac{\zeta'}{\zeta}(\alpha + it)\,d\alpha.$$

Here $\arg\zeta(2 + it) \ll 1$ uniformly in t, by Corollary 1.11. Thus by Lemma 12.1, the right-hand side above is

$$-\sum_{|\gamma - t| \le 1} \int_\sigma^2 \Im\frac{1}{\alpha + it - \rho}\,d\alpha + O(\log\tau).$$

Here the summand is

$$\arctan\frac{\sigma - \beta}{t - \gamma} - \arctan\frac{2 - \beta}{t - \gamma}.$$

If $t > \gamma$, then this lies between $-\pi$ and 0, while if $t < \gamma$, then the above lies between 0 and π. Thus in any case the quantity is bounded, and by Theorem 10.13 the number of summands is $\ll \log\tau$, so we have the result when t is not the ordinate of a zero. Since the ordinates of zeros have no finite limit point, we obtain the same bound when t is the ordinate of a zero, since in that case $\arg\zeta(s) = (\arg\zeta(\sigma + it^+) + \arg\zeta(\sigma - it^-))/2$. □

Lemma 12.4 *Let $\mathcal{A}$ denote the set of those points $s \in \mathbb{C}$ such that $\sigma \le -1$ and $|s + 2k| \ge 1/4$ for every positive integer k. Then*

$$\frac{\zeta'}{\zeta}(s) \ll \log(|s| + 1)$$

uniformly for $s \in \mathcal{A}$.

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Proof We recall (10.27), in which the first two terms are bounded for $s \in \mathcal{A}$. Also,

$$\frac{\Gamma'}{\Gamma}(1-s) \ll \log(|s|+1)$$

by Theorem C.1. Finally

$$\cot\frac{\pi s}{2} = i + \frac{2i}{e^{i\pi s}-1} \ll 1$$

since s is bounded away from even integers, so we have the result. □

We are now in a position to prove the explicit formula (12.1) in a quantitative form.

Theorem 12.5 *Let c be a constant, $c > 1$, suppose that $x \ge c$, that $T \ge 2$, and let $\langle x\rangle$ denote the distance from x to the nearest prime power, other than x itself. Then*

$$\psi_0(x) = x - \sum_{\substack{\rho \\ |\gamma|\le T}} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1-1/x^2) + R(x,T) \qquad (12.3)$$

where

$$R(x,T) \ll (\log x)\min\left(1, \frac{x}{T\langle x\rangle}\right) + \frac{x}{T}(\log xT)^2. \qquad (12.4)$$

Since $\langle x\rangle > 0$ for all x, we obtain (12.1) by letting $T \to \infty$ in the above. Moreover, if $n_1 < n_2$ are two consecutive prime powers, then from the above we see that $\sum_{|\gamma|\le T} x^\rho/\rho$ converges uniformly for x in an interval of the form $[n_1+\delta, n_2-\delta]$. This sum, of course, cannot be uniformly convergent for x in a neighbourhood of a prime power, since $\psi_0(x)$ has jump discontinuities at such points, but we see from the above that it is boundedly convergent in the neighbourhood of a prime power. The sum over ρ is also convergent when $x = 1$, but it is not boundedly convergent near 1, since $\log(1-1/x^2) \to -\infty$ as $x \to 1^+$.

Proof Let T_1 be the number supplied by Lemma 12.2. Then by Theorem 5.2 and its Corollary 5.3, with $\sigma_0 = 1 + 1/\log x$, we see that

$$\psi_0(x) = \frac{-1}{2\pi i}\int_{\sigma_0-iT_1}^{\sigma_0+iT_1} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds + R_1$$

where

$$R_1 \ll \sum_{\substack{x/2<n<2x \\ n\ne x}} \Lambda(n)\min\left(1, \frac{x}{T|x-n|}\right) + \frac{x}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_0}}.$$

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Here the second sum is $-\frac{\zeta'}{\zeta}(\sigma_0) \asymp 1/(\sigma_0 - 1) = \log x$. In the first sum, the terms for which $x + 1 \le n < 2x$ contribute an amount

$$\ll \sum_{x+1\le n<2x} \frac{x \log x}{T(n-x)} \ll \frac{x}{T}(\log x)^2.$$

The terms for which $x/2 < n \le x - 1$ are handled similarly. Finally, any terms for which $x - 1 < n < x + 1$ contribute an amount

$$\ll (\log x) \min\left(1, \frac{x}{T\langle x\rangle}\right),$$

so

$$R_1 \ll (\log x) \min\left(1, \frac{x}{T\langle x\rangle}\right) + \frac{x}{T}(\log x)^2.$$

Let K denote an odd positive integer, and let $\mathcal{C}$ denote the contour consisting of line segments connecting $\sigma_0 - iT_1$, $-K - iT_1$, $-K + iT_1$, $\sigma_0 + iT_1$. Then by Cauchy's residue theorem,

$$\psi_0(x) = x - \sum_{\substack{\rho \\ |\gamma|<T_1}} \frac{x^\rho}{\rho} + \sum_{1\le k<K/2} \frac{x^{-2k}}{2k} - \frac{\zeta'}{\zeta}(0) + R_1 + R_2$$

where

$$R_2 = \frac{-1}{2\pi i} \int_{\mathcal{C}} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s}\, ds.$$

Since $|\sigma \pm iT_1| \ge T$, we see by Lemma 12.2 that

$$\int_{-1\pm iT_1}^{\sigma_0\pm iT_1} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\, ds \ll \frac{(\log T)^2}{T} \int_{-1}^{\sigma_0} x^\sigma\, d\sigma \ll \frac{x(\log T)^2}{T \log x} \ll \frac{x(\log T)^2}{T}.$$

Similarly, since $(\log|\sigma \pm iT_1|)/|\sigma \pm iT_1| \ll (\log T)/T$, we see by Lemma 12.4 that

$$\int_{-K\pm iT_1}^{-1\pm iT_1} \frac{\zeta'}{\zeta}(s)x^s\, ds \ll \frac{\log T}{T} \int_{-\infty}^{-1} x^\sigma\, d\sigma \ll \frac{\log T}{xT \log x} \ll \frac{\log T}{T}.$$

As $|-K + it| \ge K$, by Lemma 12.4 we also see that

$$\int_{-K-iT_1}^{-K+iT_1} \frac{\zeta'}{\zeta}(s)\, \frac{x^s}{s}\, ds \ll \frac{\log KT}{K} x^{-K} \int_{-T_1}^{T_1} 1\, dt \ll \frac{T \log KT}{Kx^K}.$$

This tends to 0 as $K \to \infty$, so we obtain the stated result. □

Let $\psi_0(x, \chi) = (\psi(x^+, \chi) + \psi(x^-, \chi))/2$. Not surprisingly, our treatment of $\psi_0(x)$ extends readily to provide explicit formulæ for $\psi_0(x, \chi)$.

Lemma 12.6 *Let χ be a primitive character modulo q with $q > 1$. Then*

$$\frac{L'}{L}(s,\chi) = \sum_{\substack{\rho \\ |\gamma - t| \le 1}} \frac{1}{s-\rho} + O(\log q\tau) \tag{12.5}$$

uniformly for $-1 \le \sigma \le 2$.

Proof By combining (10.37) and Theorem C.1, it is immediate that

$$\frac{L'}{L}(s,\chi) = B(\chi) + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + O(\log q\tau).$$

On applying this at $\sigma + it$ and $2 + it$, and differencing, it follows that

$$\frac{L'}{L}(s,\chi) = \sum_{\rho} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) + O(\log q\tau).$$

By Theorem 10.17 it is clear that

$$\sum_{\substack{\rho \\ |\gamma - t| \le 1}} \frac{1}{2+it-\rho} \ll \sum_{\substack{\rho \\ |\gamma - t| \le 1}} 1 \ll \log q\tau.$$

Now suppose that n is a positive integer, and consider those zeros ρ for which $n \le |\gamma - t| \le n + 1$. Since

$$\frac{1}{s-\rho} - \frac{1}{2+it-\rho} = \frac{2-\sigma}{(s-\rho)(2+it-\rho)} \ll \frac{1}{n^2},$$

it follows that such zeros contribute an amount

$$\ll \frac{\log q + \log(|t+n|+2) + \log(|t-n|+2)}{n^2} \ll \frac{\log q(\tau + n)}{n^2}.$$

On summing over n we obtain the stated estimate. □

Lemma 12.7 *Let χ be a primitive character modulo q, and suppose that $T \ge 2$. Then there is a T_1, $T \le T_1 \le T + 1$, such that*

$$\frac{L'}{L}(\sigma \pm iT_1, \chi) \ll (\log qT)^2$$

uniformly for $-1 \le \sigma \le 2$.

Proof By Theorem 10.17, there is a $T_1 \in [T, T+1]$ such that both $|T_1 - \gamma| \gg 1/\log qT$ and $|T_1 + \gamma| \gg 1/\log qT$ for all zeros ρ of $L(s,\chi)$. Since each summand in (12.5) is $\ll \log qT$, and there are $\ll \log qT$ summands, the estimate is immediate. □

Lemma 12.8 *Let χ be a primitive character modulo q, $q > 1$. Then*

$$\arg L(s,\chi) \ll \log q\tau$$

uniformly for $-1 \le \sigma \le 2$.

Proof Suppose that $-1 \le \sigma \le 2$, and that t is not the ordinate of a zero. Then

$$\arg L(\sigma + it, \chi) = \arg L(2 + it, \chi) - \int_\sigma^2 \Im \frac{L'}{L}(\alpha + it, \chi)\, d\alpha.$$

Here $\arg L(2 + it, \chi) \ll 1$ uniformly in t, by Theorem 4.8. Thus by Lemma 12.6, the right-hand side above is

$$-\sum_{|\gamma - t| \le 1} \int_\sigma^2 \Im \frac{1}{\alpha + it - \rho}\, d\alpha + O(\log q\tau).$$

Here the summand is

$$\arctan \frac{\sigma - \beta}{t - \gamma} - \arctan \frac{2 - \beta}{t - \gamma}.$$

If $t > \gamma$, then this lies between $-\pi$ and 0, while if $t < \gamma$, then the above lies between 0 and π. Thus in any case the quantity is bounded, and by Theorem 10.17 the number of summands is $\ll \log \tau$, so we have the result when t is not the ordinate of a zero. Since the ordinates of zeros have no finite limit point, we obtain the same bound when t is the ordinate of a zero, since in that case $\arg L(s, \chi) = (\arg L(\sigma + it^+, \chi) + \arg L(\sigma - it^-, \chi))/2$. □

Lemma 12.9 *Let χ be a primitive character modulo q with $q > 1$, put $\kappa = 0$ or 1 according as $\chi(-1) = 1$ or -1, and let $\mathcal{A}(\kappa)$ denote the set of points $s \in \mathbb{C}$ such that $\sigma \le -1$ and $|s + 2n - \kappa| \ge 1/4$ for each positive integer n. Then*

$$\frac{L'}{L}(s, \chi) \ll \log(2q|s|)$$

uniformly for $s \in \mathcal{A}(\kappa)$.

Proof By (10.35) and Theorem C.1 we see that

$$\frac{L'}{L}(s, \chi) = \frac{\pi}{2} \cot \frac{\pi}{2}(s + \kappa) + O(\log q) + O(\log(|s| + 2)).$$

Here

$$\cot \frac{\pi}{2}(s + \kappa) = i + \frac{2i}{e^{i\pi(s+\kappa)} - 1} \ll 1$$

since s is bounded away from integers with the parity of κ. □

Theorem 12.10 *Let c be a constant, $c > 1$. Suppose that $x \ge c$, that $T \ge 2$, and that χ is a primitive character modulo q with $q > 1$. Then*

$$\psi_0(x, \chi) = -\sum_{\substack{\rho \\ |\gamma| \le T}} \frac{x^\rho}{\rho} - \frac{1}{2}\log(x - 1) - \frac{\chi(-1)}{2}\log(x + 1) + C(\chi) + R(x, T; \chi) \qquad (12.6)$$

where

$$C(\chi) = \frac{L'}{L}(1, \overline{\chi}) + \log \frac{q}{2\pi} - C_0 \tag{12.7}$$

and

$$R(x, T; \chi) \ll (\log x) \min\left(1, \frac{x}{T\langle x\rangle}\right) + \frac{x}{T}(\log qxT)^2. \tag{12.8}$$

Here $\langle x\rangle$ denotes the distance from x to the nearest prime power, other than x itself.

Proof Put $\sigma_0 = 1 + 1/\log x$. By arguing as in the proof of Theorem 12.5, we see that

$$\psi_0(x, \chi) = \frac{-1}{2\pi i}\int_{\sigma_0 - iT_1}^{\sigma_0 + iT_1} \frac{L'}{L}(s, \chi)\frac{x^s}{s}\,ds + R_1$$

where

$$R_1 \ll (\log x) \min\left(1, \frac{x}{T\langle x\rangle}\right) + \frac{x}{T}(\log x)^2.$$

Let K be chosen so that $K - \kappa$ is an odd positive integer, and let $\mathcal{C}$ denote the contour consisting of the line segments connecting $\sigma_0 - iT_1$, $-K - iT_1$, $-K + iT_1$, $\sigma_0 + iT_1$ where T_1 is chosen as in Lemma 12.7. Since K and κ have opposite parity, the line segment from $-K - iT_1$ to $-K + iT_1$ lies in the region $\mathcal{A}(\kappa)$ of Lemma 12.9. Thus by Cauchy's residue theorem,

$$\psi_0(x, \chi) = -\sum_{\substack{\rho \\ |\gamma| < T_1}} \frac{x^\rho}{\rho} + \sum_{1 \le k < (K+\kappa)/2} \frac{x^{\kappa - 2k}}{2k - \kappa} + E + R_1 + R_2$$

where $\kappa = 0$ if $\chi(-1) = 1$ and $\kappa = 1$ if $\chi(-1) = -1$, E is the residue of

$$-\frac{L'}{L}(s, \chi)\frac{x^s}{s}$$

at $s = 0$, and

$$R_2 = \frac{-1}{2\pi i}\int_{\mathcal{C}} \frac{L'}{L}(s, \chi)\frac{x^s}{s}\,ds.$$

By proceeding as in the latter part of the proof of Theorem 12.5, but using now Lemma 12.7 and Lemma 12.9 in place of Lemma 12.2 and Lemma 12.4, we see that

$$R_2 \ll \frac{x}{T}(\log qT)^2 + \frac{T \log qK}{Kx^K}.$$

This last term tends to 0 as $K \to \infty$. Put

$$R_3 = -\sum_{\substack{\rho \\ T<|\gamma|<T_1}} \frac{x^\rho}{\rho}.$$

Then $R(x, T) = R_1 + R_2 + R_3$, and $R_3 \ll xT^{-1} \log qT$ by Theorem 10.17.

It remains to compute the residue E. By logarithmic differentiation of the functional equation in the asymmetric form of Corollary 10.9, we find that

$$\frac{L'}{L}(s, \chi) = -\frac{L'}{L}(1 - s, \overline{\chi}) - \log \frac{q}{2\pi} - \frac{\Gamma'}{\Gamma}(1 - s) + \frac{\pi}{2} \cot \frac{\pi}{2}(s + \kappa) \tag{12.9}$$

If $\chi(-1) = -1$, then $\frac{L'}{L}(s, \chi)$ is analytic at $s = 0$, so

$$E = -\frac{L'}{L}(0, \chi) = \frac{L'}{L}(1, \overline{\chi}) + \log \frac{q}{2\pi} - C_0,$$

in view of (C.11). Since $\cot z$ is an odd function, its Laurent expansion about $z = 0$ is of the form $\cot z = 1/z + \sum_{k=1}^{\infty} c_k z^{2k-1}$. Hence if $\chi(-1) = 1$, we see by (12.8) that the Laurent expansion of $\frac{L'}{L}(s, \chi)$ begins

$$\frac{L'}{L}(s, \chi) = \frac{1}{s} - \frac{L'}{L}(1, \overline{\chi}) - \log \frac{q}{2\pi} + C_0 + \cdots$$

Hence

$$E = -\log x + \frac{L'}{L}(1, \overline{\chi}) + \log \frac{q}{2\pi} - C_0$$

in this case.

Finally, we note that

$$\sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} = -\frac{1}{2} \log(1 - x^{-2}), \qquad \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k-1} = \frac{1}{2} \log \frac{x+1}{x-1}.$$

This completes the proof. □

By letting $T \to \infty$ we immediately obtain

Corollary 12.11 *Suppose that χ is a primitive character modulo q, $q > 1$, and that $x > 1$. Then*

$$\psi_0(x, \chi) = -\sum_{\rho} \frac{x^\rho}{\rho} - \frac{1}{2} \log(x - 1) - \frac{\chi(-1)}{2} \log(x + 1) + C(\chi). \tag{12.10}$$

By Theorem 11.4 we see that $C(\chi) \ll \log q$ if $L(s, \chi)$ has no exceptional zero, and that

$$C(\chi) = \frac{1}{1 - \beta_1} + O(\log q)$$

if $L(s,\chi)$ has the exceptional zero β_1. In this latter case, the sum over ρ includes a large term due to $\rho = 1-\beta_1$. This, however, is largely cancelled by $C(\chi)$, since

$$-\frac{x^{1-\beta_1}-1}{1-\beta_1} = -\frac{\log x}{1-\beta_1}\int_0^{1-\beta_1} x^\sigma\, d\sigma \ll x^{1-\beta_1}\log x. \tag{12.11}$$

This is quite small compared with the contribution $-x^{\beta_1}/\beta_1$ made by $\rho=\beta_1$, not to mention the contributions of other zeros with $\beta \ge 1/2$.

In principle, we could derive an explicit formula for $\psi_0(x,\chi)$ when χ is imprimitive, by taking into account the contributions made by zeros on the imaginary axis. However, we find it simpler to pass from $\psi_0(x,\chi^\star)$ to $\psi_0(x,\chi)$ by elementary reasoning. Suppose that χ is a character modulo q induced by the primitive character $\chi^\star$ modulo d, where $d|q$. (The possibility that $d=1$ is not excluded here.) Then

$$\begin{aligned}\psi_0(x,\chi^\star)-\psi_0(x,\chi) &= \sum_{\substack{p|q\\ p\nmid d}}\ \sum_{\substack{k\\ 1<p^k\le x}} \chi^\star\left(p^k\right)\log p\\ &\ll \sum_{\substack{p|q\\ p\nmid d}}\left[\frac{\log x}{\log p}\right]\log p \\ &\le \omega(q/d)\log x\\ &\ll (\log q/d)(\log x).\end{aligned} \tag{12.12}$$

Note that the distinction between $\psi_0(x,\chi)$ and $\psi(x,\chi)$ can be dropped at this point:

$$\psi(x,\chi) = \psi_0(x,\chi^\star) + O((\log 2q)(\log x)). \tag{12.13}$$

This estimate, though somewhat crude, suffices for most purposes.

The explicit formulæ that we have established thus far arise from Perron's formula. We may similarly derive other explicit formulæ using other kernels in the inverse Mellin transform. Examples of such formulæ are found in Exercises 12.1.5–10. In some cases it may not be so easy to apply complex variable techniques, but for such weighted sums over primes we may use the formulæ above, with integration by parts. For example, from Theorem 12.5 we see that

$$\begin{aligned}\sum_{n\le x} w(n)\Lambda(n) &= \int_{2^-}^x w(u)\,d\psi(u)\\ &= \int_2^x w(u)\,du - \sum_{\substack{\rho\\ |\gamma|\le T}}\int_2^x w(u)u^{\rho-1}\,du + \text{ smaller terms.}\end{aligned}$$

To facilitate the estimation of these 'smaller terms' it is useful to record a little more information concerning the error terms in the truncated explicit formula.

Theorem 12.12 *Suppose that c is a constant, $c > 1$, and let χ be a character modulo q. For $x \ge c$ and $T \ge 2$ there exist functions $E_1(x, \chi)$ and $E_2(x, T, \chi)$ with the following properties:*

$$\psi(x, \chi) = E_0(\chi)x - \sum_{\substack{\rho \\ |\gamma| \le T}} \frac{x^\rho}{\rho} + E_1(x, \chi) + E_2(x, T, \chi); \quad (12.14)$$

$$\int_c^x 1 \, |dE_1(u, \chi)| \ll (\log xq)^2; \quad (12.15)$$

$$E_2(x, T, \chi) \ll \log x + \frac{x}{T}(\log xTq)^2 ; \quad (12.16)$$

$$\int_c^x |E_2(u, T, \chi)| \, du \ll \frac{x^2}{T}(\log xTq)^2. \quad (12.17)$$

Proof Suppose first that χ is non-principal. Thus χ is induced by a primitive character $\chi^\star$ (mod d) where $1 < d \le q$. Put

$$E_1(x, \chi) = \psi_0(x, \chi) - \psi_0(x, \chi^\star) - \frac{1}{2}\log(x - 1) - \frac{\chi(-1)}{2}\log(x + 1) + C(\chi^\star), \quad (12.18)$$

$$E_2(x, T, \chi) = \psi(x, \chi) - \psi_0(x, \chi) + R(x, T; \chi^\star) \quad (12.19)$$

where $R(x, T; \chi^\star)$ is defined by taking $\chi = \chi^\star$ in (12.6). Thus (12.6) gives (12.14). By (12.12) we see that

$$\int_c^x 1 \, |d(\psi_0(u, \chi) - \psi_0(u, \chi^\star))| \ll \sum_{\substack{p|q \\ p\nmid d}} \left[\frac{\log x}{\log p}\right] \log p \ll (\log x)(\log q).$$

Thus we have (12.15). It is also clear that (12.8) gives (12.16). To obtain (12.17), we note that

$$\int_c^x \min\left(1, \frac{u}{T\langle u\rangle}\right) du \le \frac{x}{T} \sum_{p^k \le 2x} \left(1 + \int_{x/T}^x \frac{1}{u}\, du\right) \ll \frac{x^2 \log T}{T \log x}.$$

Since $\psi(x, \chi) - \psi_0(x, \chi) = 0$ except for jump discontinuities at the prime powers, this term makes no contribution to the integral (12.17). Thus we have (12.17).

Now suppose that χ is principal. Put

$$E_1(x, \chi_0) = \psi(x, \chi_0) - \psi_0(x) - \log 2\pi - \frac{1}{2}\log(1 - 1/x^2),$$
$$E_2(x, T, \chi_0) = \psi(x, \chi_0) - \psi_0(x, \chi_0) + R(x, T)$$

where $R(x, T)$ is defined by (12.3). Then the desired assertions follow from (12.3) and (12.4) in the same way as in the former case, so the proof is complete. □

12.1.1 Exercises

1. Suppose that $|s-1| \ge 1$. Show that

$$\log \zeta(s) = \sum_{\substack{\rho \\ |\gamma - t| \le 1}} \log(s-\rho) + O(\log \tau)$$

uniformly for $-1 \le \sigma \le 2$, where $\log \zeta(s)$ is defined by continuous variation along the ray from $\sigma + it$ to $\infty + it$, with $\log \zeta(\infty + it) = 0$, and $|\Im \log(s - \rho)| < \pi$.

2. (a) By using the Brun–Titchmarsh inequality, show that

$$\sum_{x+1 \le n \le 2x} \frac{\Lambda(n)}{n-x} \ll (\log x)(\log\log x).$$

(b) Let R_1 be defined as in the proof of Theorem 12.5. Show that

$$R_1 \ll (\log x) \min\left(1, \frac{x}{T\langle x\rangle}\right) + \frac{x}{T}(\log x)(\log\log x).$$

3. Let δ be a small positive number. For a given $T \ge 4$, let $\mathcal{S} = \{t \in [T, T+1] : \min_\gamma |t-\gamma| \ge \delta/\log T\}$, and for $T \le t \le T+1$ define

$$f(t) = \log T + \sum_{T-1 \le \gamma \le T+2} \frac{1}{|t-\gamma|}$$

where the sum is over ordinates γ of zeros of the zeta function.

(a) Show that if $T \le t \le T+1$, then

$$\max_{-1 \le \sigma \le 2} \left|\frac{\zeta'}{\zeta}(s)\right| \ll f(t).$$

(b) Show that meas $\mathcal{S} \asymp 1$ whenever δ is a sufficiently small positive constant.

(c) Show that

$$\int_{\mathcal{S}} f(t)\,dt \ll (\log T)\log\log T.$$

(d) Deduce that for every $T \ge 4$ there is a $T_1 \in [T, T+1]$ such that

$$\max_{-1 \le \sigma \le 2} \left|\frac{\zeta'}{\zeta}(\sigma + iT_1)\right| \ll (\log T)\log\log T.$$

4. Show that if $s \ne 1$, and $\zeta(s) \ne 0$, then

$$\sum_{n \le x} \frac{\Lambda(n)}{n^s} = \frac{x^{1-s}}{1-s} - \frac{\zeta'}{\zeta}(s) - \sum_\rho \frac{x^{\rho-s}}{\rho - s} + \sum_{k=1}^\infty \frac{x^{-2k-s}}{2k+s}$$

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where it is understood that the term $n = x$ is counted with weight $1/2$ if x is a prime power, and the sum over ρ is calculated as $\lim_{T\to\infty} \sum_{|\gamma|\le T}$.

5. (cf. Ingham 1932, p. 81) By (12.1) we know that

$$\sum_\rho \frac{x^\rho}{\rho} = x - \psi_0(x) - \log 2\pi - \frac{1}{2}\log(1 - 1/x^2)$$

for $x > 1$. Show that if $0 < x < 1$, then

$$\sum_\rho \frac{x^\rho}{\rho} = \sum_{n\le 1/x} \frac{\Lambda(n)}{n} + \log x + C_0 + x + \frac{1}{2}\log\frac{1-x}{1+x}.$$

6. (de la Vallée Poussin 1896) Show that if $x > 1$, then

$$\sum_{n\le x}\Lambda(n)(x-n) = \frac{1}{2}x^2 - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - (\log 2\pi)x + \frac{\zeta'}{\zeta}(-1) - \sum_{k=1}^\infty \frac{x^{-2k+1}}{2k(2k-1)}.$$

7. Show that if $x > 1$, then

$$\sum_{n\le x}\Lambda(n)\log x/n = x - \sum_\rho \frac{x^\rho}{\rho^2} - (\log 2\pi)\log x - \left(\frac{\zeta'}{\zeta}\right)'(0) - \frac{1}{4}\sum_{k=1}^\infty \frac{x^{-2k}}{k^2}.$$

8. (Hardy & Littlewood 1918; Wigert 1920) (a) Let k be a non-negative integer. Show that for s near $-k$, the Laurent expansion of $\Gamma(s)$ begins

$$\Gamma(s) = \frac{(-1)^k}{k!(s+k)} + \frac{(-1)^k}{k!}\frac{\Gamma'}{\Gamma}(k+1) + \cdots .$$

(b) Let k be a positive integer. Show that for s near $-2k$, the Laurent expansion of $\frac{\zeta'}{\zeta}(s)$ begins

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{s+2k} - \frac{\zeta'}{\zeta}(2k+1) + \log 2\pi - \frac{\Gamma'}{\Gamma}(2k+1) + \cdots .$$

(c) Show that if $\Re z > 0$, then

$$\sum_{n=1}^\infty \Lambda(n)e^{-n/z} = z - \sum_\rho \Gamma(\rho)z^\rho - e^{-1/z}\log 2\pi + (-1+\cosh 1/z)\log z + \sum_{k=1}^\infty (-1)^k\frac{\zeta'}{\zeta}(k+1)\frac{z^{-k}}{k!} - \sum_{k=0}^\infty \frac{\Gamma'}{\Gamma}(2k+2)\frac{z^{-2k-1}}{(2k+1)!}.$$

9. Suppose that $a > 0$, that $x \ge 1$, and that x is not of the form e^{2a^2k} where k is a positive integer. Show that

$$\frac{1}{\sqrt{2\pi}\,a}\sum_{n=1}^{\infty}\Lambda(n)\exp\left(\frac{-(\log x/n)^2}{2a^2}\right)$$
$$= e^{a^2/2}x - \sum_{\rho} e^{a^2\rho^2/2}x^{\rho} + \sum_{0<k<\frac{\log x}{2a^2}} e^{2a^2k^2}x^{-2k}$$
$$-\frac{1}{2\pi}\exp\left(\frac{-(\log x)^2}{2a^2}\right)\int_{-\infty}^{\infty}\frac{\zeta'}{\zeta}(-(\log x)/a^2+it)e^{-a^2t^2/2}\,dt.$$

12.2 Weil's explicit formula

In order to see better the relationship between a sum over zeros and a corresponding sum over primes, we now derive an explicit formula that applies to a general class of kernels. (The next theorem is not used later, and can be omitted on a first reading.)

Theorem 12.13 (Weil) *Let $F(x)$ be a measurable function such that*

$$\int_{-\infty}^{\infty} e^{(\frac{1}{2}+\delta_0)2\pi|x|}|F(x)|\,dx < \infty, \tag{12.20}$$

and

$$\int_{-\infty}^{\infty} e^{(\frac{1}{2}+\delta_0)2\pi|x|}\,|dF(x)| < \infty \tag{12.21}$$

where $\delta_0 > 0$ is fixed. Suppose that $F(x) = \frac{1}{2}(F(x^-)+F(x^+))$ for all x, and that $F(x)+F(-x) = 2F(0)+O(|x|)$. Put

$$\Phi(s) = \int_{-\infty}^{\infty} F(x)e^{-(s-1/2)2\pi x}\,dx$$

for $-\delta_0 < \sigma < 1+\delta_0$. Let χ be a primitive character modulo q. Then

$$\lim_{T\to\infty}\sum_{|\gamma|\le T}\Phi(\rho) = E_0(\chi)(\Phi(0)+\Phi(1)) + \frac{1}{2\pi}\left(\log q/\pi + \frac{\Gamma'}{\Gamma}(1/4+\kappa/2)\right)F(0)$$
$$-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2}}\left(\chi(n)F\left(\frac{-1}{2\pi}\log n\right)+\overline{\chi}(n)F\left(\frac{1}{2\pi}\log n\right)\right)$$
$$+\int_0^{\infty}\frac{e^{-(1+2\kappa)\pi x}}{1-e^{-4\pi x}}(2F(0)-F(x)-F(-x))\,dx. \tag{12.22}$$

Here $E_0(\chi) = 1$ if $\chi = \chi_0$, $E_0(\chi) = 0$ otherwise, and $\kappa = 0$ if $\chi(-1) = 1$, $\kappa = 1$ if $\chi(-1) = -1$.

We note that if $\rho = 1/2 + i\gamma$, then

$$\Phi(\rho) = \int_{-\infty}^{\infty} F(x)e(-\gamma x)\,dx = \widehat{F}(\gamma).$$

The values of Γ'/Γ can be evaluated explicitly; from Appendix C we see that

$$\frac{\Gamma'}{\Gamma}(1/4) = -C_0 - 3\log 2 - \pi/2$$

and

$$\frac{\Gamma'}{\Gamma}(3/4) = -C_0 - 3\log 2 + \pi/2.$$

Here C_0 is Euler's constant. Since $\int |d f g| \le \int |f|\,|dg| + \int |g|\,|df|$, from (12.20) and (12.21) we see that $e^{a|x|}F(x)$ is of bounded variation for any a, $0 \le a \le (1/2+\delta_0)2\pi$. Hence $F(x) \ll \exp(-(1/2+\delta_0)2\pi|x|)$, and $\Phi(s)$ is analytic in the strip $-\delta_0 < \sigma < 1+\delta_0$. For $|t| \le 1$ we note that $\phi(s) \ll 1$. For $|t| \ge 1$ we integrate by parts to see that

$$\Phi(s) = \frac{1}{2\pi i t}\int_{-\infty}^{\infty} e(-tx)\,d\left(F(x)\exp((1-2\sigma)\pi x)\right);$$

hence $\Phi(s) \ll 1/(|t|+1)$ uniformly for $-\delta_0 \le \sigma \le 1+\delta_0$. In these estimates, and in the proof below, implicit constants may depend on F and on δ_0.

Proof We note that

$$\sum_{|\gamma| \le T_1} \Phi(\rho) = \frac{1}{2\pi i}\int_{\mathcal{C}} \Phi(s)\frac{\xi'}{\xi}(s,\chi)\,ds$$

where $\mathcal{C}$ is the closed polygonal contour with vertices $-\delta_1 + iT_1$, $-\delta_1 - iT_1$, $1+\delta_1 - iT_1$, $1+\delta_1+iT_1$. Here $0 < \delta_1 < \delta_0$, and T_1 is chosen so that $|T - T_1| \le 1$, and so that

$$\frac{\xi'}{\xi}(\sigma \pm iT_1, \chi) \ll (\log qT)^2$$

uniformly for $-1 \le \sigma \le 2$. Thus

$$\sum_{|\gamma| \le T} \Phi(\rho) = \frac{1}{2\pi i}\left(\int_{1+\delta_1-iT}^{1+\delta_1+iT} + \int_{-\delta_1+iT}^{-\delta_1-iT}\right)\Phi(s)\frac{\xi'}{\xi}(s,\chi)\,ds + O\left(\frac{(\log T)^2}{T}\right).$$

By the functional equation for $\xi(s,\chi)$, we see that

$$\frac{\xi'}{\xi}(s,\chi) = -\frac{\xi'}{\xi}(1-s,\overline{\chi}).$$

Hence the integral above is

$$\frac{1}{2\pi i}\int_{1+\delta_1-iT}^{1+\delta_1+iT}\Phi(s)\frac{\xi'}{\xi}(s,\chi)+\Phi(1-s)\frac{\xi'}{\xi}(s,\overline{\chi})\,ds. \qquad (12.23)$$

From (10.25) and (10.33) we see that

$$\frac{\xi'}{\xi}(s,\chi)=E_0(\chi)\left(\frac{1}{s}+\frac{1}{s-1}\right)+\frac{1}{2}\log\frac{q}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}((s+\kappa)/2)+\frac{L'}{L}(s,\chi). \qquad (12.24)$$

For $1<\sigma<1+\delta_0$,

$$\begin{aligned}\Phi(s)\frac{L'}{L}(s,\chi)&=-\Phi(s)\sum_{n=1}^{\infty}\Lambda(n)\chi(n)n^{-s}\\&=-\sum_{n=1}^{\infty}\Lambda(n)\chi(n)n^{-1/2}\int_{-\infty}^{\infty}F\left(x-\frac{1}{2\pi}\log n\right)e^{-(s-1/2)2\pi x}\,dx,\end{aligned} \qquad (12.25)$$

and similarly

$$\begin{aligned}\Phi(1-s)\frac{L'}{L}(s,\overline{\chi})=-\sum_{n=1}^{\infty}\Lambda(n)\overline{\chi}(n)n^{-1/2}&\\ \times\int_{-\infty}^{\infty}F\left(-x+\frac{1}{2\pi}\log n\right)e^{-(s-1/2)2\pi x}\,dx.&\end{aligned} \qquad (12.26)$$

From the estimate $F(x)\ll e^{-(1/2+\delta_0)2\pi|x|}$ we see that

$$\begin{aligned}&\sum_n\Lambda(n)n^{-1/2}\int_{-\infty}^{\infty}\left|F\left(x-\tfrac{1}{2\pi}\log n\right)\right|e^{-(1/2+\delta_1)2\pi x}\,dx\\&\ll\sum_{n=1}^{\infty}\Lambda(n)n^{-1/2}\left(\int_{(\log n)/(2\pi)}^{\infty}e^{-(1+\delta_0+\delta_1)2\pi x}n^{1/2+\delta_0}\,dx\right.\\&\qquad\left.+\int_{-\infty}^{(\log n)/(2\pi)}e^{(\delta_0-\delta_1)2\pi x}n^{-1/2-\delta_0}\,dx\right)\\&\ll\sum_n\Lambda(n)n^{-1-\delta_1}\ll 1.\end{aligned}$$

A similar calculation relates to the second term (12.26), and hence for $s=1+\delta_1+it$,

$$\Phi(s)\frac{L'}{L}(s,\chi)+\Phi(1-s)\frac{L'}{L}(s,\overline{\chi})=\int_{-\infty}^{\infty}H(x)e(-tx)\,dx=\widehat{H}(t)$$

where

$$H(x) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2}} \bigg(\chi(n) F\left(x - \frac{\log n}{2\pi}\right) + \overline{\chi}(n) F\left(-x + \frac{\log n}{2\pi}\right) \bigg) e^{-(1/2+\delta_1)2\pi x}.$$

Now $H(x)$ is of bounded variation, since

$$\begin{aligned} \operatorname{Var} H &\le \sum_n \frac{\Lambda(n)}{n^{1/2}} \operatorname{Var}\left(F\left(x - \frac{\log n}{2\pi}\right) e^{-(1/2+\delta_1)2\pi x} \right) \\ &\quad + \sum_n \frac{\Lambda(n)}{n^{1/2}} \operatorname{Var}\left(F\left(-x + \frac{\log n}{2\pi}\right) e^{-(1/2+\delta_1)2\pi x} \right) \\ &= 2 \left(\sum_n \Lambda(n) n^{-1-\delta_1} \right) \operatorname{Var}\left(F(x) e^{-(1/2+\delta_1)2\pi x} \right) \ll 1. \end{aligned}$$

Moreover, $H(x) = (H(x^+) + H(x^-))/2$, and thus by the Fourier integral theorem,

$$\lim_{T\to\infty} \int_{-T}^{T} \widehat{H}(t)\, dt = H(0).$$

That is,

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{2\pi i} \int_{1+\delta_1-iT}^{1+\delta_1+iT} \Phi(s) \frac{L'}{L}(s,\chi) + \Phi(1-s) \frac{L'}{L}(s,\overline{\chi})\, ds \\ &\quad = \frac{-1}{2\pi} \sum_n \frac{\Lambda(n)}{n^{1/2}} \left(\chi(n) F\left(\frac{-\log n}{2\pi}\right) + \overline{\chi}(n) F\left(\frac{\log n}{2\pi}\right) \right). \end{aligned}$$

The remaining terms from (12.24) contribute to the integral (12.23) an amount

$$\frac{1}{2\pi i} \int_{1+\delta_1-iT}^{1+\delta_1+iT} G(s)\, ds.$$

where

$$G(s) = \left(E_0(\chi)\left(\frac{1}{s} + \frac{1}{s-1}\right) + \frac{1}{2}\log\frac{q}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa}{2}\right) \right) (\Phi(s) + \Phi(1-s))$$

By Cauchy's theorem this is

$$\frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} G(s)\, ds + E_0(\chi)(\Phi(0) + \Phi(1)) + O\left(\frac{\log^2 qT}{T}\right).$$

To treat this latter integral we note that

$$\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\left(\frac{1}{s}+\frac{1}{s-1}\right)(\Phi(s)+\Phi(1-s))\,ds$$
$$=\frac{-4i}{\pi}\int_{-T}^{T}\frac{t}{1+4t^2}\left(\Phi\left(\frac{1}{2}+it\right)+\Phi\left(\frac{1}{2}-it\right)\right)dt=0.$$

Now $\Phi(1/2+it)=\widehat{F}(t)$, and hence

$$\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\frac{1}{2}(\log q/\pi)(\Phi(s)+\Phi(1-s))\,ds$$
$$=\frac{\log q/\pi}{4\pi}\int_{-T}^{T}\widehat{F}(t)+\widehat{F}(-t)\,dt\longrightarrow\frac{F(0)}{2\pi}\log q/\pi$$

as T tends to infinity. Thus to complete the proof of the theorem it suffices to establish

Lemma 12.14 *Let $a>0$ and $b>0$ be fixed. If $J\in L^1(\mathbb{R})$, J is of bounded variation on $\mathbb{R}$, and if $J(x)=J(0)+O(|x|)$, then*

$$\lim_{T\to\infty}\int_{-T}^{T}\frac{\Gamma'}{\Gamma}(a\pm ibt)\widehat{J}(t)\,dt$$
$$=\frac{\Gamma'}{\Gamma}(a)J(0)+\frac{2\pi}{b}\int_0^\infty\frac{e^{-2\pi ax/b}}{1-e^{-2\pi x/b}}(J(0)-J(\mp x))\,dx. \quad (12.27)$$

If G and J are in $L^1(\mathbb{R})$, then

$$\int_{-\infty}^{\infty}G(t)\widehat{J}(t)\,dt=\int_{-\infty}^{\infty}\widehat{G}(x)J(x)\,dx,$$

since both sides are

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(t)J(x)e(-tx)\,dx\,dt.$$

We cannot apply this with $G(t)=\frac{\Gamma'}{\Gamma}(a\pm ibt)$, since this function is not in $L^1(\mathbb{R})$. Nevertheless, the right-hand side of (12.27) is a linear functional of J, which thus serves as a surrogate for the Fourier transform of $\frac{\Gamma'}{\Gamma}(a\pm ibt)$, at least when the test function J is sufficiently well-behaved.

Proof It suffices to consider the $+$ sign on the left-hand side of (12.27), for if $K(x)=J(-x)$ then $\widehat{K}(t)=\widehat{J}(-t)$. We suppose first that $J(0)=0$. The integral with respect to t on the left-hand side of (12.27) is

$$\int_{-\infty}^{\infty}J(x)\left(\int_{-T}^{T}\frac{\Gamma'}{\Gamma}(a+ibt)e(-xt)\,dt\right)dx.$$

Since $\frac{\Gamma'}{\Gamma}(a+ibt) \ll \log(|t|+2)$, the inner integral above is $\ll T\log T$, uniformly in x. Put $\delta = T^{-2/3}$. The contribution to the above by those x for which $|x| \le \delta$ is

$$\ll \int_{-\delta}^{\delta} |x| T \log T \, dx \ll \delta^2 T \log T = T^{-1/3} \log T.$$

For $|x| \ge \delta$ we appeal to Theorem C.5 to estimate the inner integral. The error term in Theorem C.5 contributes an amount

$$\ll \int_{\delta}^{\infty} \min(x, 1) T^{-1} x^{-2} \, dx \ll T^{-1} \log T.$$

By integrating by parts we see that

$$\begin{aligned}\int_{\delta}^{\infty} J(x) \frac{e(-xT)}{x} \, dx &= \frac{J(\delta) e(-\delta T)}{2\pi i \delta T} - \frac{1}{2\pi i T} \int_{\delta}^{\infty} J(x) \frac{e(-xT)}{x^2} \, dx \\ &\quad + \frac{1}{2\pi i T} \int_{\delta}^{\infty} \frac{e(-xT)}{x} \, dJ(x) \\ &\ll \frac{1}{T} + \frac{1}{T} \int_{\delta}^{\infty} \min(x, 1) x^{-2} \, dx + \frac{1}{\delta T} \int_{\delta}^{\infty} |dJ| \\ &\ll T^{-1/3},\end{aligned}$$

and similarly for the three related terms. Hence

$$\int_{-T}^{T} \frac{\Gamma'}{\Gamma}(a+ibt) \widehat{J}(t) \, dt = \frac{-2\pi}{b} \int_{-\infty}^{-\delta} \frac{e^{2\pi a x/b}}{1 - e^{2\pi x/b}} J(x) \, dx + O\left(T^{-1/3} \log T\right).$$

On the right-hand side we see that $\int_{-\delta}^{0} \cdots \ll \delta$, so that

$$\lim_{T\to\infty} \int_{-T}^{T} \frac{\Gamma'}{\Gamma}(a+ibt) \widehat{J}(t) \, dt = \frac{-2\pi}{b} \int_{0}^{\infty} \frac{e^{-2\pi a x/b}}{1 - e^{-2\pi x/b}} J(-x) \, dx$$

provided that $J(0) = 0$. To obtain the general case we apply the above to the function $K(x) = J(x) - J(0) e^{-\pi x^2/A}$ where $A > 0$ is large. Then $\widehat{K}(t) = \widehat{J}(t) - J(0)\sqrt{A} e^{-\pi A t^2}$, and hence

$$\begin{aligned}\lim_{T\to\infty} \int_{-T}^{T} \frac{\Gamma'}{\Gamma}(a+ibt) \widehat{K}(t) \, dt &= \lim_{T\to\infty} \int_{-T}^{T} \frac{\Gamma'}{\Gamma}(a+ibt) \widehat{J}(t) \, dt \\ &\quad - J(0)\sqrt{A} \int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(a+ibt) e^{-\pi A t^2} \, dt.\end{aligned}$$

This last integral is

$$\int_{-\infty}^{\infty} \left(\frac{\Gamma'}{\Gamma}(a) + O(|t|) \right) e^{-\pi A t^2} \, dt = \frac{\Gamma'}{\Gamma}(a) A^{-1/2} + O(A^{-1}).$$

On the other hand,

$$
\begin{aligned}
-2\pi \int_0^\infty & \frac{e^{-2\pi ax/b}}{1-e^{-2\pi x/b}} K(-x)\,dx \\
&= 2\pi \int_0^\infty \frac{e^{-2\pi ax/b}}{1-e^{-2\pi x/b}} (J(0)-J(-x))\,dx \\
&+ 2\pi J(0) \int_0^\infty \frac{e^{-2\pi ax/b}}{1-e^{-2\pi x/b}} \left(e^{-\pi x^2/A}-1\right) dx.
\end{aligned}
$$

Now $e^{-\alpha} = 1 + O(\alpha)$ for $\alpha \ge 0$, and hence this last integral is

$$\ll \int_0^1 xA^{-1}\,dx + \int_1^\infty e^{-2\pi ax/b} x^2 A^{-1}\,dx \ll A^{-1}.$$

On combining these estimates, we see that (12.29) holds apart from an error term $O(A^{-1/2})$, and we obtain the result since A can be arbitrarily large. □

12.3 Notes

Section 12.1. Let $\Pi(x) = \sum_{n\le x} \Lambda(n)/\log n$. Riemann (1859) gave a heuristic proof that if $x > 1$, and x is not a prime power, then

$$\Pi(x) = \mathrm{Li}(x) - \sum_\rho \mathrm{Li}\,(x^\rho) - \log 2 + \int_x^\infty \frac{du}{(u^2-1)u\log u}.$$

Here the sum over the zeros is conditionally convergent, and it is to be understood that it is computed as the limit, as $T \to \infty$, of the sum over those zeros for which $|\gamma| \le T$. The above formula was first proved rigorously by von Mangoldt (1895), and additional proofs were subsequently given by Landau (1908a, b). For further discussion of the explicit formula in the form given by Riemann, see Edwards (1974, Chapter 1). von Mangoldt (1895) also proved the explicit formula (12.1). Landau (1909, Section 89) was the first to show that the limit in (12.1) is attained uniformly for x in a compact interval not containing a prime power. Cramér (1918) showed that (12.1) can be derived from the above. von Koch (1910) and Landau (1912) estimated the error term that arises when the explicit formula is truncated, as in Theorem 12.5. The explicit formula for $\psi_0(x,\chi)$ was first established by Landau (1908b), but with not so much attention to the constant term. In the customary form of this explicit formula (cf. Davenport (2000, p. 117)), the constant term is expressed in terms of the constant $B(\chi)$ that arises in the Hadamard product formula for $\xi(s,\chi)$. Our presentation, which avoids this, is that of Vorhauer (2006).

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Section 12.2. Although many specific explicit formulæ were derived by various authors for a variety of purposes, it was Guinand (1942) who first suggested that it would be possible to specify a general class of such formulæ. Guinand (1948) did this assuming the Riemann Hypothesis, but it seems that he imposed RH only in order to obtain a wider class of test functions. Theorem 12.13 is a special case of the main result of Weil (1952), who treats general L-functions associated with *Grössencharaktere* χ, which are representations of the group of idèle-classes of an algebraic number field k into the multiplicative group of non-zero complex numbers. Weil also showed that a necessary and sufficient condition for the Riemann hypothesis to hold for L is that the right-hand side corresponding to (12.22) is non-negative for all functions F of a certain class. Gallagher (1987) widened the class of test functions in Guinand's formula and gave several applications. See also Besenfelder (1977a, b), Yoshida (1982), Jorgenson, Lang & Goldfeld (1994), and Bombieri & Lagarias (1999).

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