11

Primes in arithmetic progressions: II

11.1 A zero-free region

For a given integer q, the primes not dividing q are distributed in the reduced residue classes modulo q. As there are no other obvious restrictions on the primes modulo q, we expect the primes to be uniformly distributed amongst the reduced residue classes. Let $\pi(x; q, a)$ denote the number of primes $p \le x$ such that $p \equiv a \pmod q$. We anticipate that if $(a, q) = 1$, then

$$\pi(x; q, a) \sim \frac{x}{\varphi(q) \log x} \qquad \text{as } x \longrightarrow \infty .$$

This asymptotic estimate is the *Prime Number Theorem for arithmetic progressions*; it can readily be established by adapting the methods of Chapters 4 and 6. For many purposes, however, it is important to have a quantitative form of this, from which one can tell how large x should be, as a function of q, to ensure that $\pi(x; q, a)$ is near $\mathrm{li}(x)/\varphi(q)$. To obtain such an estimate we must first derive a zero-free region for the Dirichlet L-functions $L(s, \chi)$ that is explicit in its dependence on both q and t. For the most part our arguments are natural generalizations of the analysis in Chapter 6, but we shall encounter a new difficulty in connection with the possible existence of a real zero β near 1 of $L(s, \chi)$ when χ is a quadratic character.

The approximate partial fraction expansion of $\frac{\zeta'}{\zeta}(s)$ (cf. Lemma 6.4) depends on the upper bound for $|\zeta(s)|$ provided by Corollary 1.17. By using Lemma 10.15 in a similar manner, we now derive a corresponding approximate partial fraction formula for $\frac{L'}{L}(s, \chi)$. In order to formulate a unified result for both the principal and non-principal characters, it is convenient to employ the notation

$$E_0(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases} \tag{11.1}$$

Lemma 11.1 *If χ is a character* (mod q) *and* $5/6 \le \sigma \le 2$, *then*

$$-\frac{L'}{L}(s,\chi) = \frac{E_0(\chi)}{s-1} - \sum_\rho \frac{1}{s-\rho} + O(\log q\tau)$$

where the sum is over all zeros ρ of $L(s,\chi)$ for which $\left|\rho - \left(\frac{3}{2}+it\right)\right| \le 5/6$.

Proof When χ is non-principal we apply Lemma 6.3 to the function

$$f(z) = L\left(z + \left(\frac{3}{2}+it\right), \chi\right)$$

with $R = 5/6$ and $r = 2/3$. By Lemma 10.15 we may take $M = Cq\tau$ for a suitable absolute constant C, and by the Euler product for $L(s,\chi)$ we see that

$$|f(0)| = \left|L\left(\tfrac{3}{2}+it,\chi\right)\right| = \prod_p \left|1-\chi(p)p^{-\frac{3}{2}-it}\right|^{-1} \ge \prod_p \left(1+p^{-3/2}\right)^{-1} \gg 1.$$

Now suppose that $\chi = \chi_0$. The zeros of the function $1 - p^{-s}$ form an arithmetic progression on the imaginary axis. Hence by (4.22), the zeros of $L(s,\chi_0)$ are the zeros of $\zeta(s)$ together with the union of several arithmetic progressions on the imaginary axis. Since these latter zeros all lie at a distance $\ge 3/2$ from the point $\frac{3}{2}+it$, none of them is included in the sum over ρ. Moreover, by taking logarithmic derivatives of both sides of (4.22) we see that

$$\frac{L'}{L}(s,\chi_0) = \frac{\zeta'}{\zeta}(s) + \sum_{p|q} \frac{\log p}{p^s - 1}.$$

But $(\log p)/(p^s-1) \ll 1$ for $\sigma \ge 5/6$, so the sum over p is $\ll \omega(q) \ll \log q$ by Theorem 2.10. Hence we obtain the stated identity by appealing to Lemma 6.4. □

The generalization of Lemma 6.5 is straightforward.

Lemma 11.2 *If* $\sigma > 1$, *then*

$$\Re\left(-3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2)\right) \ge 0.$$

Proof By the Dirichlet series expansion (4.25) for $\frac{L'}{L}(s,\chi)$ we see that the left-hand side above is

$$\Re \sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\Lambda(n)}{n^\sigma}(3 + 4\chi(n)n^{-it} + \chi(n)^2 n^{-2it}).$$

The quantity $\chi(n)n^{-it}$ is unimodular when $(n,q)=1$, so for such n there is a real number θ_n such that $\chi(n)n^{-it} = e^{i\theta_n}$. Thus the above is

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\Lambda(n)}{n^\sigma}(3 + 4\cos\theta_n + \cos 2\theta_n).$$

This is non-negative because $3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0$ for all θ. □

The groundwork laid above enables us to establish a variant of Theorem 6.6 for Dirichlet L-functions.

Theorem 11.3 *There is an absolute constant $c > 0$ such that if χ is a Dirichlet character modulo q, then the region*

$$R_q = \{s : \sigma > 1 - c/\log q\tau\}$$

contains no zero of $L(s, \chi)$ unless χ is a quadratic character, in which case $L(s, \chi)$ has at most one, necessarily real, zero $\beta < 1$ in R_q.

A zero lying in R_q, as described above, is called *exceptional*. No exceptional zero is known, and indeed it may be conjectured that if χ is quadratic, then $L(\sigma, \chi) > 0$ for all $\sigma > 0$. We give further study to exceptional zeros in the next section.

Proof The case $\chi = \chi_0$ is immediate from (4.22) and Theorem 6.6, so we may assume that χ is non-principal. Also, the Euler product (4.21) for $L(s, \chi)$ is absolutely convergent when $\sigma > 1$, and hence $L(s, \chi) \neq 0$ for such s. Thus it suffices to consider a zero $\rho_0 = \beta_0 + i\gamma_0$ of $L(s, \chi)$ with $12/13 \leq \beta_0 \leq 1$. We consider several cases, the first of which parallels the proof of Theorem 6.6 most closely. □

Case i. Complex χ. If $\sigma > 1$ and ρ is a zero of an L-function, then $\Re(s - \rho) > 0$ and hence $\Re(1/(s - \rho)) > 0$. Thus by Lemma 11.1, if $0 < \delta \leq 1$, then

$$-\Re\frac{L'}{L}(1 + \delta, \chi_0) \leq \frac{1}{\delta} + c_1 \log q,$$

$$-\Re\frac{L'}{L}(1 + \delta + i\gamma_0, \chi) \leq \frac{-1}{1 + \delta - \beta_0} + c_1 \log q(|\gamma_0| + 4), \qquad (11.2)$$

$$-\Re\frac{L'}{L}(1 + \delta + 2i\gamma_0, \chi^2) \leq c_1 \log q(2|\gamma_0| + 4)$$

for some absolute constant c_1. The hypothesis that χ is complex is needed for this last inequality, to ensure that $\chi^2 \neq \chi_0$ in the appeal to Lemma 11.1. We multiply both sides of the first inequality by 3, the second by 4, and sum all three. By Lemma 11.2, the resulting left-hand side is non-negative. That is,

$$\frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + c_2 \log q(|\gamma_0|+4) \ge 0$$

for some constant c_2. If $\beta_0 = 1$, then letting $\delta \to 0^+$ gives an immediate contradiction, so it may be assumed that $\beta_0 < 1$. Then, on taking $\delta = 6(1-\beta_0)$, it follows that

$$1 - \beta_0 \ge \frac{1}{14 c_2 \log q(|\gamma_0|+4)}.$$

Hence $\rho_0 \notin R_q$ if c is chosen sufficiently small.

This argument also applies with only small changes when χ is quadratic, provided that $|\gamma_0|$ is large. We can even allow $|\gamma_0|$ to be small, as long as it is large compared with $1-\beta_0$. We now consider such a case.

Case 2. Quadratic χ, $|\gamma_0| \ge 6(1-\beta_0)$. By Theorem 4.9, $L(1,\chi) \ne 0$, so $\gamma_0 \ne 0$. Hence we can proceed as above, except that as $\chi^2 = \chi_0$ the third inequality in (11.2) must be replaced by the weaker inequality

$$-\Re \frac{L'}{L}(1+\delta+2i\gamma_0, \chi^2) \le \frac{\delta}{\delta^2 + 4\gamma_0^2} + c_1 \log q(2|\gamma_0|+4).$$

Again if $\beta_0 = 1$, then taking $\delta \to 0^+$ gives a contradiction. Thus it can be supposed that $\beta_0 < 1$. Since $|\gamma_0| \ge 6(1-\beta_0)$, this implies that

$$-\Re \frac{L'}{L}(1+\delta+2i\gamma_0, \chi^2) \le \frac{\delta}{\delta^2 + 144(1-\beta_0)^2} + c_1 \log q(2|\gamma_0|+4).$$

We combine this inequality with the first two inequalities in (11.2) and apply Lemma 11.2 with $\sigma = 1+\delta = 1+6(1-\beta_0)$ to see that

$$\frac{1}{1-\beta_0}\left(\frac{3}{6} - \frac{4}{7} + \frac{6}{180}\right) + c_2 \log q(|\gamma_0|+4) \ge 0.$$

The factor in large parentheses above is $-4/105 < -1/27$, so

$$1 - \beta_0 \ge \frac{1}{27 c_2 \log q(|\gamma_0|+4)}.$$

Case 3. Quadratic χ, $0 < |\gamma_0| \le 6(1-\beta_0)$. Since $L(s,\chi)$ is real when s is real, it follows by the Schwarz reflection principle that $L(\beta_0 - i\gamma_0, \chi) = 0$. Hence by Lemma 11.1 we see that if $1 < \sigma \le 2$, then

$$\begin{aligned} -\Re \frac{L'}{L}(\sigma,\chi) &\le -\Re \frac{1}{\sigma-\rho_0} - \Re \frac{1}{\sigma - \overline{\rho_0}} + c_1 \log 4q \\ &= \frac{-2(\sigma-\beta_0)}{(\sigma-\beta_0)^2 + \gamma_0^2} + c_1 \log 4q \\ &\le \frac{-2(\sigma-\beta_0)}{(\sigma-\beta_0)^2 + 36(1-\beta_0)^2} + c_1 \log 4q. \end{aligned} \qquad (11.3)$$

Rather than apply Lemma 11.2 we simply observe that if $\sigma > 1$, then

$$-\frac{L'}{L}(\sigma, \chi_0) - \frac{L'}{L}(\sigma, \chi) = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\Lambda(n)}{n^\sigma}(1 + \chi(n)) \geq 0. \tag{11.4}$$

We put $\sigma = 1 + \delta = 1 + a(1 - \beta_0)$ and combine the first inequality in (11.2) and (11.3) in the above to deduce that

$$\frac{1}{1 - \beta_0}\left(\frac{1}{a} - \frac{2(a+1)}{(a+1)^2 + 36}\right) + c_2 \log 4q \geq 0.$$

The factor in large parentheses is $\sim -1/a$ as $a \to \infty$, so it is certainly possible to choose a value of a so that this factor is negative. Indeed, when $a = 13$ this factor is $-33/754 < -1/27$, and hence

$$1 - \beta_0 \geq \frac{1}{27 c_2 \log 4q}.$$

(We note that our supposition that $\beta_0 \geq 12/13$ implies that $\sigma = 1 + 13(1 - \beta_0) \leq 2$, so that Lemma 11.1 is applicable.)

Case 4. Quadratic χ, real zeros. If β_0 is a real zero of $L(s, \chi)$, then $\beta_0 < 1$ by Theorem 4.9. Suppose that $\beta_0 \leq \beta_1 < 1$ are two such zeros. Then by Lemma 11.1,

$$\begin{aligned} -\Re\frac{L'}{L}(\sigma, \chi) &\leq -\frac{1}{\sigma - \beta_0} - \frac{1}{\sigma - \beta_1} + c_1 \log 4q \\ &\leq -\frac{2}{\sigma - \beta_0} + c_1 \log 4q. \end{aligned}$$

On combining the first part of (11.2) and the above in (11.4) with $\sigma = 1 + \delta = 1 + a(1 - \beta_0)$, we find that

$$\frac{1}{1 - \beta_0}\left(\frac{1}{a} - \frac{2}{a+1}\right) + c_2 \log 4q \geq 0.$$

On taking $a = 2$ we deduce that

$$1 - \beta_0 \geq \frac{1}{6 c_2 \log 4q}.$$

This completes the proof. □

In the same way that Theorem 6.7 was derived from Theorem 6.6, we now derive estimates for $\frac{L'}{L}(s, \chi)$ and $\log L(s, \chi)$ in a portion of the critical strip.

Theorem 11.4 *Let χ be a non-principal character modulo q, let c be the constant in* Theorem 3, *and suppose that $\sigma \geq 1 - c/(2 \log q\tau)$. If $L(s, \chi)$ has no exceptional zero, or if β_1 is an exceptional zero of $L(s, \chi)$ but $|s - \beta_1| \geq$*

$1/\log q$, *then*

$$\frac{L'}{L}(s,\chi) \ll \log q\tau, \tag{11.5}$$

$$|\log L(s,\chi)| \le \log\log q\tau + O(1), \tag{11.6}$$

and

$$\frac{1}{L(s,\chi)} \ll \log q\tau. \tag{11.7}$$

Alternatively, if β_1 *is an exceptional zero of* $L(s,\chi)$ *and* $|s-\beta_1| \le 1/\log q$, *then*

$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + O(\log q) \quad (s \ne \beta_1), \tag{11.8}$$

$$|\arg L(s,\chi)| \le \log\log q + O(1) \quad (s \ne \beta_1), \tag{11.9}$$

and

$$|s-\beta_1| \ll |L(s,\chi)| \ll |s-\beta_1|(\log q)^2. \tag{11.10}$$

Proof If $\sigma > 1$, then by Corollary 1.11 we see that

$$\left|\frac{L'}{L}(s,\chi)\right| \le \sum_{n=1}^{\infty} \Lambda(n)n^{-\sigma} = -\frac{\zeta'}{\zeta}(\sigma) \ll \frac{1}{\sigma-1}.$$

Hence (11.5) is obvious if $\sigma \ge 1 + 1/\log q\tau$. Let $s_1 = 1 + 1/\log q\tau + it$. Then

$$\frac{L'}{L}(s_1,\chi) \ll \log q\tau.$$

From this and Lemma 11.1 it follows that

$$\sum_{\rho} \frac{1}{s_1-\rho} \ll \log q\tau \tag{11.11}$$

where the sum is over those zeros of $L(s,\chi)$ for which $|\rho - (3/2+it)| \le 5/6$. Hence

$$\sum_{\rho} \frac{1}{s-\rho} = \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho}\right) + O(\log q\tau). \tag{11.12}$$

Suppose that $1 - c/(2\log q\tau) \le \sigma \le 1 + 1/\log q\tau$ and that $|s-\beta_1| \ge 1/\log q$ if $L(s,\chi)$ has an exceptional zero β_1. Since $|s-\rho| \asymp |s_1-\rho|$ for all zeros ρ, it follows that

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} = \frac{1+1/\log q\tau - \sigma}{(s-\rho)(s_1-\rho)} \ll \frac{1}{|s_1-\rho|^2\log q\tau} \ll \Re\frac{1}{s_1-\rho}.$$

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On summing this over ρ and appealing to (11.11) we find that

$$\sum_\rho \frac{1}{s-\rho} \ll \log q\tau, \tag{11.13}$$

and (11.5) follows by Lemma 11.1.

To derive (11.6) we first note that if $\sigma > 1$, then

$$|\log L(s,\chi)| \le \sum_{n=2}^\infty \frac{\Lambda(n)}{\log n} n^{-\sigma} = \log \zeta(\sigma).$$

Since $\zeta(\sigma) \le \sigma/(\sigma-1)$ by Corollary 1.14, we see that (11.6) holds when $\sigma \ge 1 + 1/\log q\tau$. In particular, (11.6) holds at the point $s_1 = 1 + 1/\log q\tau + it$. To treat the remaining s it suffices to note that

$$\log L(s,\chi) - \log L(s_1,\chi) = \int_{s_1}^{s} \frac{L'}{L}(w,\chi)\,dw \ll |s_1 - s| \log q\tau \ll 1$$

by (11.5). The estimate (11.6) trivially implies (11.7) since $\log 1/|L(s,\chi)| = -\Re \log L(s,\chi)$.

Now suppose that $L(s,\chi)$ has an exceptional zero β_1 such that $|s-\beta_1| \le 1/\log q$. Then $1 - c/(2\log 4q) \le \sigma \le 1 + 1/\log q$, so by Lemma 11.1,

$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + {\sum_\rho}' \frac{1}{s-\rho} + O(\log q)$$

where $\sum_\rho'$ denotes a sum over all zeros ρ such that $|\rho - (3/2 + it)| \le 5/6$ *except* for the exceptional zero β_1. The proof of (11.13) applies to $\sum_\rho'$, so we have (11.8). Proceeding as in the proof of (11.6), we find that

$$\log L(s,\chi) = \log \frac{s-\beta_1}{s_1-\beta_1} + \log L(s_1,\chi) + O(1),$$

which implies that

$$\left|\log L(s,\chi) - \log \frac{s-\beta_1}{s_1-\beta_1}\right| \le |\log L(s_1,\chi)| + O(1) \le \log\log q + O(1).$$

But $\arg(s-\beta_1) \ll 1$, $\arg(s_1-\beta_1) \ll 1$, and $\log|s_1-\beta_1| = -\log\log q + O(1)$, so we have (11.9) and (11.10). □

Our methods yield not only a zero-free region, but also enable us to bound the number of zeros ρ of $L(s,\chi)$ that might lie near $1+it$.

Theorem 11.5 *Let $n(r;t,\chi)$ denote the number of zeros ρ of $L(s,\chi)$ in the disc $|\rho-(1+it)| \le r$. Then $n(r;t,\chi) \ll r\log q\tau$ uniformly for $1/\log q\tau \le r \le 3/4$.*

Here the constraint $r \ge 1/\log q\tau$ is needed because $L(s,\chi)$ might have an exceptional zero. If $L(s,\chi)$ has no exceptional zero, then the bound holds uniformly for $0 \le r \le 3/4$, in view of the zero-free region of Theorem 11.3.

Proof In view of Theorem 6.8, we may suppose that χ is non-principal. Suppose first that $1/\log q\tau \le r \le 1/3$. Take $s_1 = 1 + r + it$. Then $\Re(s_1 - \rho)^{-1} \ge 0$ for all zeros ρ, and $\Re(s_1-\rho)^{-1} \gg 1/r$ if ρ is counted by $n(r;t,\chi)$. Hence

$$\frac{1}{r} n(r;t,\chi) \ll \Re \sum_\rho \frac{1}{s_1-\rho}$$

where the sum is over all zeros ρ such that $|\rho - (3/2+it)| \le 5/6$. By Lemma 11.1 we see that the above is $\ll \log q\tau$, since

$$\left|\frac{L'}{L}(s_1)\right| \le -\frac{\zeta'}{\zeta}(1+r) \asymp \frac{1}{r} \ll \log q\tau.$$

If $1/3 \le r \le 3/4$, then it suffices to apply Jensen's inequality to $L(s,\chi)$ on a disc with centre $3/2+it$, with $R = 4/3$ and $r = 5/4$, in view of the estimates provided by Lemma 10.15. □

11.1.1 Exercises

1. Let $S(x;q)$ denote the number of integers n, $0 < n \le x$, such that $(n,q) = 1$, and put $R(x;q) = S(x;q) - (\varphi(q)/q)x$.
   (a) Show that if $\sigma > 0$, $x > 0$, and $s \ne 1$, then
   $$L(s,\chi_0) = \sum_{n \le x} \chi_0(n) n^{-s} + \frac{\varphi(q)}{q} \cdot \frac{x^{1-s}}{s-1} - \frac{R(x;q)}{x^s} + s\int_x^\infty R(u;q)u^{-s-1}\,du.$$
   Show that this includes Theorem 1.12 as a special case.
   (b) Let $\delta > 0$ be fixed. Show that if $\sigma \ge \delta$, then
   $$L(s,\chi_0) = \frac{\varphi(q)}{q} \cdot \frac{x^{1-s}}{s-1} + \sum_{n\le x} \chi_0(n) n^{-s} + O(d(q)|s|x^{-\sigma}).$$
2. Suppose that δ is fixed, $0 < \delta < 1$. Show that
   $$\sum_{p|q} \frac{\log p}{p^s - 1} \ll (\log q)^{1-\delta}$$
   uniformly for $\sigma \ge \delta$. (This improves on the estimate used in the latter part of the proof of Lemma 11.1.)
3. (a) Show that if $\sigma > 0$, then
   $$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty (\{x\} - 1/2)x^{-s-1}\,dx.$$

(b) Show that if $f(x)$ is a monotonically decreasing function, then
$$\int_0^1 (x - 1/2) f(x)\,dx \le 0.$$
(c) Show that
$$\zeta(\sigma) > \frac{1}{\sigma - 1} + \frac{1}{2}$$
for $\sigma > 0$.

(d) Show that
$$-\zeta'(s) = \frac{1}{(s-1)^2} + \int_1^\infty (\{x\} - 1/2)(1 - s\log x)x^{-s-1}\,dx$$
for $\sigma > 0$.

(e) Show that if $\sigma > 0$, then
$$\left|\zeta'(\sigma) + \frac{1}{(\sigma-1)^2}\right| < \frac{1}{2}\int_1^\infty |1 - \sigma\log x| x^{-\sigma-1}\,dx = \frac{1}{e\sigma}.$$
(f) Justify the following chain of inequalities for $\sigma > 1$:
$$-\frac{\zeta'}{\zeta}(\sigma) < \frac{\frac{1}{(\sigma-1)^2} + \frac{1}{e\sigma}}{\frac{1}{\sigma-1} + \frac{1}{2}} = \frac{1}{\sigma-1}\cdot\frac{1 + \frac{(\sigma-1)^2}{e\sigma}}{1 + \frac{\sigma-1}{2}} < \frac{1}{\sigma-1}.$$
(g) Show that if χ_0 is the principal character (mod q), then
$$-\frac{L'}{L}(\sigma, \chi_0) < \frac{1}{\sigma - 1}$$
for $\sigma > 1$. (This improves on the first inequality in (11.2), in the proof of Theorem 11.3.)

4. Let χ be a character (mod q), and suppose that the order d of χ is odd.
   (a) Show that $\Re\chi(n) \ge -\cos\pi/d$ for all integers n.
   (b) Show that if $\sigma > 1$, then $\log|L(\sigma,\chi)| \ge -(\cos\pi/d)\log\zeta(\sigma)$.
   (c) Show that $L(1,\chi) \asymp L(1 + 1/\log q, \chi)$.
   (d) Show that $|L(1,\chi)| \gg (\log q)^{-\cos\pi/d}$.
   (e) Deduce in particular that if χ is a cubic character (mod q), then $|L(1,\chi)| \gg 1/\sqrt{\log q}$.
5. *Grössencharaktere* for $\mathbb{Q}(\sqrt{-1})$, continued from Exercise 10.1.28. For an ideal $\mathfrak{a} = (a + ib)$ in the ring $\mathcal{O}\{a + ib : a, b \in \mathbb{Z}\}$ of Gaussian integers, put $\chi_m(\mathfrak{a}) = e^{4mi\arg(a+ib)}$. The ideal $\mathfrak{a}$ is the set of (Gaussian integer) multiples of the number $a + ib$, but it can equally well be expressed as the set of Gaussian integer multiples of $(a + ib)i^k$ for $k = 0, 1, 2, 3$. Note that the stated value of $\chi_m(\mathfrak{a})$ is independent of the choice of k.

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(a) Show that

$$L(s,\chi_m)=\prod_{\mathfrak{p}}\left(1-\frac{\chi_m(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}$$

for $\sigma>1$, where the product is over all prime ideals $\mathfrak{p}$ in the ring.

(b) Let $\Lambda(\mathfrak{a})=\log(a^2+b^2)$ if $\mathfrak{a}=(a+ib)^k$ for some positive integer k and $a+ib$ is a Gaussian prime, and $\Lambda(\mathfrak{a})=0$ otherwise. Show that

$$\frac{L'}{L}(s,\chi_m)=-\sum_{\mathfrak{a}}\frac{\Lambda(\mathfrak{a})\chi_m(\mathfrak{a})}{N(\mathfrak{a})^s}$$

for $\sigma>1$.

(c) Show that there is an absolute constant $c>0$ such that $L(s,\chi_m)\neq 0$ for $\sigma>1-c/\log m\tau$ for every positive integer m.

11.2 Exceptional zeros

Although there is no known quadratic character χ for which $L(s,\chi)$ has an exceptional real zero, the possible existence of such zeros is a recurring issue in the theory in its current stage of development. The techniques of the preceding section do not seem to offer a means of eliminating exceptional zeros entirely, but nevertheless they may be used to show that such zeros occur at most rarely. To this end we introduce a variant of Lemma 11.5 that allows us to consider two different quadratic characters.

Lemma 11.6 (Landau) *Suppose that χ_1 and χ_2 are quadratic characters. If $\sigma>1$, then*

$$-\frac{\zeta'}{\zeta}(\sigma)-\frac{L'}{L}(\sigma,\chi_1)-\frac{L'}{L}(\sigma,\chi_2)-\frac{L'}{L}(\sigma,\chi_1\chi_2)\ \geq\ 0.$$

Proof It suffices to express the left-hand side as a Dirichlet series and to note that

$$1+\chi_1(n)+\chi_2(n)+\chi_1\chi_2(n)=(1+\chi_1(n))(1+\chi_2(n))\geq 0$$

for all n. □

Theorem 11.7 (Landau) *There is a constant $c>0$ such that if χ_1 and χ_2 are quadratic characters modulo q_1 and q_2, respectively, and if $\chi_1\chi_2$ is non-principal, then $L(s,\chi_1)L(s,\chi_2)$ has at most one real zero β such that $1-c/\log q_1q_2<\beta<1$.*

Proof Since any given L-function can have at most one such zero, if there are two zeros, then one of them, say β_1, is a zero of $L(s, \chi_1)$, and the other, β_2, is a zero of $L(s, \chi_2)$. We may assume that c is so small that $5/6 \le \beta_i < 1$. Also, we note that $\chi_1\chi_2$ is a non-principal character (mod q_1q_2). Hence by four applications of Lemma 11.1 we see that if $0 < \delta \le 1$, then

$$-\frac{\zeta'}{\zeta}(1+\delta) \le \frac{1}{\delta} + c_1 \log 4,$$

$$-\frac{L'}{L}(1+\delta, \chi_i) \le \frac{-1}{1+\delta-\beta_i} + c_1 \log q_i,$$

$$-\frac{L'}{L}(1+\delta, \chi_1\chi_2) \le c_1 \log q_1q_2.$$

We sum these inequalities and apply Lemma 11.4 to see that

$$\frac{1}{\delta} - \frac{1}{1+\delta-\beta_1} - \frac{1}{1+\delta-\beta_2} + c_2 \log q_1q_2 \ge 0.$$

Without loss of generality we may suppose that $\beta_1 \le \beta_2$. Then

$$\frac{1}{\delta} - \frac{2}{1+\delta-\beta_1} + c_2 \log q_1q_2 \ge 0,$$

and by taking $\delta = 2(1-\beta_1)$ we deduce that

$$1 - \beta_1 \ge \frac{1}{6c_2 \log q_1q_2}.$$

□

The following corollaries are immediate.

Corollary 11.8 (Landau) *There is a positive constant $c > 0$ such that $\prod_\chi L(s, \chi)$ has at most one zero in the region $\sigma > 1 - c/\log q\tau$. Here the product is over all Dirichlet characters χ* (mod q). *If such a zero exists then it is necessarily real and the associated character χ is quadratic.*

Corollary 11.9 (Landau) *For each positive number A there is a $c(A) > 0$ such that if $\{q_i\}$ is a strictly increasing sequence of natural numbers with the property that for each q_i there is a primitive quadratic character χ_i* (mod q_i) *for which $L(s, \chi_i)$ has a zero β_i satisfying*

$$\beta_i > 1 - \frac{c(A)}{\log q_i},$$

then

$$q_{i+1} > q_i^A.$$

Corollary 11.10 (Page) *There is a constant $c > 0$ such that for every $Q \ge 1$ the region $\sigma \ge 1 - c/\log Q\tau$ contains at most one zero of the function*

$$\prod_{q \le Q} \prod_{\chi}{}^{*} L(s, \chi)$$

where $\prod_{\chi}^{*}$ denotes a product over all primitive characters χ (mod q). If such a zero exists, then it is necessarily real and the associated character χ is quadratic.

We now turn to the problem of showing that even an exceptional zero cannot be too close to 1. By taking $s = 1$ in (11.10) we see that this is equivalent to showing that $L(1, \chi)$ cannot be too small. Suppose that χ is a primitive quadratic character modulo q, and let $r(n) = \sum_{d|n} \chi(d)$. Then $r(n) \ge 0$ for all n and $r(n) \ge 1$ when n is a perfect square. Since $\sum_{n=1}^{\infty} r(n)n^{-s} = \zeta(s)L(s, \chi)$ for $\sigma > 1$, we find that

$$\sum_{n \le x} r(n)n^{-s} = \frac{L(1, \chi)x^{1-s}}{1 - s} + \zeta(s)L(s, \chi) + \text{error terms}. \qquad (11.14)$$

Here the error terms are small if x is sufficiently large in terms of q. Estimates of this kind can be derived from Corollary 1.15 by the method of the hyperbola, or else by employing an inverse Mellin transform. Suppose that $0 \le s < 1$ in the above. We can give a lower bound for the left-hand side, which yields a lower bound for $L(1, \chi)$ if the second term on the right-hand side does not interfere. Since $\zeta(s) < 0$ for $0 < s < 1$ (cf. Corollary 1.14), this term is harmless if $L(s, \chi) \ge 0$. If this cannot be arranged, we may alternatively eliminate this term by taking two values of x and differencing. Since the method of the hyperbola leads to tedious details, we use an inverse Mellin transform to derive a more precise version of (11.14). To make the estimates easier we introduce an Abelian weighting of the sum. By (5.23) with x replaced by $1/x$ we see that

$$\sum_{n=1}^{\infty} r(n)e^{n/x} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s)L(s, \chi)\Gamma(s)x^s \, ds.$$

We move the contour of integration to the line $\Re s = -1/2$, which gives rise to residues at the poles at $s = 1$ and $s = 0$. Thus the above is

$$= L(1, \chi)x + \zeta(0)L(0, \chi) + \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \zeta(s)L(s, \chi)\Gamma(s)x^s \, ds.$$

By Corollary 10.5 we know that $\zeta(-1/2 + it) \ll \tau$, by Corollary 10.10 we know that $L(-1/2 + it, \chi) \ll q\tau$, and by (C.19) we know that $\Gamma(-1/2 + it) \ll \tau^{-1}e^{-\pi\tau/2}$. Hence the integral is $\ll qx^{-1/2}$. By (10.11) we know that $\zeta(0) = -1/2$, and by Corollary 10.9 we know that $L(0, \chi) \ge 0$. (More precisely, $L(0,\chi)=0$ if $\chi(-1)=1$, and $L(0,\chi)\asymp q^{1/2}L(1,\chi)$ if $\chi(-1)=-1$.) Since the perfect squares on the left-hand side contribute an amount $\gg x^{1/2}$, we deduce that

$$x^{1/2}\ll xL(1,\chi)+qx^{-1/2}.$$

On taking $x=Cq$ with C a large constant we deduce that $L(1,\chi)\gg q^{-1/2}$. Now consider the possibility that χ is an imprimitive quadratic character. Then there is a primitive quadratic character $\chi^\star$ modulo d, with $d|q$, that induces χ. Thus $L(1,\chi)=L(1,\chi^\star)\prod_{p|q/d}(1-\chi^\star(p)/p)\ge L(1,\chi^\star)\varphi(q/d)d/q\gg d^{-1/2}(\log\log 3q/d)^{-1}\gg q^{-1/2}$, by Theorem 2.9, so we have

Theorem 11.11 *If χ is a quadratic character modulo q, then $L(1,\chi)\gg q^{-1/2}$.*

By (11.10) the following corollary is immediate.

Corollary 11.12 *There is an absolute constant $c>0$ such that if χ is a quadratic character modulo q and $L(s,\chi)$ has an exceptional zero β_1, then*

$$\beta_1\le 1-\frac{c}{q^{1/2}(\log q)^2}.$$

By elaborating on the above argument we can obtain better lower bounds for $1-\beta_1$. To facilitate this we first establish a convenient inequality that depends only on the analyticity and size of the relevant Dirichlet series in the immediate vicinity of the real axis.

Lemma 11.13 (Estermann) *Suppose that $f(s)$ is analytic for $|s-2|\le 3/2$, and that $|f(s)|\le M$ for s in this disc. Suppose also that*

$$F(s)=\zeta(s)f(s)=\sum_{n=1}^{\infty}r(n)n^{-s}$$

for $\sigma>1$, that $r(1)=1$, and that $r(n)\ge 0$ for all n. If there is a $\sigma\in[19/20,1)$ such that $f(\sigma)\ge 0$, then

$$f(1)\ge\frac{1}{4}(1-\sigma)M^{-3(1-\sigma)}.$$

To put this in perspective, we recall that our proof in Chapter 4 that $L(1,\chi)\ne 0$ depended on Landau's theorem (Theorem 1.7). The above amounts to a quantitative elaboration of Landau's theorem, for if $f(1)$ were 0, then $F(s)$ would be analytic for $s>1/2$, so by Landau's theorem the Dirichlet series would converge when $\sigma>1/2$. This would imply that $F(\sigma)>0$ for $\sigma>1/2$. But $\zeta(\sigma)<0$ for $1/2<\sigma<1$ (cf. Corollary 1.14), so it would follow that

$f(\sigma) < 0$ in this interval. Thus the hypothesis above that $f(\sigma) \ge 0$ implies – by Landau's theorem – that $f(1) > 0$. In the above we obtain not just this qualitative information but a quantitative lower bound for $f(1)$ in terms of the size of σ and the size of $f(s)$ in a surrounding disc.

Proof As in the proof of Landau's theorem we begin by expanding $F(s)$ in powers of $2 - s$,

$$F(s) = \sum_{k=0}^{\infty} b_k (2-s)^k \tag{11.15}$$

for $|s-2| < 1$. By Cauchy's coefficient formula we know that

$$b_k = \frac{(-1)^k}{k!} F^{(k)}(2) = \frac{1}{k!} \sum_{n=1}^{\infty} r(n) n^{-2} (\log n)^k.$$

Thus $b_k \ge 0$ for all k, and $b_0 = \sum_{n=1}^{\infty} r(n) n^{-2} \ge 1$. For $|s-2| < 1$ we may write

$$\frac{1}{s-1} = \frac{1}{1-(2-s)} = \sum_{k=0}^{\infty} (2-s)^k.$$

On multiplying this by $f(1)$ and subtracting from (11.15) we deduce that

$$F(s) - \frac{f(1)}{s-1} = \sum_{k=0}^{\infty} (b_k - f(1))(2-s)^k \tag{11.16}$$

for $|s-2| < 1$. But the left-hand side is analytic for $|s-2| \le 3/2$, so the series converges in this larger disc. In order to estimate the coefficients on the right-hand side we bound the left-hand side when s lies on the circle $|s-2| = 3/2$. To this end, we note by (1.24) that

$$\begin{aligned} |\zeta(s)| &= \left| 1 + \frac{1}{s-1} + s \int_1^{\infty} \frac{[u]-u}{u^{s+1}}\, du \right| \\ &\le 1 + \frac{1}{|s-1|} + \frac{|s|}{\sigma}. \end{aligned}$$

The relation $|s-2| = 3/2$ implies that $|s-1| \ge 1/2$, that $|s| \le 7/2$, and that $\sigma \ge 1/2$. Hence $|\zeta(s)| \le 10$ for the s under consideration. Since $|f(1)/(s-1)| \le 2M$, it follows that the left-hand side of (11.16) has modulus $\le 12M$ for $|s-2| \le 3/2$. By the Cauchy coefficient inequalities we deduce that $|b_k - f(1)| \le 12M(2/3)^k$. We apply this bound for all $k > K$ where K is a parameter to be chosen later. Thus from (11.16) we see that if $1/2 < \sigma \le 2$, then

$$\zeta(\sigma) f(\sigma) - \frac{f(1)}{\sigma - 1} \ge \sum_{k=0}^{K} (b_k - f(1))(2-\sigma)^k - 12M \sum_{k>K} \left(\tfrac{2}{3}(2-\sigma)\right)^k.$$

We observe that if $19/20 \le \sigma < 1$, then $\frac{2}{3}(2-\sigma) \le 7/10$. We also recall that $b_0 \ge 1$ and that $b_k \ge 0$ for all k. Hence the above is

$$\ge 1 - f(1)\frac{1-(2-\sigma)^{K+1}}{1-(2-\sigma)} - 40M(7/10)^{K+1}.$$

On cancelling the common term $f(1)/(1-\sigma)$ from both sides, and rearranging, we find that

$$1 \le \frac{f(1)(2-\sigma)^{K+1}}{1-\sigma} + \zeta(\sigma)f(\sigma) + 40M(7/10)^{K+1},$$

a relation comparable to (11.14). To ensure that the last term on the right does not overwhelm the left-hand side, we take $K = [(\log 80M)/\log 10/7]$. Then the last term on the right is $\le 1/2$. Since $\zeta(\sigma) < 0$ by Corollary 1.14, and $f(\sigma) \ge 0$ by hypothesis, it follows that

$$f(1) \ge \frac{1}{2}(1-\sigma)(2-\sigma)^{-K-1} \ge \frac{10}{21}(1-\sigma)(2-\sigma)^{-K}. \tag{11.17}$$

But

$$\begin{aligned}(2-\sigma)^K &\le (2-\sigma)^{(\log 80M)/\log 10/7} = (80M)^{(\log(2-\sigma))/\log 10/7}\\ &\le 80^{(\log 21/20)/\log 10/7} M^{(\log(2-\sigma))/\log 10/7}.\end{aligned}$$

Here the first factor is $< 13/7$. Since $\log(1+\delta) \le \delta$ for any $\delta \ge 0$, on taking $\delta = 1-\sigma$ we see that $\log(2-\sigma) \le 1-\sigma$. Also, $\log 10/7 > 1/3$ and it can certainly be supposed that $M \ge 1$, so the expression above is $< (13/7)M^{3(1-\sigma)}$. This with (11.17) gives the desired lower bound for $f(1)$. □

We are now prepared to prove an important strengthening of Theorem 11.11.

Theorem 11.14 (Siegel) *For each positive number ε there is a positive constant $C(\varepsilon)$ such that if χ is a quadratic character modulo q, then*

$$L(1,\chi) > C(\varepsilon)q^{-\varepsilon}.$$

Proof We assume, as we may, that $\varepsilon \le 1/5$. For the present we restrict our attention to primitive characters. We consider two cases, according to whether there exists a primitive quadratic character χ_1 such that $L(s,\chi_1)$ has a real zero β_1 in the interval $[1-\varepsilon/4, 1)$, or not. Suppose first that there is no such zero. We take $f(s) = L(s,\chi)$, $\sigma = 1-\varepsilon/4$. Then $f(\sigma) > 0$ and by Lemma 10.15 we may take $M \ll q^{1/2}$. Hence by Lemma 11.13, $f(1) \gg \varepsilon q^{-3\varepsilon/8}$. Thus there is a constant $C_1(\varepsilon) > 0$ such that $L(1,\chi) \ge C_1(\varepsilon)q^{-\varepsilon}$.

Now consider the contrary case, in which there is a primitive quadratic character χ_1 modulo q_1 such that $L(s,\chi_1)$ has a real zero $\beta_1 \ge 1-\varepsilon/4$. Since $L(1,\chi_1) > 0$ there is a constant $C_2(\varepsilon) > 0$ such that $L(1,\chi_1) \ge C_2(\varepsilon)q_1^{-\varepsilon}$.

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Now suppose that χ is a primitive quadratic character, $\chi \neq \chi_1$. We apply Lemma 11.13 with $f(s) = L(s, \chi)L(s, \chi_1)L(s, \chi\chi_1)$. To see that the Dirichlet series coefficients of $\zeta(s)f(s)$ are non-negative, we note first that if $g(s)$ is a Dirichlet series with non-negative coefficients, then $\exp g(s)$ is also a Dirichlet series with non-negative coefficients, since the power series coefficients of the exponential function are non-negative. Then it suffices to apply this observation with

$$g(s) = \log \zeta(s)f(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n}(1 + \chi(n))(1 + \chi_1(n))n^{-s}.$$

In view of Lemma 10.15 we may take $M = C_3 q q_1$. On taking $\sigma = \beta_1$, we find that

$$f(1) \geq \frac{1}{4}(C_3 q q_1)^{-3(1-\beta_1)} \geq \frac{1}{4}(C_3 q q_1)^{-3\varepsilon/4} \geq C_4(\varepsilon)q^{-\varepsilon}.$$

Now

$$f(1) = L(1, \chi)L(1, \chi_1)L(1, \chi\chi_1) \ll L(1, \chi)(\log q q_1)^2$$

by Lemma 10.15, and hence we deduce that

$$L(1, \chi) \geq C_5(\varepsilon)q^{-2\varepsilon}. \tag{11.18}$$

We may assume that $C_5 \leq C_1$, so that (11.18) holds in either case.

We now extend to imprimitive characters. Suppose that χ is induced by a primitive character χ^* (mod d), so that $q = dr$ for some r. Then

$$L(1, \chi) = L(1, \chi^*) \prod_{p|r} \left(1 - \frac{\chi^*(p)}{p}\right) \geq L(1, \chi^*)\frac{\varphi(r)}{r} \geq C_5(\varepsilon)d^{-2\varepsilon}\frac{\varphi(r)}{r}.$$

By Theorem 2.9 the above is

$$\geq C_6(\varepsilon)(dr)^{-2\varepsilon} = C_6(\varepsilon)q^{-2\varepsilon},$$

and hence the proof is complete. □

We are unable to compute the value of the constant $C(\varepsilon)$ in Siegel's theorem when $\varepsilon < 1/2$, because we have no way of estimating the size of the smallest possible q_1 when the second case arises in the proof. Such a constant is called 'non-effective.' This is our first encounter with a non-effective constant, so the distinction between effectively computable constants and non-effective constants arises here for the first time.

Corollary 11.15 *For any $\varepsilon > 0$ there is a positive number $C(\varepsilon)$ such that if χ is a quadratic character modulo q and β is a real zero of $L(s, \chi)$, then $\beta < 1 - C(\varepsilon)q^{-\varepsilon}$.*

Proof We may certainly suppose that $\beta > 1 - c/\log 4q > 1 - \frac{1}{\log q}$, where c is the number appearing in Theorem 11.3, so that β is an exceptional zero by the criterion following that theorem. By taking $s = 1$ in (10) we see that

$$L(1, \chi) \ll (1 - \beta)(\log q)^2$$

and the corollary follows easily from the theorem. □

11.2.1 Exercises

1. Call a modulus q 'exceptional' if there is a primitive quadratic character $\chi \pmod q$ such that $L(s, \chi)$ has a real zero β such that $\beta > 1 - c/\log q$. Show that if c is sufficiently small, then the number of exceptional q not exceeding Q is $\ll \log\log Q$.
2. Use the last part of Theorem 4 to show that if $L(s, \chi)$ has an exceptional zero β_1, then $L'(\beta_1, \chi) \gg 1$.
3. (cf. Mahler 1934, Davenport 1966, Haneke 1973, Goldfeld & Schinzel 1975) Suppose that χ is a quadratic character, and put $r(n) = \sum_{d|n} \chi(d)$.
   (a) Show that
$$\sum_{n \le y} \frac{\chi(n)}{n} = L(1, \chi) + O\left(q^{1/2} y^{-1} \log q\right).$$
   (b) Show that
$$\sum_{n \le y} \frac{\chi(n) \log n}{n} = -L'\left(1, \chi\right) + O(q^{1/2} y^{-1} (\log qy)^2)\right).$$
   (c) Verify that
$$\begin{aligned}\sum_{n \le x} \frac{r(n)}{n} &= \sum_{d \le y} \frac{\chi(d)}{d} \sum_{m \le x/d} \frac{1}{m} + \sum_{m \le x/y} \frac{1}{m} \sum_{d \le x/m} \frac{\chi(d)}{d} \\ &\quad - \left(\sum_{d \le y} \frac{\chi(d)}{d}\right)\left(\sum_{m \le x/y} \frac{1}{m}\right) \\ &= \Sigma_1 + \Sigma_2 - \Sigma_3,\end{aligned}$$
   say.
   (d) Show that
$$\Sigma_1 = (\log x + C_0) L(1, \chi) + L'(1, \chi) + O\left(q^{1/2} y^{-1} (\log qy)^2\right) + O(yx^{-1}).$$
   (e) Show that
$$\Sigma_2 = (\log x/y + C_0) L(1, \chi) + O(yx^{-1} \log q) + O\left(q^{1/2} y^{-1} \log q\right).$$

(f) Show that

$$\Sigma_3 = (\log x/y + C_0)L(1, \chi) + O(yx^{-1}\log q) + O\big(q^{1/2}y^{-1}(\log qx)^2\big).$$

(g) Show that

$$\sum_{n\le x}\frac{r(n)}{n} = (\log x + C_0)L(1, \chi) + L'(1, \chi) + O\big(q^{1/4}x^{-1/2}(\log qx)^{3/2}\big).$$

(h) Show that for each $c < 1/2$ there is a constant $q_0(c)$ such that if $q \ge q_0(c)$ and $L(1, \chi) < c/\log q$, then

$$L'(1, \chi) \asymp \sum_{n\le q}\frac{r(n)}{n}.$$

(i) Show that $L''(\sigma, \chi) \ll (\log q)^3$ for $\sigma \ge 1 - 1/\log q$.

(j) Show that there is an absolute constant $c > 0$ such that if $L(s, \chi)$ has an exceptional zero β_1 for which $\beta_1 \ge 1 - c/(\log q)^3$, then

$$L(1, \chi) \asymp (1-\beta_1)\sum_{n\le q}\frac{r(n)}{n}.$$

4. Use Estermann's lemma (Lemma 11.13) to give a second proof that if $L(s, \chi)$ has an exceptional zero β_1, then $L(1, \chi) \gg 1 - \beta_1$ (cf. (11.10) of Theorem 11.4).
5. Use Estermann's lemma (Lemma 11.13) to give a second proof that if χ is a cubic character $(\bmod\, q)$, then $L(1, \chi) \gg (\log q)^{-1/2}$ (cf. Exercise 11.1.4(e)).
6. (Tatuzawa 1951) Let χ_1 and χ_2 be distinct primitive quadratic characters, modulo q_1 and q_2, respectively, and suppose that $L(1, \chi_i) < C\varepsilon q_i^{-\varepsilon}$ for $i = 1, 2$ where $0 < \varepsilon \le 1$ and $C > 0$.
   (a) Show that $\min_{x>1}\frac{x}{\log x} = e$. By a change of variables, deduce that if $\varepsilon > 0$, then $\min_{x>1} x^\varepsilon/\log x = e\varepsilon$. Use this to show that $\min_{x>1} x^\varepsilon/(\log x)^2 = e^2\varepsilon^2/4$.
   (b) Explain why there exists a constant $c_1 > 0$ such that $L(1, \chi) \ge c_1/\log q$ whenever $L(s, \chi)$ has no exceptional zero. Let $C_1 = ec_1$. Show that if $C < C_1$, then $L(s, \chi_1)$ and $L(s, \chi_2)$ have exceptional zeros, say β_1 and β_2. (From now on, suppose that $C < C_1$.)
   (c) Explain why there is a positive constant c_2 such that $L(1, \chi) \ge c_2(1-\beta)$ whenever β is an exceptional zero of $L(s, \chi)$. Let $C_2 = c_2/6$. Show that if $C < C_2$, then $\beta > 1 - \varepsilon/6$. Let $C_3 = c_2/20$. Show that if $C < C_3$, then $\beta > 19/20$. (From now on, suppose that $C < C_i$ for $i = 1, 2, 3$.)
   (d) Explain why there is a constant $c_3 > 0$ such that at most one of $L(s, \chi_1)$, $L(s, \chi_2)$ has a zero in the interval $[1 - c_3/\log q_1q_2, 1]$.
   (e) Show that $L(s, \chi_1)L(s, \chi_2)$ has a zero β that satisfies the three inequalities $\beta \ge 19/20$, $\beta \ge 1 - \varepsilon/6$, $\beta \le 1 - c_3/\log q_1q_2$.

(f) Let $f(s) = L(s, \chi_1)L(s, \chi_2)L(s, \chi_1\chi_2)$. Show that there is an absolute constant $c_4 > 0$ such that $f(1) \ge c_4(\log q_1q_2)^{-1}(q_1q_2)^{-\varepsilon/2}$.

(g) Explain why there is a constant $c_5 > 0$ such that $L(1, \chi_1\chi_2) \le c_5 \log q_1q_2$.

(h) Show that $C \ge c_4^{1/2}c_5^{-1/2}e/4$.

(i) Conclude that there is a positive effectively computable absolute C such that if $0 < \varepsilon \le 1$, then the inequality $L(1, \chi) > C\varepsilon q^{-\varepsilon}$ holds for all primitive quadratic characters, with at most one exception.

7. (Fekete & Pólya 1912, Pólya & Szegö 1925, p. 44, Heilbronn 1937) Let $S_1(x, \chi) = \sum_{1\le n\le x} \chi(n)$.

(a) Show that if χ is a quadratic character such that $S_1(x, \chi) \ge 0$ for all $x \ge 1$, then $L(\sigma, \chi) > 0$ for all $\sigma > 0$.

(b) Let $\chi_d(n) = \left(\frac{d}{n}\right)$. Show that the hypothesis above holds for $d = -3, -4, -7, -8$, but not for $d = 5,\ 8$.

(c) For $k > 1$ let $S_k(N, \chi) = \sum_{n=1}^{N} S_{k-1}(n, \chi)$. Show that

$$S_k(N, \chi) = \sum_{n=1}^{N} \binom{N - n + k - 1}{k - 1} \chi(n).$$

(d) Let $\Delta f(x) = f(x + 1) - f(x)$ and $\Delta_k f(x) = \Delta(\Delta_{k-1} f(x))$. Show that $\Delta_k f(x) = \sum_{r=0}^{k}(-1)^r \binom{k}{r} f(x + k - r)$, and that if $f^{(k)}(x)$ is continuous then

$$\Delta_k f(x) = \int_x^{x+1} \int_{u_1}^{u_1+1} \cdots \int_{u_{k-1}}^{u_{k-1}+1} f^{(k)}(u_k)\, du_k du_{k-1} \cdots du_1.$$

(e) Show that if $\sigma > 0$, then $(-1)^k \Delta_k(x^{-\sigma}) > 0$ for all $x > 0$.

(f) Show that $L(s, \chi) = (-1)^k \sum_{n=1}^{\infty} S_k(n, \chi)\Delta_k(n^{-s})$ for $\sigma > 0$.

(g) Show that if χ is a quadratic character and k is an integer such that $S_k(N, \chi) \ge 0$ for all integers $N \ge 1$, then $L(\sigma, \chi) > 0$ for all $\sigma > 0$.

(h) For $\chi_5(n) = \left(\frac{5}{n}\right)$ and $\chi_8(n) = \left(\frac{8}{n}\right)$ find the least k such that the hypothesis above is satisfied.

(i) Let $P(z, \chi) = \sum_{n=1}^{\infty} \chi(n)z^n$ for $|z| < 1$. Show that $P(z, \chi)(1 - z)^{-k} = \sum_{n=1}^{\infty} S_k(n, \chi)z^n$ for $|z| < 1$.

(j) Show that if χ is a quadratic character for which $S_k(N, \chi) \ge 0$ for all positive integers N, then $P(z, \chi) > 0$ for $0 < z < 1$.

(k) Show that $\sum_{n=1}^{12} \left(\frac{n}{163}\right)(7/10)^n = -0.0483$, and that $\sum_{n=13}^{\infty}(7/10)^n = 0.0323$. Deduce that $P(0.7, \chi_{-163}) < 0$, and hence that for any k there is an N for which $S_k(N, \chi_{-163}) < 0$.

8. S. Chowla (1972) conjectured that for any primitive quadratic character χ^* there is a character χ induced by χ^* such that $S_1(x, \chi) \ge 0$ for all $x \ge 1$ (in the notation of the preceding exercise). Show that Chowla's conjecture implies that $L(\sigma, \chi) > 0$ when χ is a quadratic character and $\sigma > 0$. See also Rosser (1950).
9. (Bateman & Chowla 1953) Suppose that k is a positive integer such that
$$\sum_{1\le n\le x} \frac{\lambda(n)}{n}\Big(1-\frac{n}{x}\Big)^k \ge 0 \tag{11.19}$$
for all $x \ge 1$. (It is not known whether there is such a k.) (a) Show that if χ is a quadratic character, then
$$\sum_{1\le n\le x} \frac{\chi(n)}{n}\Big(1-\frac{n}{x}\Big)^k \ge \sum_{1\le n\le x} \frac{\lambda(n)}{n}\Big(1-\frac{n}{x}\Big)^k$$
for all $x \ge 1$.
(b) Show that if there is a k such that (11.19) holds for all $x \ge 1$, then $L(\sigma, \chi) > 0$ when χ is a quadratic character and $\sigma > 0$.

11.3 The Prime Number Theorem for arithmetic progressions

The various inequalities for zeros of Dirichlet L-functions established above are motivated by a desire to imitate for primes in arithmetic progressions the quantitative form of the Prime Number Theorem achieved in Theorem 6.9. For $(a, q) = 1$ we set
$$\pi(x; q, a) = \sum_{\substack{p\le x\\ p\equiv a\,(q)}} 1, \quad \vartheta(x; q, a) = \sum_{\substack{p\le x\\ p\equiv a\,(q)}} \log p, \quad \psi(x; q, a) = \sum_{\substack{n\le x\\ n\equiv a\,(q)}} \Lambda(n), \tag{11.20}$$
and correspondingly for any Dirichlet character χ we put
$$\pi(x, \chi) = \sum_{p\le x} \chi(p), \quad \vartheta(x, \chi) = \sum_{p\le x} \chi(p)\log p, \quad \psi(x, \chi) = \sum_{n\le x} \chi(n)\Lambda(n). \tag{11.21}$$

By multiplying both sides of (4.27) by $\Lambda(n)$, and summing over $n \le x$, we see that
$$\psi(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a)\psi(x, \chi), \tag{11.22}$$
and similarly for $\pi(x; q, a)$ and $\vartheta(x; q, a)$. We deal with $\psi(x, \chi)$ in much the same way that we dealt with $\psi(x)$ in Chapter 6.

Theorem 11.16 *There is a constant* $c_1 > 0$ *such that if* $q \le \exp(2c_1\sqrt{\log x})$, *then*

$$\psi(x,\chi) = E_0(\chi)x + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big) \qquad (11.23)$$

when $L(s,\chi)$ *has no exceptional zero, but*

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big) \qquad (11.24)$$

when $L(s,\chi)$ *has an exceptional zero* β_1. *Here* $E_0(\chi) = 1$ *if* $\chi = \chi_0$, *and* $E_0(\chi) = 0$ *otherwise.*

Proof By Theorems 4.8 and 5.2 we see that

$$\psi(x,\chi) = \frac{-1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{L'}{L}(s,\chi)\frac{x^s}{s}\,ds + R$$

where $\sigma_0 > 1$ and

$$R \ll \sum_{x/2<n<2x} \Lambda(n)\min\left(1, \frac{x}{T|x-n|}\right) + \frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_0}}$$

by Corollary 5.3. As in the proof of Theorem 6.9 we suppose that $2 \le T \le x$ and set $\sigma_0 = 1 + 1/\log x$. Thus

$$R \ll \frac{x}{T}(\log x)^2,$$

as before. As in the proof of Theorem 6.9, we let $\mathcal{C}$ denote a closed contour that consists of line segments joining the points $\sigma_0 - iT$, $\sigma_0 + iT$, $\sigma_1 + iT$, $\sigma_1 - iT$, but now the choice of σ_1 is a little more complicated, since we want to ensure that $\mathcal{C}$ does not pass too closely to an exceptional zero.

Case 1. *There is no exceptional zero.* In this case we take $\sigma_1 = 1 - c/(5\log qT)$ where c is the constant in Theorem 11.3. If χ is non-principal, then the integrand is analytic on and inside $\mathcal{C}$, but if $\chi = \chi_0$, then it has a pole at $s = 1$ with residue x. Hence

$$\frac{-1}{2\pi i}\int_{\mathcal{C}} \frac{L'}{L}(s,\chi)\frac{x^s}{s}\,ds = E_0(\chi)x. \qquad (11.25)$$

We estimate the integrals from $\sigma_0 + iT$ to $\sigma_1 + iT$, from $\sigma_1 + iT$ to $\sigma_1 - iT$, and from $\sigma_1 - iT$ to $\sigma_0 - iT$ as in the proof of Theorem 6.9, using the estimate (11.5) of Theorem 11.4. Thus we find that

$$\psi(x,\chi) - E_0(\chi)x \ll x(\log x)^2\left(\frac{1}{T} + \exp\left(\frac{-c\log x}{5\log qT}\right)\right). \qquad (11.26)$$

Case 2. *There is an exceptional zero* β_1, *and it satisfies* $\beta_1 \ge 1 - c/(4\log qT)$. In this case we take $\sigma_1 = 1 - c/(3\log qT)$. The integrand in (11.25) now has a pole inside $\mathcal{C}$ at β_1, so the left-hand side of (11.25) has the value $-x^{\beta_1}/\beta_1$. Otherwise, the estimates proceed as before, and we find that

$$\psi(x, \chi) = -\frac{x^{\beta_1}}{\beta_1} + O\left(x(\log x)^2\left(\frac{1}{T} + \exp\left(\frac{-c\log x}{5\log qT}\right)\right)\right). \quad (11.27)$$

Case 3. *There is an exceptional zero β_1, but it satisfies $\beta_1 < 1 - c/(4\log qT)$.* We proceed exactly as in Case 1, and so we obtain (11.26). To pass to (11.27) it suffices to note that

$$\frac{x^{\beta_1}}{\beta_1} \ll x\exp\left(\frac{-c\log x}{5\log qT}\right)$$

in the current case.

We have established (11.26) if there is no exceptional zero, and (11.27) if there is one. To complete our argument, we need only observe that if $c_1 = \sqrt{c/20}$, if $q \le \exp(2c_1\sqrt{\log x})$, and if $T = \exp(2c_1\sqrt{\log x})$, then (11.26) gives (11.23) and (11.27) gives (11.24). □

We are now in a position to prove

Corollary 11.17 (Page) *Let c_1 be the same constant as in Theorem* 11.16. *If* $(a, q) = 1$, *then*

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big) \quad (11.28)$$

when there is no exceptional character modulo q, and

$$\psi(x; q, a) = \frac{x}{\varphi(q)} - \frac{\chi_1(a)x^{\beta_1}}{\varphi(q)\beta_1} + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big) \quad (11.29)$$

when there is an exceptional character χ_1 modulo q and β_1 is the concomitant zero.

Proof If $q \le \exp\big(2c_1\sqrt{\log x}\big)$, then we have only to insert the estimates of Theorem 11.16 into (11.22). If q is larger, then the stated estimates are still valid, but are worse than trivial. To see this, note first that the largest term in $\psi(x; q, a)$ is $\le \log x$, and the number of terms is $\le x/q + 1$, so it is immediate that

$$\psi(x; q, a) \le (x/q + 1)\log x \ll x\exp(-c_1\sqrt{\log x})$$

when $q \ge \exp(2c_1\sqrt{\log x})$. □

Presumably, exceptional zeros do not exist. However, if such a zero does exist, then we have a second main term in (11.29) that is bigger than the error term when $x < \exp(c_1^2/(1-\beta_1)^2)$. If β_1 is extremely close to 1, then one might have $\beta_1 \ge 1 - 1/\log x$, and in such a situation the second main term is of the same order of magnitude as the first main term, since

$$x - \frac{x^{\beta_1}}{\beta_1} = (\beta_1 - 1)x^{\beta_1}/\beta_1 + (\log x)\int_{\beta_1}^1 x^\sigma \, d\sigma \asymp (1-\beta_1)x\log x. \quad (11.30)$$

Thus if $1-\beta_1$ is small compared with $1/\log x$, then the main term is nearly doubled if $\chi_1(a) = -1$, and it is nearly annihilated if $\chi_1(a) = 1$. Unfortunately, the upper bound provided by the Brun–Titchmarsh theorem (Theorem 3.9) is not quite strong enough to refute such a possibility.

The constants c and c_1 in Theorems 11.3, 11.4, 11.16 and Corollary 11.17 are effectively computable. However, if we are willing to accept non-effective constants, then by Siegel's theorem (Theorem 11.14), or more precisely by its corollary (Corollary 11.15), we can eliminate the second main term, provided that q is more sharply limited.

Corollary 11.18 *Let c_1 be the same constant as in Theorem* 11.16. *For any positive A there is an $x_0(A)$ such that if $q \le (\log x)^A$, then*

$$\psi(x,\chi) = E_0(\chi)x + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big) \quad (11.31)$$

for $x \ge x_0(A)$.

Proof Suppose that χ is quadratic and that $L(s,\chi)$ has an exceptional zero β_1. Then

$$x^{\beta_1} = x\exp(-(1-\beta_1)\log x) \le x\exp(-C(\varepsilon)q^{-\varepsilon}\log x)$$

by Siegel's theorem (Corollary 11.15). Since $q \le (\log x)^A$, the above is

$$\le x\exp(-C(\varepsilon)(\log x)^{1-A\varepsilon}).$$

In order to reach (11.31) we need to take ε a little smaller than $1/(2A)$, say $\varepsilon = 1/(3A)$. Then the above is

$$\le x\exp\big(-c_1\sqrt{\log x}\big)$$

provided that $x \ge x_0 = \exp((c_1/C(\varepsilon))^6)$. □

The constraint $q \le (\log x)^A$ can be rewritten as $x \ge \exp(q^{1/A})$. This implies the constraint $x \ge x_0(A)$ if q is sufficiently large, say $q \ge q_0(A)$. We note also that the implicit constant in (11.31) is absolute. If we were to allow the implicit constant to depend on A, e.g. to be as large as $\exp((c_1/C(\varepsilon))^3)$, then we would obtain an estimate

$$\psi(x,\chi) \ll_A x\exp\big(-c_1\sqrt{\log x}\big)$$

that is valid for all q and all $x \ge \exp\big(q^{1/A}\big)$, though of course the implicit constant is so large that the bound is worse than the trivial $\psi(x,\chi) \ll x$ when $x < x_0$. By applying (11.22) and (11.28), we obtain

Corollary 11.19 (The Siegel–Walfisz theorem) *Let c_1 be the constant in Theorem* 11.16*, and suppose that A is given, $A > 0$. If $q \le (\log x)^A$ and $(a,q) = 1$, then*

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O_A\big(x\exp\big(-c_1\sqrt{\log x}\big)\big).$$

Pertaining to $\vartheta(x;q,a)$ and $\pi(x;q,a)$ we have estimates similar to those of Corollary 11.17.

Corollary 11.20 *Let c_1 be the constant in Theorem* 11.16*. If $(a,q) = 1$, then*

$$\vartheta(x;q,a) = \frac{x}{\varphi(q)} + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big) \tag{11.32}$$

and

$$\pi(x;q,a) = \frac{\mathrm{li}(x)}{\varphi(q)} + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big) \tag{11.33}$$

when there is no exceptional character modulo q, but

$$\vartheta(x;q,a) = \frac{x}{\varphi(q)} - \frac{\chi_1(a)x^{\beta_1}}{\varphi(q)\beta_1} + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big) \tag{11.34}$$

and

$$\pi(x;q,a) = \frac{\mathrm{li}(x)}{\varphi(q)} - \frac{\chi_1(a)\mathrm{li}\left(x^{\beta_1}\right)}{\varphi(q)} + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big) \tag{11.35}$$

when there is an exceptional character χ_1 modulo q and β_1 is the concomitant zero.

Proof Since

$$0 \le \psi(x;q,a) - \vartheta(x;q,a) \le \psi(x) - \vartheta(x) \ll x^{1/2},$$

the assertions concerning $\vartheta(x;q,a)$ follow immediately from Corollary 11.17. As for $\pi(x;q,a)$, we write

$$\pi(x;q,a) = \int_{2-}^{x} \frac{1}{\log u}\,d\vartheta(u;q,a) = \frac{\mathrm{li}(x)}{\varphi(q)} + \int_{2-}^{x} \frac{1}{\log u}\,d(\vartheta(u;q,a) - u/\varphi(q)).$$

This last integral we integrate by parts (as in the proof of Theorem 6.9), and

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find that it is

$$\frac{\vartheta(u;q,a)-u/\varphi(q)}{\log u}\Big|_{2^-}^{x} - \int_2^x \frac{\vartheta(u;q,a)-u/\varphi(q)}{u(\log u)^2}\,du.$$

If there is no exceptional zero, then the numerator in the integrand is $\ll u\exp(-c_1\sqrt{\log u}) \ll x\exp(-c_1\sqrt{\log x})$, so we obtain (11.33). If there is an exceptional character χ_1, then the main term is reduced by $\chi_1(a)/\varphi(q)$ times the amount

$$\int_2^x \frac{1}{\log u}\,d\frac{u^{\beta_1}}{\beta_1} = \int_2^x \frac{u^{\beta_1-1}}{\log u}\,du = \int_{2^{\beta_1}}^{x^{\beta_1}} \frac{1}{\log v}\,dv = \mathrm{li}(x^{\beta_1}) + O(1).$$

The error term is still treated in the same way, so we obtain (11.35). □

By arguing in the same manner from Corollary 11.19, we obtain

Corollary 11.21 *Let c_1 be the constant in Theorem* 11.16*, and suppose that A is given, $A > 0$. If $q \le (\log x)^A$ and $(a,q) = 1$, then*

$$\vartheta(x;q,a) = \frac{x}{\varphi(q)} + O_A\big(x\exp\big(-c_1\sqrt{\log x}\big)\big) \tag{11.36}$$

and

$$\pi(x;q,a) = \frac{\mathrm{li}(x)}{\varphi(q)} + O_A\big(x\exp\big(-c_1\sqrt{\log x}\big)\big). \tag{11.37}$$

11.3.1 Exercises

1. Suppose that χ is a character modulo q. Explain why

$$\psi(x,\chi) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \chi(a)\psi(x;q,a).$$

2. Suppose that $\exp(2c_1\sqrt{\log x}) \le q \le x$. Show that there is a positive constant c_2 such that

$$\psi(x,\chi) = E_0(\chi)x + O\left(x\exp\left(\frac{-c_2\log x}{\log q}\right)\right)$$

if $L(s,\chi)$ has no exceptional zero, and that

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + \left(x\exp\left(\frac{-c_2\log x}{\log q}\right)\right)$$

if $L(s,\chi)$ has the exceptional zero β_1.

3. Show that if $q \le \exp(2c_1\sqrt{\log x})$, then

$$\vartheta(x,\chi) = E_0(\chi)x + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big)$$

when $L(s, \chi)$ has no exceptional zero, and that

$$\vartheta(x, \chi) = -\frac{x^{\beta_1}}{\beta_1} + O\big(x \exp\big(-c_1\sqrt{\log x}\big)\big)$$

when $L(s, \chi)$ has an exceptional zero β_1.

4. Suppose that $q \le \exp(c_1\sqrt{\log x})$, and put $x_0 = \exp\big(\big(\frac{\log q}{2c_1}\big)^2\big)$.
   (a) Explain why $\pi(x_0; \chi) \ll x_0 \le x^{1/4}$.
   (b) Treat $\pi(x, \chi) - \pi(x_0, \chi)$ as in the proof of Corollary 11.20 to show that

$$\pi(x, \chi) \ll x \exp\big(-c_1\sqrt{\log x}\big)$$

   if $L(s, \chi)$ has no exceptional zero, and that

$$\pi(x, \chi) = -\operatorname{li}(x^{\beta_1}) + O\big(x \exp\big(-c_1\sqrt{\log x}\big)\big)$$

   if $L(s, \chi)$ has the exceptional zero β_1.

5. Suppose that A is given, $A > 0$. Show that if $q \le (\log x)^A$, then

$$\vartheta(x, \chi) = E_0(x)x + O\big(x \exp\big(-c_1\sqrt{\log x}\big)\big),$$

and that

$$\pi(x, \chi) = E_0(\chi)\operatorname{li}(x) + O\big(x \exp\big(-c_1\sqrt{\log x}\big)\big).$$

By analogy with (11.20) we set

$$\Lambda(x; q, a) = \sum_{\substack{n \le x \\ n \equiv a(q)}} \lambda(n), \qquad M(x; q, a) = \sum_{\substack{n \le x \\ n \equiv a(q)}} \mu(n). \tag{11.38}$$

Here it is no longer natural to restrict to $(a, q) = 1$. Correspondingly, if χ is a character modulo q, we put

$$\Lambda(x, \chi) = \sum_{n \le x} \chi(n)\lambda(n), \qquad M(x, \chi) = \sum_{n \le x} \chi(n)\mu(n). \tag{11.39}$$

6. Let c_1 be the constant of Theorem 11.16, suppose that $q \le \exp(2c_1\sqrt{\log x})$ and that χ is a character modulo q. Show that

$$\Lambda(x, \chi) \ll x \exp\big(-c_1\sqrt{\log x}\big)$$

when $L(s, \chi)$ has no exceptional zero, and that

$$\Lambda(x, \chi) = \frac{L(2\beta_1, \chi_0)x^{\beta_1}}{L'(\beta_1, \chi)\beta_1} + O\big(x \exp\big(-c_1\sqrt{\log x}\big)\big)$$

when $L(s, \chi)$ has an exceptional zero β_1. (Note that in this latter case, the result of Exercise 11.1.2 is useful.)

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7. Let c_1 be the constant of Theorem 11.16, suppose that $q \le \exp(2c_1\sqrt{\log x})$ and that χ is a character modulo q. Show that

$$M(x,\chi) \ll x\exp\big(-c_1\sqrt{\log x}\big)$$

when $L(s,\chi)$ has no exceptional zero, and that

$$M(x,\chi) = \frac{x^{\beta_1}}{L'(\beta_1,\chi)\beta_1} + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big)$$

when $L(s,\chi)$ has an exceptional zero β_1.

8. Let c_1 be the constant in Theorem 11.16, and suppose that A is given, $A > 0$. Show that if $q \le (\log x)^A$ and χ is a character modulo q, then

$$\Lambda(x,\chi) \ll_A \exp\big(-c_1\sqrt{\log x}\big),$$

and that

$$M(x,\chi) \ll_A x\exp\big(-c_1\sqrt{\log x}\big).$$

9. Show that if $(a,q) = 1$, then

$$\Lambda(x;q,a) = \frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\Lambda(x,\chi),$$

and that

$$M(x;q,a) = \frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)M(x,\chi).$$

10. Let c_1 be the constant in Theorem 11.16. Show that if $(a,q) = 1$, then

$$\Lambda(x;q,a) \ll x\exp\big(-c_1\sqrt{\log x}\big)$$

if there is no exceptional χ modulo q, and that

$$\Lambda(x;q,a) = \frac{\chi_1(a)L(2\beta_1,\chi_0)x^{\beta_1}}{\varphi(q)L'(\beta_1,\chi_1)\beta_1} + O\big(x\exp\big(-c_1\sqrt{\log x}\big)\big)$$

if there is an exceptional character χ_1 modulo q with associated zero β_1.

11. Suppose that $(a,q) = d$, and write $a = db$, $q = dr$.
   (a) Show that $\Lambda(x;q,a) = \lambda(d)\Lambda(x/d;r,b)$.
   (c) Show that

$$\Lambda(x;q,a) \ll \frac{x}{d}\exp\big(-c_1\sqrt{\log x/d}\big)$$

   if no L-function modulo r has an exceptional zero, and that

$$\Lambda(x;q,a) = \frac{\lambda(d)\chi_1(b)L(2\beta_1,\chi_0)(x/d)^{\beta_1}}{\varphi(r)L'(\beta_1,\chi_1)\beta_1} + O\left(\frac{x}{d}\exp\big(-c_1\sqrt{\log x/d}\big)\right)$$

if there is an exceptional character χ_1 modulo r with associated zero β_1. Here χ_0 is the principal character modulo r.

(d) Show that if $q \le (\log x)^A$, then

$$\Lambda(x; q, a) \ll_A x \exp\big(-c_1\sqrt{\log x}\big)$$

for all a.

12. Suppose that $(a, q) = 1$. Show that

$$M(x; q, a) \ll x \exp\big(-c_1\sqrt{\log x}\big)$$

if there is no exceptional character χ modulo q, and that

$$M(x; q, a) = \frac{\chi_1(a)x^{\beta_1}}{\varphi(q)L'(\beta_1, \chi_1)\beta_1} + O\big(x \exp\big(-c_1\sqrt{\log x}\big)\big)$$

if there is an exceptional character χ_1 modulo q with associated zero β_1.

13. Suppose that $d = (a, q)$, and write $q = dr$, $a = bd$.

(a) Show that if d is not square-free, then $M(x; q, a) = 0$.

(b) Explain why one does not expect that $M(x; q, a) = \mu(d)M(x/d; r, b)$ is true in general.

(c) Show instead that

$$M(x; q, a) = \mu(d) \sum_{\substack{k|d \\ (k,r)=1}} \mu(k)M(x/(dk); r, b\overline{k})$$

where $k\overline{k} \equiv 1 \pmod r$.

(d) Show that $M(x; q, a) \ll x/q$ in any case.

(e) Deduce that $M(x; q, a) \ll x \exp(-c\sqrt{\log x})$ if there is no exceptional character modulo r, and that

$$M(x; q, a) = \frac{\mu(d)\chi_1(b)(x/d)^{\beta_1}}{\varphi(r)L'(\beta_1, \chi_1)\beta_1} \prod_{\substack{p|d \\ p\nmid r}} \left(1 - \frac{\chi_1(p)}{p^{\beta_1}}\right) + O\big(x \exp\big(-c\sqrt{\log x}\big)\big)$$

if there is an exceptional character χ_1 with associated zero β_1.

(f) Show that if $q \le (\log x)^A$, then $M(x; q, a) \ll_A x \exp(-c\sqrt{\log x})$ for all a.

14. *Grössencharaktere* for $\mathbb{Q}(\sqrt{-1})$, continued from Exercise 11.1.5. Put $\psi(x, \chi_m) = \sum_{N(\mathfrak{a})\le x} \Lambda(\mathfrak{a})\chi_m(\mathfrak{a})$. Show that if $1 \le m \le \exp(\sqrt{\log x})$, then $\psi(x, \chi_m) \ll x \exp(-c\sqrt{\log x})$ where $c > 0$ is a suitable absolute constant.

11.4 Applications

The fundamental estimates of the preceding section can be applied to a wide variety of counting problems, of which the following are representative examples.

Theorem 11.22 (Walfisz) *Let $A > 0$ be fixed, and let $R(n)$ denote the number of ways of writing n as a sum of a prime and a square-free number. Then*

$$R(n) = c(n)\mathrm{li}(n) + O\left(n/(\log n)^A\right)$$

where

$$c(n) = \prod_{p \nmid n}\left(1 - \frac{1}{p(p-1)}\right) = \left(\prod_{p|n}\left(1 + \frac{1}{p^2 - p - 1}\right)\right)\left(\prod_{p}\left(1 - \frac{1}{p(p-1)}\right)\right).$$

Proof Clearly

$$\begin{aligned} R(n) &= \sum_{p<n} \mu(n-p)^2 \\ &= \sum_{p<n} \sum_{d^2|(n-p)} \mu(d) \end{aligned}$$

by (2.4). Here the divisibility relation is equivalent to asserting that $p \equiv n \pmod{d^2}$. Hence on inverting the order of summations we see that the above is

$$= \sum_{d \le \sqrt{n}} \mu(d)\pi(n-1; d^2, n).$$

If $(d, n) > 1$, then the summand is $O(1)$, and hence such $d \le \sqrt{n}$ contribute an amount that is $O(\sqrt{n})$. We now restrict our attention to those d for which $(d, n) = 1$. For small d, say $d \le y = (\log x)^A$ we can apply the Siegel–Walfisz theorem (Corollary 11.19). Thus we see that

$$\sum_{\substack{d \le y \\ (d,n)=1}} \mu(d)\pi(n-1; d^2, n) = \mathrm{li}(x) \sum_{\substack{d \le y \\ (d,n)=1}} \frac{\mu(d)}{\varphi(d^2)} + O\big(xy\exp\big(-c\sqrt{\log x}\big)\big).$$

Since $\varphi(d^2) = d\varphi(d)$, we see that the sum in the main term is

$$\sum_{\substack{d=1 \\ (d,n)=1}}^{\infty} \frac{\mu(d)}{d\varphi(d)} + O\left(\sum_{d>y} \frac{1}{d\varphi(d)}\right) = \prod_{p \nmid n}\left(1 - \frac{1}{p(p-1)}\right) + O(1/y)$$

by (1.31). To treat $d > y$ we could appeal to the Brun–Titchmarsh theorem (Theorem 3.9), but the moduli d^2 are increasing so rapidly that the trivial

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estimate $\pi(x; q, a) \ll 1 + x/q$ is enough:

$$\sum_{y<d<\sqrt{n}} \pi(n-1; d^2, n) \ll \sum_{y<d<\sqrt{n}} \frac{n}{d^2} \ll \frac{n}{y}.$$

On combining our estimates we obtain the stated result. □

In some situations, as below, we find it fruitful to use the Prime Number Theorem for arithmetic progressions in conjunction with sieve estimates.

Theorem 11.23 *Let $N(x)$ denote the number of integers $n \le x$ for which $(n, \varphi(n)) = 1$. Then*

$$N(x) \sim \frac{e^{-C_0} x}{\log\log\log x}$$

as $x \to \infty$.

Proof We note that $(n, \varphi(n)) = 1$ if and only if n has the following two properties: (i) n is square-free, and (ii) there do not exist prime factors p, p' of n such that $p' \equiv 1 \pmod p$. Let $p(n)$ denote the least prime factor of n. We shall show that if $p(n)$ is small compared with $\log\log x$ then n is unlikely to have the property (ii). We also show that n is likely to have both properties (i) and (ii) if $p(n)$ is large compared with $\log\log x$. Thus $N(x)$ is approximately the number of integers $n \le x$ for which $p(n) > \log\log x$.

Let $A_p(x)$ denote the number of $n \le x$ that satisfy (i) and (ii) and for which $p(n) = p$. Thus

$$N(x) = \sum_{p \le x} A_p(x).$$

We begin by estimating $A_p(x)$ when $p \le \log\log x$. Let p be given, and suppose that n is an integer such that $p(n) = p$ and for which (ii) holds. Write $n = pm$; then m is relatively prime to all prime numbers $< p$ and also to all primes $\equiv 1 \pmod p$. Thus by the sieve estimate (3.20) we see that

$$A_p(x) \ll \frac{x}{p} \left(\prod_{p'<p} \left(1 - \frac{1}{p'}\right)\right) \prod_{\substack{p' \le x/p \\ p' \equiv 1 (p)}} \left(1 - \frac{1}{p'}\right).$$

Here the first product is $\asymp 1/\log p$ by Mertens' estimate (Theorem 2.7(e)). By Theorem 4.12(d) we know that the second product is $\asymp (\log x)^{-1/(p-1)}$ for any fixed prime p. To derive a bound that is uniform in p we appeal to the Siegel–Walfisz theorem (Corollary 11.19), by which we see that $\pi(u; p, 1) \asymp$

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$u/(p \log u)$ uniformly for $u \geq e^p$. Hence by integrating by parts we deduce that

$$\sum_{\substack{e^p \leq p' \leq x/p \\ p' \equiv 1 (p)}} \frac{1}{p'} \asymp \frac{1}{p}(\log \log x/p - \log p) \asymp \frac{\log \log x}{p}$$

uniformly for $p \leq \log \log x$. Hence there is a constant $c > 0$ such that in this range,

$$A_p(x) \ll \frac{x}{p \log p} \exp(-c(\log \log x)/p).$$

Now it is not hard to show that the number of integers $n \leq x$ such that $p(n) = p$ is $\asymp x/(p \log p)$ uniformly for $p \leq x/2$. Hence the exponential above reflects the relative improbability that n satisfies condition (ii). On summing, we find that

$$\sum_{\frac{1}{2}U < p \leq U} A_p(x) \ll \frac{x}{(\log U)^2} \exp(-c(\log \log x)/U).$$

We take $U = 2^{-k} \log \log x$ and sum over k to see that

$$\sum_{p \leq \log \log x} A_p(x) \ll \frac{x}{(\log \log \log x)^2}.$$

We now consider n for which $p(n)$ is large, say $p(n) \geq y$ where y, to be chosen later, is somewhat larger than $\log \log x$. Let $\Phi(x, y)$ denote the number of integers $n \leq x$ composed entirely of prime numbers $> y$. By the sieve of Eratosthenes (Theorem 3.1) and Mertens' estimate (Theorem 2.7(e)) we see that

$$\sum_{y < p \leq x} A_p(x) \leq \Phi(x, y) = \frac{e^{-C_0} x}{\log y} + O\left(\frac{x}{(\log y)^2}\right) + O\left(e^{y/\log y}\right).$$

To derive a corresponding lower bound for the left-hand side we start with the numbers counted by $\Phi(x, y)$ and then delete those that do not satisfy (i) or (ii). If n does not satisfy (i), then there is a prime number p such that $p^2 | n$. The number of such $n \leq x$ is not more than $[x/p^2] \leq x/p^2$. Hence the total number of n counted in $\Phi(x, y)$ for which (i) fails is not more than $x \sum_{p > y} p^{-2} \ll x/(y \log y)$. Similarly, if n does not satisfy (ii), then there exist primes p, p' with $pp' | n$ such that $p' \equiv 1 \pmod{p}$. If p and p' are given, then the number of $n \leq x$ for which $pp' | n$ is $\leq x/(pp')$. Hence the total number of n counted in $\Phi(x, y)$ for which (ii) fails is not more than

$$x \sum_{y \leq p \leq \sqrt{x}} \frac{1}{p} \sum_{\substack{p' \leq x/p \\ p' \equiv 1 (p)}} \frac{1}{p'}. \tag{11.40}$$

By the Brun–Titchmarsh inequality (Theorem 3.9) we see that

$$\sum_{\substack{U<p'\le 2U\\ p'\equiv 1(p)}} \frac{1}{p'} \ll \frac{1}{p\log 2U/p}$$

uniformly for $U \ge p$. We take $U = 2^k p$ and sum over k to see that the inner sum in (11.40) is $\ll (\log\log 4x/p^2)/p$. Hence the expression (11.40) is

$$\ll x(\log\log x)\sum_{p>y}\frac{1}{p^2} \ll \frac{x\log\log x}{y\log y}.$$

On combining our estimates we see that

$$\sum_{y\le p\le x} A_p(x) \ge \frac{e^{C_0}x}{\log y} - O\left(\frac{x}{(\log y)^2}\right) - O\left(e^{y/\log y}\right) - O\left(\frac{x}{y\log y}\right) - O\left(\frac{x\log\log x}{y\log y}\right).$$

In order that the last error term above is of a smaller order of magnitude than the main term, it is necessary to choose y so that $y/\log\log x \to \infty$. Thus there is necessarily a remaining range $\log\log x < p \le y$ to be treated. By using the sieve (i.e., (3.20)) as in our treatment of small p we see that the number of integers $n \le x$ for which $p(n) = p$ is $\ll x/(p\log p)$, uniformly for $p \le \sqrt{x}$. Hence $A_p(x) \ll x/(p\log p)$, and consequently

$$\sum_{U\le p\le 2U} A_p(x) \ll \frac{x}{(\log U)^2}.$$

We put $U = 2^k\log\log x$ and sum over $1 \le k \le K$ where $K \ll \log\frac{y}{\log\log x}$ to see that

$$\sum_{\log\log x\le p\le y} A_p(x) \ll \frac{x}{(\log\log\log x)^2}\log\frac{y}{\log\log x}.$$

In order that this is a smaller order of magnitude than the main term, it is necessary to take $y \le (\log\log x)^{(1+\varepsilon)}$ with $\varepsilon \to 0$ as $x \to \infty$. By taking y to be of this form with ε tending to 0 slowly, we obtain the stated result. □

11.4.1 Exercises

1. Let $R(n)$ be defined as in Theorem 11.22.
   (a) Show that if there is a primitive quadratic character $\chi_1 \pmod{q_1}$, $q_1 \le \exp(\sqrt{\log x})$, for which $L(s,\chi_1)$ has a real zero $\beta_1 > 1 - c(\log x)^{-1/2}$, then

$$R(n) = c(n)\mathrm{li}(n) - \chi_1(n)c_1(n)\mathrm{li}(n^{\beta_1}) + O\big(n\exp\big(-c\sqrt{\log n}\big)\big)$$

where

$$c_1(n) = \sum_{\substack{d=1 \\ (d,n)=1 \\ q_1|d^2}}^{\infty} \frac{\mu(d)}{d\varphi(d)}.$$

(b) Show that $c_1(n) = 0$ if $8|q_1$.

(c) Show that if q_1 is odd, then

$$c_1(n) = \frac{\mu(q_1)c(q_1 n)}{q_1\varphi(q_1)}.$$

(d) Show that if $4\|q_1$, then

$$c_1(n) = \frac{4\mu(q_1/2)c(q_1 n)}{q_1\varphi(q_1)}$$

2. In the proof of Theorem 11.23, specify ε as an explicit function of x to show that

$$N(x) = \frac{x}{\log\log\log x}\left(e^{-C_0} + O\left(\frac{\log\log\log\log x}{\log\log\log x}\right)\right).$$

3. Let a be a fixed non-zero integer. Show that the number of primes $p \le x$ such that $p + a$ is square-free is $c(a)\mathrm{li}(x) + O_A(x(\log x)^{-A})$ where $c(a)$ is defined as in Theorem 11.22.

4. Show that the appeal to the Siegel–Walfisz theorem in the proof of Theorem 11.23 can be replaced by an appeal to Page's theorem in conjunction with Corollary 11.12.

5. (Vaughan 1973) Let A and B be positive numbers. Show that

$$\sum_{p\le x}\left(\frac{\varphi(p-1)}{p-1}\right)^B = C\,\mathrm{li}(x) + O_{A,B}(x/(\log x)^A)$$

where

$$C = \prod_p \left(1 - \frac{1-(1-1/p)^B}{p-1}\right).$$

6. (Erdős 1951)

(a) Let $r(n)$ denote the number of solutions of $p + 2^k = n$ with p prime and $k \ge 1$, and let $y = c\sqrt{\log x}$ where c is a sufficiently small positive constant. Define $q' = \prod_{2<p\le y} p$. If there is a primitive character χ^* modulo q^* with $q^*|q'$ for which $L(s,\chi^*)$ has an exceptional zero, then let p be any prime divisor of q^* and define $q = q'/p$. Otherwise let $q = q'$. Prove that

$$\sum_{m\le x/q} r(qm) = \frac{x}{\varphi(q)\log 2} + O\left(\frac{x}{\varphi(q)\log x}\right).$$

(b) Show that $r(n) = \Omega(\log\log n)$.

11.5 Notes

Section 11.1. Theorem 11.3 is a combination of work by Gronwall (1913) and Titchmarsh (1930).

Section 11.2. Lemma 11.6, Theorem 11.7, and Corollaries 11.8, 11.9 originate in Landau (1918a, b), while Corollary 11.10 is from Page (1935). Theorem 11.11 can also be proved by appealing to the Dirichlet class number formula, which asserts that if d is a quadratic discriminant and $\chi_d(n) = \left(\frac{d}{n}\right)_K$ is the associated quadratic character, then

$$L(1, \chi_d) = \begin{cases} \dfrac{2\pi h}{w\sqrt{-d}} & (d < 0), \\ \dfrac{h \log \varepsilon}{\sqrt{d}} & (d > 0); \end{cases}$$

see Davenport (2000, Section 6). If $d < 0$, then $\chi_d(-1) = -1$, $\mathbb{Q}(\sqrt{d})$ is an imaginary quadratic field with class number h, and w denotes the number of roots of unity in the field (which is to say that $w = 6$ if $d = -3$, $w = 4$ if $d = -4$, and $w = 2$ otherwise). If $d > 0$, then $\chi_d(-1) = 1$, $\mathbb{Q}(\sqrt{d})$ is a real quadratic field with class number h and fundamental unit ε. Since $\varepsilon \gg \sqrt{d}$, it follows that if χ is a quadratic character with $\chi(-1) = 1$, then $L(1, \chi) \gg (\log q)/q^{1/2}$.

Corollary 11.12 has been sharpened by Davenport (1966), Haneke (1973), and by Goldfeld & Schinzel (1975).

Section 11.3. Let $h(d)$ denote the number of equivalence classes of primitive binary quadratic forms of discriminant d. Gauss (1801, Section 303) conjectured that $h(d) \to \infty$ as $d \to -\infty$. (The behaviour for $d > 0$ is quite different – the heuristics of Cohen & Lenstra (1984a, b) predict that $h(p) = 1$ for a positive proportion of primes $p \equiv 1 \pmod 4$.) For Gauss, the generic binary quadratic form was written $ax^2 + 2bxy + cy^2$, which is to say that the middle coefficient is even. Put $\Delta = b^2 - ac$. In Gauss's notation, Landau (1903) found that if $\Delta < 0$, then the class number is 1 precisely when $\Delta = -1, -2, -3, -4, -7$. Binary quadratic forms $ax^2 + bxy + cy^2$ with $d = b^2 - 4ac$ correspond, when d is a fundamental quadratic discriminant, to ideals in the ring $\mathcal{O}_K$ of integers in the quadratic number field $K = \mathbb{Q}(\sqrt{d})$. In this notation, $h(d) = 1$ if and only if $\mathcal{O}_K$ is a unique factorization domain. The problem of determining all $d < 0$ for which $h(d) = 1$ is now solved, but historically it was enormously more difficult than the class number 1 problem settled by Landau. Landau (1918b) recorded Hecke's observation that if $d < 0$ is a quadratic discriminant and $L(s, \chi_d) > 0$ for $1 - c/\log |d| < s < 1$, then $h(d) \gg_c |d|^{1/2}/\log |d|$. In view of Dirichlet's class number formula (4.36), we have obtained Hecke's result – by a different method – in Theorem 11.4. Thus we have a good lower bound for $h(d)$ when $d < 0$, except for those d for which $L(s, \chi_d)$ has an exceptional real zero. Deuring (1933) showed that if $h(d) = 1$ has infinitely many solutions with $d < 0$, then the Riemann Hypothesis is true. Mordell (1934) showed that the same conclusion can be derived from the weaker hypothesis that $h(d)$ does not tend to infinity as $d \to -\infty$. Heilbronn (1934) found that instead of arguing from a hypothetical zero ρ of the zeta function with $\beta > 1/2$ one could just as well argue from an exceptional zero of a quadratic L-function, and thus proved Gauss's conjecture that $h(d) \to \infty$ as $d \to -\infty$. Landau (1935) put Heilbronn's theorem in a quantitative form: $h(d) > |d|^{3/8-\varepsilon}$ as $d \to -\infty$. Through a different arrangement of the technical details, Siegel (1935) sharpened Landau's argument to show that $h(d) > |d|^{1/2-\varepsilon}$, which by (4.36) is the case $d < 0$ of Theorem 11.14. To achieve his result, Siegel first generalized to algebraic number fields the formula (found in Exercise 10.1.10) that Riemann used to prove the functional equation for $\zeta(s)$. Then Siegel applied this to the quartic number field $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ whose Dedekind zeta function is $\zeta_K(s) = \zeta(s)L(s, \chi_{d_1})L(s, \chi_{d_2})L(s, \chi_{d_1 d_2})$. It is now recognized that Siegel's formula arises through the choice of the kernel in a Mellin transform, and that many other choices work just as well; see Goldfeld(1974). Our exposition is based on that of Estermann (1948).

It is easy to show that the complex quadratic field of discriminant $d < 0$ has unique factorization in the nine cases $d = -3, -4, -7, -8, -11, -19, -43, -67, -163$. Heilbronn & Linfoot (1934) showed that there could exist at most one more such discriminant. The 'problem of the tenth discriminant' was solved first by Heegner (1952). However, Heegner's paper contained many assertions for which proofs were not provided, and Heegner also used results from Weber's *Algebra* which were known not to be trustworthy. Consequently, for many years Heegner's paper was thought to be incorrect. Baker (1966) proved a fundamental lower bound for linear forms in logarithms of algebraic numbers, which by means of a result of Gel'fond & Linnik (1948) reduced the class number 1 problem to a finite calculation. Meanwhile, Stark (1967) showed that there is no tenth discriminant by translating Heegner's argument into parallel language where it could be checked. After a reexamination of Heegner's work, Deuring (1968), Birch (1969), and Stark (1969) all concluded that Heegner's paper was after all correct. Gel'fond & Linnik reduced the class number problem to a question concerning linear forms in three logarithms, which Baker treated successfully. However, with a small modification of their argument, Gel'fond & Linnik could have reduced the problem to linear forms in two logarithms, which Gel'fond had already treated. Thus one could say that Gel'fond & Linnik 'should' have solved the problem in 1948.

Baker (1971) and Stark (1971b, 1972) reduced the complete determination of complex quadratic fields with $h(d) = 2$ to a finite calculation which was provided by Bundschuh & Hock (1969), Ellison *et al.* (1971), Montgomery & Weinberger (1973), and by Stark (1975).

The effective determination of all quadratic discriminants $d < 0$ for which $h(d)$ takes specific larger values became possible only with the addition of further ideas. Goldfeld (1976) showed that a zero at $s = 1/2$ of the L-function of an elliptic curve would be useful if it is of sufficiently high multiplicity. In particular, if (i) the Birch–Swinnerton-Dyer conjectures are true, and if (ii) there exist elliptic curves of arbitrarily high rank, then $h(d) \gg_A (\log |d|)^A$ for arbitrarily large A, with an effectively computable implicit constant. Although these conjectures remain unproved, Gross & Zagier (1986) were able to establish enough to give an effective lower bound for $h(d)$ tending to infinity. For accounts of this, see Zagier (1984), Goldfeld (1985), Coates (1986), and finally Oesterlé (1988), who developed the Goldfeld and Gross–Zagier work to show that

$$h(d) \ge \frac{1}{55}(\log |d|) \prod_{\substack{p|d \\ p<|d|}} \left(1 - \frac{[2\sqrt{p}]}{p+1}\right).$$

By means of this inequality, Arno (1992), Wagner (1996), and Arno, Robinson & Wheeler (1998) treated progressively larger collections of class numbers. Most recently, Watkins (2004) settled the complete determination of all discriminants $d < 0$ for which $h(d) \le 100$.

With regard to Corollary 11.17, Page (1935) states the final conclusion in a less precise form in which the term corresponding to the exceptional zero is replaced by $O(x^{\beta_1}/\phi(q))$.

The deduction of Corollaries 11.18 and 11.19 from Siegel's theorem was first recorded by Walfisz (1936).

Section 11.4. Theorem 11.22 is due to Walfisz (1936). In a weaker form it occurs first in Estermann (1931), and is given in a somewhat refined form but without the benefit of Siegel's theorem in Page (1935). For similar theorems see see Mirsky (1949).

Theorem 11.23 is due to Erdős (1948).

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