5

Dirichlet series: II

5.1 The inverse Mellin transform

In Chapter 1 we saw that we can express a Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in terms of the coefficient sum $A(x) = \sum_{n \le x} a_n$, by means of the formula

$$\alpha(s) = s \int_1^{\infty} A(x) x^{-s-1}\, dx, \tag{5.1}$$

which holds for $\sigma > \max(0, \sigma_c)$. This is an example of a Mellin transform. In the reverse direction, Perron's formula asserts that

$$A(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s) \frac{x^s}{s}\, ds \tag{5.2}$$

for $\sigma_0 > \max(0, \sigma_c)$. This is an example of an inverse Mellin transform.

To understand why we might expect that (2) should be true, note that if $\sigma_0 > 0$, then by the calculus of residues

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} y^s \frac{ds}{s} = \begin{cases} 1 & \text{if } y > 1, \\ 0 & \text{if } 0 < y < 1. \end{cases} \tag{5.3}$$

Thus we would expect that

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s) \frac{x^s}{s}\, ds = \sum_n \frac{a_n}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \sum_{n \le x} a_n. \tag{5.4}$$

The interchange of limits here is difficult to justify, since $\alpha(s)$ may not be uniformly convergent, and because the integral in (5.3) is neither uniformly nor absolutely convergent. Moreover, if x is an integer, then the term $n = x$ in (5.4) gives rise to the integral (5.3) with $y = 1$, and this integral does not converge, although its Cauchy principal value exists:

$$\lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{ds}{s} = \frac{1}{2} \tag{5.5}$$

for $\sigma_0 > 0$. We now give a rigorous form of Perron's formula.

Theorem 5.1 (Perron's formula) *If* $\sigma_0 > \max(0, \sigma_c)$ *and* $x > 0$, *then*

$$\sum_{n\le x}{}' a_n = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha(s)\frac{x^s}{s}\,ds.$$

Here $\sum'$ *indicates that if x is an integer, then the last term is to be counted with weight* $1/2$.

Proof Choose N so large that $N > 2x + 2$, and write

$$\alpha(s) = \sum_{n\le N} a_n n^{-s} + \sum_{n>N} a_n n^{-s} = \alpha_1(s) + \alpha_2(s),$$

say. By (5.4), modified in recognition of (5.5), we see that

$$\sum_{n\le x}{}' a_n = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha_1(s)\frac{x^s}{s}\,ds;$$

here the justification is trivial since there are only finitely many terms. As for $\alpha_2(s)$, we observe that

$$\alpha_2(s) = \int_N^\infty u^{-s}\,d(A(u) - A(N)) = s\int_N^\infty (A(u) - A(N))u^{-s-1}\,du.$$

But $A(u) - A(N) \ll u^\theta$ for $\theta > \max(0, \sigma_c)$, and hence

$$\alpha_2(s) \ll \left(1 + \frac{|s|}{\sigma - \theta}\right) N^{\theta-\sigma}$$

for $\sigma > \theta > \max(0, \sigma_c)$. Implicit constants here and in the rest of this proof may depend on the a_n. Hence

$$\int_{\sigma_0\pm iT}^{T\pm iT} \alpha_2(s)\frac{x^s}{s}\,ds \ll \frac{N^\theta}{\sigma_0-\theta}\int_{\sigma_0}^\infty \left(\frac{x}{N}\right)^\sigma d\sigma \ll \frac{N^\theta}{\sigma_0-\theta}\,\frac{(x/N)^{\sigma_0}}{\log N/x},$$

and

$$\int_{T-iT}^{T+iT} \alpha_2(s)\frac{x^s}{s}\,ds \ll N^\theta (x/N)^{\sigma_0}$$

for large T. We take θ so that $\sigma_0 > \theta > \max(0, \sigma_c)$. Hence by Cauchy's theorem

$$\int_{\sigma_0-iT}^{\sigma_0+iT} = \int_{\sigma_0-iT}^{T-iT} + \int_{T-iT}^{T+iT} + \int_{T+iT}^{\sigma_0+iT} \ll x^{\sigma_0} N^{\theta-\sigma_0}.$$

On combining our estimates, we see that

$$\limsup_{T\to\infty}\left|\sum_{n\le x}{}' a_n - \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha(s)\frac{x^s}{s}\,ds\right| \ll x_0^\sigma N^{\theta-\sigma_0}.$$

Since this holds for arbitrarily large N, it follows that the lim sup is 0, and the proof is complete. □

We have now established a precise relationship between (5.1) and (5.2), but Theorem 5.1 is not sufficiently quantitative to be useful in practice. We express the error term more explicitly in terms of the *sine integral*

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin u}{u}\,du.$$

By integration by parts we see that $\mathrm{si}(x) \ll 1/x$ for $x \geq 1$, and hence that

$$\mathrm{si}(x) \ll \min(1, 1/x) \tag{5.6}$$

for $x > 0$. We also note that

$$\mathrm{si}(x) + \mathrm{si}(-x) = -\int_{-\infty}^{+\infty} \frac{\sin u}{u}\,du = -\pi. \tag{5.7}$$

Theorem 5.2 *If* $\sigma_0 > \max(0, \sigma_a)$ *and* $x > 0$*, then*

$$\sideset{}{'}\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha(s)\frac{x^s}{s}\,ds + R \tag{5.8}$$

where

$$R = \frac{1}{\pi}\sum_{x/2<n<x} a_n\, si\left(T\log\frac{x}{n}\right) - \frac{1}{\pi}\sum_{x<n<2x} a_n\, si\left(T\log\frac{n}{x}\right) + O\left(\frac{4^{\sigma_0}+x^{\sigma_0}}{T}\sum_n \frac{|a_n|}{n^{\sigma_0}}\right).$$

Proof Since the series $\alpha(s)$ is absolutely convergent on the interval $[\sigma_0 - iT, \sigma_0 + iT]$, we see that

$$\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha(s)\frac{x^s}{s}\,ds = \sum_n a_n \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\left(\frac{x}{n}\right)^s \frac{ds}{s}.$$

Thus it suffices to show that

$$\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} y^s\,\frac{ds}{s} = \begin{cases} 1 + O(y^{\sigma_0}/T) & \text{if } y \geq 2,\\ 1 + \frac{1}{\pi}\mathrm{si}(T\log y) + O(2^{\sigma_0}/T) & \text{if } 1 \leq y \leq 2,\\ -\frac{1}{\pi}\mathrm{si}(T\log 1/y) + O(2^{\sigma_0}/T) & \text{if } 1/2 \leq y \leq 1,\\ O(y^{\sigma_0}/T) & \text{if } y \leq 1/2 \end{cases} \tag{5.9}$$

for $\sigma_0 > 0$.

To establish the first part of this formula, suppose that $y \geq 2$, and let $\mathcal{C}$ be the piecewise linear path from $-\infty - iT$ to $\sigma_0 - iT$ to $\sigma_0 + iT$ to $-\infty + iT$. Then by the calculus of residues we see that

$$\frac{1}{2\pi i}\int_{\mathcal{C}} y^s\,\frac{ds}{s} = 1,$$

since the integrand has a pole with residue 1 at $s = 0$. In addition,

$$\int_{-\infty\pm iT}^{\sigma_0\pm iT} y^s \frac{ds}{s} = \int_{-\infty}^{\sigma_0} \frac{y^{\sigma\pm iT}}{\sigma \pm iT}\, d\sigma \ll \frac{1}{T}\int_{-\infty}^{\sigma_0} y^\sigma\, d\sigma = \frac{y^{\sigma_0}}{T\log y} \ll \frac{y^{\sigma_0}}{T},$$

so we have (5.9) in the case $y \ge 2$. The case $y \le 1/2$ is treated similarly, but the contour is taken to the right, and there is no residue.

Suppose now that $1 \le y \le 2$, and take $\mathcal{C}$ to be the closed rectangular path from $\sigma_0 - iT$ to $\sigma_0 + iT$ to iT to $-iT$ to $\sigma_0 - iT$, with a semicircular indentation of radius ε at $s = 0$. Then by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{\mathcal{C}} y^s \frac{ds}{s} = 0.$$

We note that

$$\int_{\pm iT}^{\sigma_0\pm iT} y^s \frac{ds}{s} \ll \frac{1}{T}\int_0^{\sigma_0} y^\sigma\, d\sigma \le \frac{1}{T}\int_0^{\sigma_0} 2^\sigma\, d\sigma \ll \frac{2^{\sigma_0}}{T}.$$

The integral around the semicircle tends to $1/2$ as $\varepsilon \to 0$, and the remaining integral is

$$\begin{aligned}\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left(\int_{i\varepsilon}^{iT} + \int_{-iT}^{-i\varepsilon}\right) y^s \frac{ds}{s} &= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_\varepsilon^T \left(y^{it} - y^{-it}\right)\frac{dt}{t}\\ &= \frac{1}{\pi}\int_0^{T\log y} \sin v\, \frac{dv}{v}\\ &= \frac{1}{2} + \frac{1}{\pi}\mathrm{si}(T\log y)\end{aligned}$$

by (5.7). This gives (5.9) when $1 \le y \le 2$ and the case $1/2 \le y \le 1$ is treated similarly. □

In many situations, Theorem 5.2 contains more information than is really needed – it is often more convenient to appeal to the following less precise result.

Corollary 5.3 *In the situation of* Theorem 5.2,

$$R \ll \sum_{\substack{x/2<n<2x\\ n\ne x}} |a_n| \min\left(1, \frac{x}{T|x-n|}\right) + \frac{4^{\sigma_0} + x^{\sigma_0}}{T}\sum_{n=1}^\infty \frac{|a_n|}{n^{\sigma_0}}.$$

Proof From (5.6) we see that

$$\mathrm{si}(T|\log n/x|) \ll \min\left(1, \frac{1}{T|\log n/x|}\right).$$

But $n/x = 1 + (n-x)/x$ and $|\log(1+\delta)| \asymp |\delta|$ uniformly for $-1/2 \le \delta \le 1$, so the above is

$$\asymp \min\left(1, \frac{x}{T|x-n|}\right)$$

if $x/2 \le n \le 2x$. Thus the stated bound follows from Theorem 5.2. □

In classical harmonic analysis, for $f \in \mathrm{L}^1(\mathbb{T})$ we define Fourier coefficients $\widehat{f}(k) = \int_0^1 f(x)e(-k\alpha)\,d\alpha$, and we expect that the Fourier series $\sum \widehat{f}(k)e(k\alpha)$ provides a useful formula for $f(\alpha)$. As it happens, the Fourier series may diverge, or converge to a value other than $f(\alpha)$, but for most f a satisfactory alternative can be found. For example, if f is of bounded variation, then

$$\frac{f(\alpha^-) + f(\alpha^+)}{2} = \lim_{K\to\infty} \sum_{-K}^{K} \widehat{f}(k)e(k\alpha).$$

A sharp quantitative form of this is established in Appendix D.1. Analogously, if $f \in L^1(\mathbb{R})$, then we can define the Fourier transform of f,

$$\widehat{f}(t) = \int_{-\infty}^{+\infty} f(x)e(-tx)\,dx, \tag{5.10}$$

and we expect that

$$f(x) = \int_{-\infty}^{+\infty} \widehat{f}(t)e(tx)\,dt. \tag{5.11}$$

As in the case of Fourier series, this may fail, but it is not difficult to show that if f is of bounded variation on $[-A, A]$ for every A, then

$$\frac{f(\alpha^-) + f(\alpha^+)}{2} = \lim_{T\to\infty} \int_{-T}^{T} \widehat{f}(t)e(tx)\,dt. \tag{5.12}$$

The relationship between (5.1) and (5.2) is precisely the same as between (5.10) and (5.11). Indeed, if we take $f(x) = A(e^{2\pi x})e^{-2\pi\sigma x}$, then $f \in L^1(\mathbb{R})$ by Theorem 1.3, and by changing variables in (5.1) we find that

$$\widehat{f}(t) = \frac{\alpha(\sigma + it)}{2\pi(\sigma + it)}.$$

Thus (5.2) is equivalent to (5.11), and an appeal to (5.12) provides a second (real variable) proof of Theorem 5.1.

In general, if

$$F(s) = \int_0^\infty f(x)x^{s-1}\,dx, \tag{5.13}$$

then we say that $F(s)$ is the *Mellin transform* of $f(x)$. By (5.10) and (5.11) we expect that

$$f(x) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s)x^{-s}\,ds, \tag{5.14}$$

and when this latter formula holds we say that f is the *inverse Mellin transform* of F. Thus if $A(x)$ is the summatory function of a Dirichlet series $\alpha(s)$, then $\alpha(s)/s$ is the Mellin transform of $A(1/x)$ for $\sigma > \max(0, \sigma_c)$, and Perron's formula (Theorem 5.1) asserts that if $\sigma_0 > \max(0, \sigma_c)$, then $A(1/x)$ is the inverse

Mellin transform of $\alpha(s)/s$. Further instances of this pairing arise if we take a *weight function* $w(x)$, and form a *weighted summatory function*

$$A_w(x) = \sum_{n=1}^{\infty} a_n w(n/x).$$

Let $K(s)$ denote the Mellin transform of $w(x)$,

$$K(s) = \int_0^{\infty} w(x)x^{s-1}\, dx.$$

Then we expect that

$$\alpha(s)K(s) = \int_0^{\infty} A_w(x)x^{-s-1}\, dx, \tag{5.15}$$

and that

$$A_w(x) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \alpha(s)K(s)x^s\, ds. \tag{5.16}$$

Alternatively, we may start with a *kernel* $K(s)$, and define the weight $w(x)$ to be its inverse Mellin transform. The precise conditions under which these identities hold depends on the weight or kernel; we mention several important examples.

1. Cesàro weights. For a positive integer k, put

$$C_k(x) = \frac{1}{k!}\sum_{n\le x} a_n(x-n)^k. \tag{5.17}$$

Then $C_k(x) = \int_0^x C_{k-1}(u)\, du$ for $k \ge 1$ where $C_0(x) = A(x)$, and hence $C_k(x) \ll x^{\theta}$ for $\theta > k + \max(0, \sigma_c)$. (The implicit constant here may depend on k, on θ, and on the a_n.) By integrating (5.1) by parts repeatedly, we see that

$$\alpha(s) = s(s+1)\cdots(s+k)\int_1^{\infty} C_k(x)x^{-s-k-1}\, dx \tag{5.18}$$

for $\sigma > \max(0, \sigma_c)$. By following the method used to prove Theorem 5.1, it may also be shown that

$$C_k(x) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \alpha(s)\frac{x^{s+k}}{s(s+1)\cdots(s+k)}\, ds \tag{5.19}$$

when $x > 0$ and $\sigma_0 > \max(0, \sigma_c)$. Here the critical step is to show that if $y \ge 1$ and $\sigma_0 > 0$, then

$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{y^s}{s(s+1)\cdots(s+k)}\, ds = \sum_{j=0}^{k} \operatorname{Res}\left(\frac{y^s}{s(s+1)\cdots(s+k)}\right|_{s=-j}$$

by the calculus of residues; this is

$$= \sum_{j=0}^{k} \frac{(-1)^j y^{-j}}{j!(k-j)!} = \frac{1}{k!}(1 - 1/y)^k$$

by the binomial theorem.

2. Riesz typical means. For positive integers k and positive real x put

$$R_k(x) = \frac{1}{k!} \sum_{n \le x} a_n (\log x/n)^k. \tag{5.20}$$

Then $R_k(x) = \int_0^x R_{k-1}(u)/u \, du$ where $R_0(x) = A(x)$, so that $R_k(x) \ll x^\theta$ for $\theta > \max(0, \sigma_c)$. (The implicit constant here may depend on k, on θ, and on the a_n.) By integrating (5.1) by parts repeatedly we see that

$$\alpha(s) = s^{k+1} \int_1^\infty R_k(x) x^{-s-1} \, dx \tag{5.21}$$

for $\sigma > \max(0, \sigma_c)$. By following the method used to prove Theorem 5.1 we also find that

$$R_k(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s) \frac{x^s}{s^{k+1}} \, ds \tag{5.22}$$

when $x > 0$ and $\sigma_0 > \max(0, \sigma_c)$. Here the critical observation is that if $y \ge 1$ and $\sigma_0 > 0$, then

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{y^s}{s^{k+1}} \, ds = \operatorname{Res}\left(\frac{y^s}{s^{k+1}} \right|_{s=0} = \frac{1}{k!} (\log y)^k.$$

3. Abelian weights. For $\sigma > 0$ we have

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} \, du = n^s \int_0^\infty e^{-nx} x^{s-1} \, dx.$$

We multiply by $a_n n^{-s}$ and sum, to find that

$$\alpha(s)\Gamma(s) = \int_0^\infty P(x) x^{s-1} \, dx \tag{5.23}$$

where

$$P(x) = \sum_{n=1}^\infty a_n e^{-nx}. \tag{5.24}$$

These operations are valid for $\sigma > \max(0, \sigma_a)$, but by partial summation $P(x) \ll x^{-\theta}$ as $x \to 0^+$ for $\theta > \max(0, \sigma_c)$, so that the integral in (5.23) is absolutely convergent in the half-plane $\sigma > \max(0, \sigma_c)$. Hence the integral is an analytic function in this half-plane, so that by the principle of uniqueness of analytic continuation it follows that (5.23) holds for $\sigma > \max(0, \sigma_c)$. In the opposite direction,

$$P(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s)\Gamma(s)x^{-s}\,ds \tag{5.25}$$

for $x > 0$, $\sigma > \max(0, \sigma_c)$. To prove this we recall from Theorem 1.5 that $\alpha(s) \ll \tau$ uniformly for $\sigma \geq \varepsilon + \max(0, \sigma_c)$, and from Stirling's formula (Theorem C.1) we see that $|\Gamma(s)| \asymp e^{-\frac{\pi}{2}|t|}|t|^{\sigma-1/2}$ as $|t| \to \infty$ with σ bounded. Thus the value of the integral is independent of σ_0, and in particular we may assume that $\sigma_0 > \max(0, \sigma_a)$. Consequently the terms in $\alpha(s)$ can be integrated individually, and it suffices to appeal to Theorem C.4.

The formulæ (5.23) and (5.25) provide an important link between the Dirichlet series $\alpha(s)$ and the power series generating function $P(x)$. Indeed, these formulæ hold for complex x, provided that $\Re x > 0$. In particular, by taking $x = \delta - 2\pi i\alpha$ we find that

$$\sum_{n=1}^{\infty} a_n e(n\alpha)e^{-n\delta} = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s)\Gamma(s)(\delta - 2\pi i\alpha)^{-s}\,ds.$$

It may be noted in the above examples that smoother weights $w(x)$ give rise to kernels $K(s)$ that tend to 0 rapidly as $|t| \to \infty$. Further useful kernels can be constructed as linear combinations of the above kernels.

Since the Mellin transform is a Fourier transform with altered variables, all results pertaining to Fourier transforms can be reformulated in terms of Mellin transforms. Particularly useful is Plancherel's identity, which asserts that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\|f\|_2 = \|\widehat{f}\|_2$. This is the analogue for Fourier transforms of Parseval's identity for Fourier series, which asserts that $\sum_k |\widehat{f}(k)|^2 = \|f\|_2^2$. By the changes of variables we noted before, we obtain

Theorem 5.4 (Plancherel's identity) *Suppose that* $\int_0^\infty |w(x)|x^{-\sigma-1}\,dx < \infty$, *and also that* $\int_0^\infty |w(x)|^2 x^{-2\sigma-1}\,dx < \infty$. *Put* $K(s) = \int_0^\infty w(x)x^{-s-1}\,dx$. *Then*

$$2\pi \int_0^\infty |w(x)|^2 x^{-2\sigma-1}\,dx = \int_{-\infty}^{+\infty} |K(\sigma + it)|^2\,dt.$$

Among the many possible applications of this theorem, we note in particular that

$$2\pi \int_0^\infty |A(x)|^2 x^{-2\sigma-1}\,dx = \int_{-\infty}^{+\infty} \left|\frac{\alpha(\sigma + it)}{\sigma + it}\right|^2 dt \tag{5.26}$$

for $\sigma > \max(0, \sigma_c)$.

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5.1.1 Exercises

1. Show that if $\sigma_c < \sigma_0 < 0$, then
$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha(s)\frac{x^s}{s}\,ds = {\sum}'_{n>x} a_n.$$
2. (a) Show that if $y \ge 0$, then
$$-\frac{\pi}{2} = \mathrm{si}(0) \le \mathrm{si}(y) \le \mathrm{si}(\pi) = 0.28114\ldots.$$

   (b) Show that if $y \ge 0$, then
$$\Im\int_y^\infty \frac{e^{iu}}{u}\,du = \Im\int_y^{y+i\infty} \frac{e^{iz}}{z}\,dz.$$
   (c) Deduce that if $y \ge 0$, then $|\mathrm{si}(y)| < 1/y$.
3. (a) Let $\beta > 0$ be fixed. Show that if $\sigma_0 > 0$, then
$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(s/\beta)y^s\,ds = \beta e^{-y^{-\beta}}.$$

   (b) Let $\beta > 0$ be fixed. Show that if $x > 0$ and $\sigma_0 > \max(0, \sigma_c)$, then
$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \alpha(s)\Gamma(s/\beta)x^s\,ds = \beta\sum_{n=1}^{\infty} a_n e^{-(n/x)^\beta}.$$
4. (a) Suppose that $a > 0$ and that b is real. Explain why
$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{a^2s^2/2+bs}\,ds = \frac{e^{-b^2/(2a^2)}}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{a^2(s+b/a^2)^2/2}\,ds\,.$$
   (b) Explain why the values of the integrals above are independent of the value of σ_0. Hence show that if $\sigma_0 = -b/a^2$, then the above is
$$= \frac{e^{-b^2/(2a^2)}}{2\pi}\int_{-\infty}^{+\infty} e^{-a^2t^2/2}\,dt = \frac{1}{\sqrt{2\pi}\,a}e^{-b^2/a^2}.$$
   (c) Show that if $a > 0$, $x > 0$ and $\sigma_0 > \sigma_c$, then
$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \alpha(s)e^{a^2s^2/2}x^s\,ds = \frac{1}{\sqrt{2\pi}\,a}\sum_{n=1}^{\infty} a_n \exp\left(-\frac{(\log x/n)^2}{2a^2}\right).$$
5. Take $k = 1$ in (5.22) for several different values of x, and form a suitable linear combination, to show that if $x \ge 0$ and and $\sigma_c < 0$, then
$$\frac{2}{\pi}\int_{-\infty}^{+\infty} \alpha(it)\left(\frac{\sin\frac{1}{2}t\log x}{t}\right)^2 dt = \sum_{n\le x} a_n \log x/n.$$

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6. Let $w(x) \nearrow$, and suppose that $w(x) \ll x^\sigma$ as $x \to \infty$ for some fixed σ. Let σ_w be the infimum of those σ such that $\int_0^\infty w(x)x^{-\sigma-1}\,dx < \infty$, and put

$$K(s) = \int_0^\infty w(x)x^{-s-1}\,dx$$

for $\sigma > \sigma_w$.
   (a) Show that $A_w(x) = \sum_{n=1}^\infty a_n w(x/n)$ satisfies $A_w(x) \ll x^\theta$ for $\theta > \max(\sigma_w, \sigma_c)$.
   (b) Show that

$$K(s)\alpha(s) = \int_0^\infty A_w(x)x^{-s-1}\,dx$$

   for $\sigma > \max(\sigma_w, \sigma_c)$.
   (c) Show that

$$\tfrac{1}{2}(A_w(x^-) + A_w(x^+)) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha(s)K(s)x^s\,ds$$

   for $\sigma_0 > \max(\sigma_w, \sigma_c)$, $x > 0$.
7. Show that

$$\zeta(s) = -s\int_0^\infty \frac{\{x\}}{x^{s+1}}\,dx$$

for $0 < \sigma < 1$, and that

$$2\pi\int_0^\infty \{x\}^2 x^{-2\sigma-1}\,dx = \int_{-\infty}^{+\infty} \left|\frac{\zeta(\sigma+it)}{\sigma+it}\right|^2 dt$$

for $0 < \sigma < 1$.
8. (a) Show that if $f \in L^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$, then $\widehat{f'}(t) = 2\pi i t \widehat{f}(t)$.
   (b) Suppose that f is a function such that $f \in L^1(\mathbb{R})$, that $xf(x) \in L^2(\mathbb{R})$, and that $f' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Show that

$$\int_{-\infty}^{+\infty} |f(x)|^2\,dx = -\int_{-\infty}^{+\infty} x\big(f'(x)\overline{f}(x) + f(x)\overline{f'}(x)\big)\,dx.$$

The Cauchy–Schwarz inequality asserts that

$$\left|\int_{-\infty}^{+\infty} a(x)b(x)\,dx\right|^2 \le \left(\int_{-\infty}^{+\infty} |a(x)|^2\,dx\right)\left(\int_{-\infty}^{+\infty} |b(x)|^2\,dx\right).$$

By means of this inequality, or otherwise, show that

$$\left(\int_{-\infty}^{+\infty} |xf(x)|^2\,dx\right)\left(\int_{-\infty}^{+\infty} |t\widehat{f}(t)|^2\,dt\right) \ge \frac{1}{16\pi^2}\left(\int_{-\infty}^{+\infty} |f(x)|^2\,dx\right)^2.$$

This is a form of the Heisenberg uncertainty principle. From it we see that if f tends to 0 rapidly outside $[-A, A]$, and if $\widehat{f}$ tends to 0 rapidly outside $[-B, B]$, then $AB \gg 1$.

9. (a) Note the identity

$$f\overline{g} = \tfrac{1}{2}|f+g|^2 - \tfrac{1}{2}|f-g|^2 + \tfrac{i}{2}|f+ig|^2 - \tfrac{i}{2}|f-ig|^2.$$

(b) Show that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and if $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$\int_{-\infty}^{+\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{+\infty} \widehat{f}(t)\overline{\widehat{g}(t)}\,dt.$$

10. Suppose that F is strictly increasing, and that for $i = 1, 2$ the functions f_i are real-valued with $f_i \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $F(f_i) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

(a) Show that

$$\int_{-\infty}^{+\infty} (f_1(x) - f_2(x))(F(f_1(x)) - F(f_2(x)))\,dx$$
$$= \int_{-\infty}^{+\infty} \big(\widehat{f_1}(t) - \widehat{f_2}(t)\big)\big(\widehat{F(f_1)}(t) - \widehat{F(f_2)}(t)\big)\,dt.$$

(b) Suppose additionally that $\widehat{f_i}(t) = 0$ for $|t| \ge T$, and that $\widehat{F(f_1)}(t) = \widehat{F(f_2)}(t)$ for $-T \le t \le T$. Show that $f_1 = f_2$ a.e.

5.2 Summability

We say that an infinite series $\sum a_n$ is Abel summable to a, and write $\sum a_n = a$ (A) if

$$\lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n r^n = a.$$

Abel proved that if a series converges, then it is A-summable to the same value. Because of this historical antecedent, we call a theorem 'Abelian' if it states that one kind of summability implies another. Perhaps the simplest Abelian theorem asserts that if $\sum_{n=1}^{\infty} a_n$ converges to a, then

$$\lim_{N\to\infty} \sum_{n=1}^{N} \left(1 - \frac{n}{N}\right) a_n = a. \qquad (5.27)$$

This is the Cesàro method of summability of order 1, and so we abbreviate the relation above as $\sum a_n = a$ (C, 1). On putting $s_N = \sum_{n=1}^{N} a_n$, we reformulate the above by saying that if $\lim_{N\to\infty} s_N = a$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s_n = a. \tag{5.28}$$

Here, as in Abel summability and in most other summabilities, each term in the second limit is a linear function of the terms in the first limit. Following Toeplitz and Schur, we characterize those linear transformations $T = [t_{mn}]$ that preserves limits of sequences. We call T *regular* if the following three conditions are satisfied:

$$\text{There is a } C = C(T) \text{ such that } \sum_{n=1}^{\infty} |t_{mn}| \le C \text{ for all } m; \tag{5.29}$$

$$\lim_{m\to\infty} t_{mn} = 0 \text{ for all } n; \tag{5.30}$$

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} t_{mn} = 1. \tag{5.31}$$

We now show that regular transformations preserve limits, and relegate the verification of the converse to exercises.

Theorem 5.5 *Suppose that T satisfies* (5.29) *above. If $\{a_n\}$ is a bounded sequence, then the sequence*

$$b_m = \sum_{n=1}^{\infty} t_{mn} a_n \tag{5.32}$$

is also bounded. If T satisfies (5.29) *and* (5.30), *and if* $\lim_{n\to\infty} a_n = 0$, *then* $\lim_{m\to\infty} b_m = 0$. *Finally, if T is regular and* $\lim_{n\to\infty} a_n = a$, *then* $\lim_{m\to\infty} b_m = a$.

The important special case (5.28) is obtained by noting that the (semi-infinite) matrix $[t_{mn}]$ with

$$t_{mn} = \begin{cases} 1/m & \text{if } 1 \le n \le m, \\ 0 & \text{if } n > m \end{cases}$$

is regular. Moreover, the proof of Theorem 5.5 requires only a straightforward elaboration of the usual proof of (5.28).

Proof If $|a_n| \le A$ and (5.29) holds, then

$$|b_m| \le \sum_{n=1}^{\infty} |t_{mn} a_n| \le A \sum_{n=1}^{\infty} |t_{mn}| \le CA.$$

To establish the second assertion, suppose that $\varepsilon > 0$ and that $|a_n| < \varepsilon$ for $n > N = N(\varepsilon)$. Now

$$|b_m| \le \sum_{n=1}^{N} |t_{mn}a_n| + \sum_{n>N} |t_{mn}a_n| = \Sigma_1 + \Sigma_2,$$

say. From (5.29) and the argument above with $A = \varepsilon$ we see that $\Sigma_2 \le C\varepsilon$. From (5.30) we see that $\lim_{m\to\infty} \Sigma_1 = 0$. Hence $\limsup_{m\to\infty} |b_m| \le C\varepsilon$, and we have the desired conclusion since ε is arbitrary. Finally, suppose that T is regular and that $\lim_{n\to\infty} a_n = a$. We write $a_n = a + \alpha_n$, so that

$$b_m = a\sum_{n=1}^{\infty} t_{mn} + \sum_{n=1}^{\infty} t_{mn}\alpha_n.$$

Since $\lim_{n\to\infty} \alpha_n = 0$, we may appeal to the preceding case to see that the second sum tends to 0 as $m \to \infty$. Hence by (5.31) we conclude that $\lim_{m\to\infty} b_m = a$, and the proof is complete. □

In Chapter 1 we used Theorem 1.1 to show that if $\mathcal{S}$ is a sector of the form $\mathcal{S} = \{s : \sigma > \sigma_0,\ |t - t_0| \le H(\sigma - \sigma_0)\}$ where H is an arbitrary positive constant, and if the Dirichlet series $\alpha(s)$ converges at the point s_0, then

$$\lim_{\substack{s\to s_0\\ s\in\mathcal{S}}} \alpha(s) = \alpha(s_0).$$

To see how this may also be derived from Theorem 5.5, let $\{s_m\}$ be an arbitrary sequence of points of $\mathcal{S}$ for which $\lim_{m\to\infty} s_m = s_0$. It suffices to show that $\lim_{m\to\infty} \alpha(s_m) = \alpha(s_0)$. Take

$$t_{mn} = n^{s_0 - s_m} - (n+1)^{s_0 - s_m},$$

so that

$$\alpha(s_m) = \sum_{n=1}^{\infty} t_{mn}\Big(\sum_{k=1}^{n} a_k k^{-s_0}\Big).$$

In view of Theorem 5.5, it suffices to show that $[t_{mn}]$ is regular. The conditions (5.30) and (5.31) are clearly satisfied, and (5.29) follows on observing that if $s \in \mathcal{S}$, then $s - s_0 \ll_H \sigma - \sigma_0$, so that

$$\begin{aligned}\left|n^{s_0-s} - (n+1)^{s_0-s}\right| &= \left|(s - s_0)\int_n^{n+1} u^{s_0-s-1}\,du\right| \\ &\ll_H (\sigma - \sigma_0)\int_n^{n+1} u^{\sigma_0-\sigma-1}\,du \\ &= n^{\sigma_0-\sigma} - (n+1)^{\sigma_0-\sigma}.\end{aligned}$$

Thus we have the result. Abel's analogous theorem on the convergence of power series can be derived similarly from Theorem 5.5.

The converse of Abel's theorem on power series is false, but Tauber (1897) proved a partial converse: If $a_n = o(1/n)$ and $\sum a_n = a$ (A), then $\sum a_n = a$. Following Hardy and Littlewood, we call a theorem 'Tauberian' if it provides a partial converse of an Abelian theorem. The qualifying hypothesis ('$a_n = o(1/n)$' in the above) is the 'Tauberian hypothesis'. For simplicity we begin with partial converses of (5.27).

Theorem 5.6 *If $\sum_{n=1}^{\infty} a_n = a$ (C, 1), then $\sum a_n = a$ provided that one of the following hypotheses holds*:
(a) $a_n \ge 0$ *for* $n \ge 1$;
(b) $a_n = O(1/n)$ *for* $n \ge 1$;
(c) *There is a constant A such that $a_n \ge -A/n$ for all $n \ge 1$.*

Proof Clearly (a) implies (c). If (b) holds, then both $\Re a_n$ and $\Im a_n$ satisfy (c). Thus it suffices to prove that $\sum a_n = a$ when (c) holds. We observe that if H is a positive integer, then

$$\begin{aligned}\sum_{n=1}^{N} a_n &= \frac{N+H}{H}\sum_{n=1}^{N+H} a_n\left(1-\frac{n}{N+H}\right) - \frac{N}{H}\sum_{n=1}^{N} a_n\left(1-\frac{n}{N}\right)\\ &\quad -\frac{1}{H}\sum_{N<n<N+H} a_n(N+H-n) \qquad (5.33)\\ &= T_1 - T_2 - T_3,\end{aligned}$$

say. Take $H = [\varepsilon N]$ for some $\varepsilon > 0$. By hypothesis, $\lim_{N\to\infty} T_1 = a(1+\varepsilon)/\varepsilon$, and $\lim_{N\to\infty} T_2 = a/\varepsilon$. From (c) we see that

$$T_3 \ge -A\sum_{N<n<N+H}\frac{1}{n} \ge -\frac{AH}{N} \ge -A\varepsilon.$$

Hence on combining these estimates in (5.33) we see that

$$\limsup_{N\to\infty}\sum_{n=1}^{N} a_n \le a + A\varepsilon.$$

Since ε can be taken arbitrarily small, it follows that

$$\limsup_{N\to\infty}\sum_{n=1}^{N} a_n \le a.$$

To obtain a corresponding lower bound we note that

$$\begin{aligned}\sum_{n=1}^{N} a_n &= \frac{N}{H}\sum_{n=1}^{N} a_n\left(1-\frac{n}{N}\right) - \frac{N-H}{H}\sum_{n=1}^{N-H} a_n\left(1-\frac{n}{N-H}\right) \qquad (5.34)\\ &\quad +\frac{1}{H}\sum_{N-H<n<N} a_n(n+H-N).\end{aligned}$$

Arguing as we did before, we find that

$$\liminf_{N\to\infty} \sum_{n=1}^{N} a_n \ge a - A\varepsilon/(1-\varepsilon),$$

so that

$$\liminf_{N\to\infty} \sum_{n=1}^{N} a_n \ge a,$$

and the proof is complete. □

If we had argued from (a) or (b), then the treatment of the term T_3 above would have been simpler, since from (a) it follows that $T_3 \ge 0$, while from (b) we have $T_3 \ll \varepsilon$.

Our next objective is to generalize and strengthen Theorem 5.6. The type of generalization we have in mind is exhibited in the following result, which can be established by adapting the above proof: Let β be fixed, $\beta \ge 0$. If

$$\sum_{n=1}^{N} a_n \left(1 - \frac{n}{N}\right) = (a + o(1))N^{\beta},$$

and if $a_n \ge -An^{\beta-1}$, then

$$\sum_{n=1}^{N} a_n = (a(\beta+1) + o(1))N^{\beta}.$$

Concerning the possibility of strengthening Theorem 5.6, we note that by an Abelian argument (or by an application of Theorem 5.5) it may be shown that $\sum a_n = a$ (C, 1) implies that $\sum a_n = a$ (A). Thus if we replace (C, 1) by (A) in Theorem 5.6, then we have weakened the hypothesis, and the result would therefore be stronger. Indeed, Hardy (1910) conjectured and Littlewood (1911) proved that if $\sum a_n = a$ (A) and $a_n = O(1/n)$, then $\sum a_n = a$. That is, the condition '$a_n = o(1/n)$' in Tauber's theorem can be replaced by the condition (b) above. In fact the still weaker condition (c) suffices, as will be seen by taking $\beta = 0$ in Corollary 5.9 below. We now formulate a general result for the Laplace transform, from which the analogues for power series and Dirichlet series follow easily.

Theorem 5.7 (Hardy–Littlewood) *Suppose that $a(u)$ is Riemann-integrable over $[0, U]$ for every $U > 0$, and that the integral*

$$I(\delta) = \int_0^\infty a(u)e^{-u\delta}\,du$$

converges for every $\delta > 0$. *Let* β *be fixed,* $\beta \ge 0$, *and suppose that*

$$I(\delta) = (\alpha + o(1))\delta^{-\beta} \tag{5.35}$$

as $\delta \to 0^+$. *If, moreover, there is a constant* $A \ge 0$ *such that*

$$a(u) \ge -A(u+1)^{\beta-1} \tag{5.36}$$

for all $u \ge 0$, *then*

$$\int_0^U a(u)\,du = \left(\frac{\alpha}{\Gamma(\beta+1)} + o(1)\right) U^\beta. \tag{5.37}$$

The basic properties of the gamma function are developed in Appendix C, but for our present purposes it suffices to put

$$\Gamma(\beta) = \int_0^\infty u^{\beta-1}e^{-u}\,du$$

for $\beta > 0$. From this it follows by integration by parts that

$$\beta\Gamma(\beta) = \Gamma(\beta+1) \tag{5.38}$$

when $\beta > 0$.

The amount of unsmoothing required in deriving (5.37) from (5.35) is now much greater than it was in the proof of Theorem 5.6. Nevertheless we follow the same line of attack. To obtain the proper perspective we review the preceding proof. Let $\mathcal{J} = [0, 1]$, let $\chi_{\mathcal{J}}(u)$ be its characteristic function, and put $K(u) = \max(0, 1-u)$ for $u \ge 0$. Thus $\sum_{n=1}^N a_n = \sum_n a_n \chi_{\mathcal{J}}(n/N)$, and $\sum_{n=1}^N a_n(1 - n/N) = \sum_n a_n K(n/N)$. Our strategy was to approximate to $\chi_{\mathcal{J}}(u)$ by linear combinations of $K(\kappa u)$ for various values of κ, $\kappa > 0$. The relation underlying (5.33) and (5.34) is both simple and explicit:

$$\frac{1}{\varepsilon}\big(K(u) - (1-\varepsilon)K(u/(1-\varepsilon))\big) \le \chi_{\mathcal{J}}(u) \le \frac{1}{\varepsilon}((1+\varepsilon)K(u/(1+\varepsilon)) - K(u)); \tag{5.39}$$

we took $\varepsilon = H/N$. In the present situation we wish to approximate to $\chi_{\mathcal{J}}(u)$ by linear combinations of $e^{-\kappa u}$, $\kappa > 0$. We make the change of variable $x = e^{-u}$, so that $0 \le x \le 1$, and we put $\mathcal{J} = [1/e, 1]$. Then we want to approximate to $\chi_{\mathcal{J}}(x)$ by a linear combination $P(x)$ of the functions x^κ, $\kappa > 0$. In fact it suffices to use only integral values of κ, so that $P(x)$ is a polynomial that vanishes at the origin. In place of (5.33), (5.34) and (5.39) we shall substitute

Lemma 5.8 *Let* ε *be given,* $0 < \varepsilon < 1/4$, *and put* $\mathcal{J} = [1/e, 1]$, $\mathcal{K} = [e^{-1-\varepsilon}, e^{-1+\varepsilon}]$. *There exist polynomials* $P_\pm(x)$ *such that for* $0 \le x \le 1$ *we have*

$$P_-(x) \le \chi_{\mathcal{J}}(x) \le P_+(x) \tag{5.40}$$

and

$$|P_\pm(x) - \chi_{\mathcal{J}}(x)| \le \varepsilon x(1-x) + 5\chi_{\mathcal{K}}(x). \tag{5.41}$$

Proof Let $g(x) = (\chi_{\mathcal{J}}(x) - x)/(x(1-x))$. Then g is continuous in [0, 1] apart from a jump discontinuity at $x = 1/e$ of height $e^2/(e-1) < 5$. Hence by Weierstrass's theorem on the uniform approximation of continuous functions by polynomials we see that there are polynomials $Q_\pm(x)$ such that $Q_-(x) \le g(x) \le Q_+(x)$ for $0 \le x \le 1$, and for which

$$|g(x) - Q_\pm(x)| \le \varepsilon + 5\chi_{\mathcal{K}}(x) \tag{5.42}$$

for $0 \le x \le 1$. Then the polynomials $P_\pm(x) = x + x(1-x)Q_\pm(x)$ have the desired properties. □

Proof of Theorem 5.7 We suppose first that $\alpha = 0$. We note that if $P(x)$ is a polynomial such that $P(0) = 0$, say $P(x) = \sum_{r=1}^R c_r x^r$, then by (5.35) we see that

$$\int_0^\infty a(u)P(e^{-u\delta})\,du = \sum_{r=1}^R c_r I(r\delta) = o(\delta^{-\beta}) \tag{5.43}$$

as $\delta \to 0^+$. In the notation of the above lemma,

$$\int_0^U a(u)\,du = \int_0^\infty a(u)\chi_{\mathcal{J}}(e^{-u/U})\,du.$$

If (5.40) holds, then by (5.36) we see that

$$\int_0^\infty a(u)\big(P_+\big(e^{-u/U}\big) - \chi_{\mathcal{J}}\big(e^{-u/U}\big)\big)\,du$$
$$\ge -A\int_0^\infty (u+1)^{\beta-1}\big(P_+\big(e^{-u/U}\big) - \chi_{\mathcal{J}}\big(e^{-u/U}\big)\big)\,du.$$

By (5.41) this latter integral is

$$\ll \varepsilon\int_0^\infty (u+1)^{\beta-1}e^{-u/U}(1-e^{-u/U})\,du + \int_{(1-\varepsilon)U}^{(1+\varepsilon)U}(u+1)^{\beta-1}\,du.$$

In the first term, the integrand is $\ll (u+1)^\beta U^{-1}$ for $0 \le u \le U$; it is $\ll u^{\beta-1}e^{-u/U}$ for $u \ge U$. Hence the first integral is $\ll U^\beta$. The second integral is $\ll \varepsilon U^\beta$. On taking $\delta = 1/U$, $P = P_+$ in (5.43) and combining our results, we find that

$$\int_0^U a(u)\,du \le A_1\varepsilon U^\beta + o(U^\beta).$$

Since ε can be arbitrarily small, we deduce that

$$\limsup_{U\to\infty} U^{-\beta}\int_0^U a(u)\,du \le 0.$$

By arguing similarly with P_- instead of P_+, we see that the corresponding liminf is ≥ 0, and so we have (5.37) in the case $\alpha = 0$.

Suppose now that $\alpha \ne 0$, $\beta > 0$. We note first that

$$\int_0^\infty (u+1)^{\beta-1}e^{-u\delta}\,du = e^\delta\int_1^\infty v^{\beta-1}e^{-v\delta}\,dv = e^\delta\int_0^\infty v^{\beta-1}e^{-v\delta}\,dv + O(e^\delta),$$

and that

$$\int_0^\infty v^{\beta-1}e^{-v\delta}\,dv = \delta^{-\beta}\int_0^\infty w^{\beta-1}e^{-w}\,dw = \delta^{-\beta}\Gamma(\beta).$$

Hence if $b(u) = a(u) - \alpha(u+1)^{\beta-1}/\,\Gamma(\beta)$, then $b(u) \ge -B(u+1)^{\beta-1}$, and

$$\int_0^\infty b(u)e^{-u\delta}\,du = o(\delta^{-\beta}).$$

Thus $\int_0^U b(u)\,du = o(U^\beta)$, so that

$$\int_0^U a(u)\,du = \frac{\alpha}{\beta\Gamma(\beta)}U^\beta + o(U^\beta),$$

and we have (5.37), in view of (5.38).

For the remaining case, $\beta = 0$, it suffices to consider $b(u) = a(u) - \alpha\chi_{[0,1]}(u)$. □

Corollary 5.9 *Suppose that $p(z) = \sum_{n=0}^\infty a_n z^n$ converges for $|z| < 1$, and that $\beta \ge 0$. If $p(x) = (\alpha + o(1))(1-x)^{-\beta}$ as $x \to 1^-$, and if $a_n \ge -An^{\beta-1}$ for $n \ge 1$, then*

$$\sum_{n=0}^N a_n = \left(\frac{\alpha}{\Gamma(\beta+1)} + o(1)\right)N^\beta.$$

Proof Put $a(u) = a_n$ for $n \le u < n+1$. Then (5.36) holds, and

$$I(\delta) = \sum_{n=0}^\infty a_n\int_n^{n+1} e^{-u\delta}\,du = \frac{1-e^{-\delta}}{\delta}p(e^{-\delta}).$$

But $1 - e^{-\delta} \sim \delta$ as $\delta \to 0^+$, so that (5.35) holds. The result now follows by taking $U = N+1$ in (5.37). □

Corollary 5.10 *If $\sum a_n = \alpha$ (A), and if the sequence $s_N = \sum_{n=0}^N a_n$ is bounded, then $\sum a_n = \alpha$ (C, 1).*

Proof Take $\beta = 1$, $p(z) = \sum_{n=0}^{\infty} s_n z^n = (1-z)^{-1} \sum_{n=0}^{\infty} a_n z^n$ in Corollary 5.9. Then $\sum_{n=0}^{N} s_n = (\alpha + o(1))N$, which is the desired result. □

For Dirichlet series we have similarly

Theorem 5.11 *Suppose that* $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *converges for* $\sigma > 1$, *and that* $\beta \ge 0$. *If* $\alpha(\sigma) = (\alpha + o(1))(\sigma - 1)^{-\beta}$ *as* $\sigma \to 1^+$, *and if* $a_n \ge -A(1 + \log n)^{\beta - 1}$, *then*

$$\sum_{n=1}^{N} \frac{a_n}{n} = \left(\frac{\alpha}{\Gamma(\beta + 1)} + o(1) \right) (\log N)^{\beta}.$$

Proof Take $a(u) = \sum_{u-1 \le \log n < u} a_n/n$. Then $I(\delta)$ converges for $\delta > 0$, and moreover

$$I(\delta) = \sum_{n=1}^{\infty} \frac{a_n}{n} \int_{\log n}^{1+\log n} e^{-u\delta}\, du = \frac{1 - e^{-\delta}}{\delta} \alpha(1 + \delta),$$

so that (5.37) follows. To obtain the desired conclusion we require a further appeal to our Tauberian hypothesis. We note that

$$\int_0^{\log N} a(u)\, du = \sum_{n \le N} \frac{a_n}{n} - \sum_{N/e < n \le N} \frac{a_n}{n} \log \frac{ne}{N}.$$

By our Tauberian hypothesis this is

$$\le \sum_{n \le N} \frac{a_n}{n} + A_1 (\log N)^{\beta - 1},$$

so that

$$\sum_{n \le N} \frac{a_n}{n} \ge \left(\frac{\alpha}{\Gamma(\beta + 1)} + o(1) \right) (\log N)^{\beta} - A_1 (\log N)^{\beta - 1}.$$

On taking $U = 1 + \log N$ in (5.37) we may derive a corresponding upper bound to complete the proof. □

The qualitative arguments we have given can be put in quantitative form as the need arises. For example, it is easy to see that if

$$\sum_{n=1}^{N} a_n = N + O\big(\sqrt{N}\big), \tag{5.44}$$

then

$$\sum_{n=1}^{N} a_n (N - n) = \frac{1}{2} N^2 + O\big(N^{3/2}\big). \tag{5.45}$$

This is best possible (take $a_n = 1 + n^{-1/2}$), but if the error term is oscillatory, then smoothing may reduce its size (consider $a_n = \cos\sqrt{n}$). Conversely if (5.45) holds and if the sequence a_n is bounded, then the method used to prove Theorem 5.6 can be used to show that

$$\sum_{n=1}^{N} a_n = N + O\big(N^{3/4}\big). \tag{5.46}$$

This conclusion, though it falls short of (5.44), is best possible (take $a_n = 1 + \cos n^{1/4}$). We can also put Theorem 5.7 in quantitative form, but here the loss in precision is much greater, and in general the importance of Theorem 5.7 and its corollaries lies in its versatility. For example, it can be shown that if $\sum_{n=0}^{\infty} a_n r^n = (1-r)^{-1} + O(1)$ as $r \to 1^-$, and if $a_n = O(1)$, then

$$\sum_{n=0}^{N} a_n = N + O\left(\frac{N}{\log N}\right).$$

This error term, though weak, is best possible (take $a_n = 1 + \cos(\log n)^2$).

For Dirichlet series it can be shown that if

$$\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \frac{1}{s-1} + O(1)$$

as $s \to 1^+$, and if the sequence a_n is bounded, then

$$\sum_{n=1}^{N} \frac{a_n}{n} = \log N + O\left(\frac{\log N}{\log\log N}\right).$$

This is also best possible (take $a_n = 1 + \cos(\log\log n)^2$), but we can obtain a sharper result by strengthening our analytic hypothesis. For example, it can be shown that if $\alpha(s)$ is analytic in a neighbourhood of 1 and if the sequence a_n is bounded, then

$$\sum_{n=1}^{N} \frac{a_n}{n} = O(1).$$

However, even this stronger assumption does not allow us to deduce that

$$\sum_{n=1}^{N} a_n = o(N),$$

as we see by considering $a_n = \cos\log n$. In Chapter 8 we shall encounter further Tauberian theorems in which the above conclusion is derived from hypotheses concerning the behaviour of $\alpha(s)$ throughout the half-plane $\sigma \ge 1$.

5.2.1 Exercises

1. Let T be a regular matrix such that $t_{mn} \ge 0$ for all m, n. Show that if $\lim_{n\to\infty} a_n = +\infty$, then $\lim_{m\to\infty} b_m = +\infty$.
2. Show that if $T = [t_{mn}]$ and $U = [u_{mn}]$ are regular matrices, then so is $TU = V = [v_{mn}]$ where
$$v_{mn} = \sum_{k=1}^{\infty} t_{mk} u_{kn}.$$
3. Show that if $\boldsymbol{b} = T\boldsymbol{a}$ and $\lim_{m\to\infty} b_m = a$ whenever $\lim_{n\to\infty} a_n = a$, then T is regular.
4. For $n = 0, 1, 2, \ldots$ let $t_n(x)$ be defined on $[0, 1)$, and suppose that the t_n satisfy the following conditions:
   (i) There is a constant C such that if $x \in [0, 1)$, then $\sum_{n=0}^{\infty} |t_n(x)| \le C$.
   (ii) For all n, $\lim_{x\to 1^-} t_n(x) = 0$.
   (iii) $\lim_{x\to 1^-} \sum_{n=0}^{\infty} t_n(x) = 1$.
   Show that if $\lim_{n\to\infty} a_n = a$ and if $b(x) = \sum_{n=0}^{\infty} a_n t_n(x)$, then $\lim_{x\to 1^-} b(x) = a$.
5. (Kojima 1917) Suppose that the numbers t_{mn} satisfy the following conditions:
   (i) There is a constant C such that $\sum_{n=1}^{\infty} |t_{mn}| \le C$ for all m.
   (ii) For all n, $\lim_{m\to\infty} t_{mn}$ exists.
   (iii) $\lim_{m\to\infty} \sum_{n=1}^{\infty} t_{mn}$ exists.
   Show that if $\lim_{n\to\infty} a_n$ exists and if $b_m = \sum_{n=1}^{\infty} t_{mn} a_n$, then $\lim_{m\to\infty} b_m$ exists.
6. For positive integers n let $K_n(x)$ be a function defined on $[0, \infty)$ such that
   (i) $\int_0^{\infty} K_n(x)\,dx \to 1$ as $n \to \infty$;
   (ii) $\int_0^{\infty} |K_n(x)|\,dx \le C$ for all n;
   (iii) $\lim_{n\to\infty} K_n(x) = 0$ uniformly for $0 \le x \le X$.
   Suppose that $a(x)$ is a bounded function, and that $b_n = \int_0^{\infty} a(x) K_n(x)\,dx$. Show that if $\lim_{x\to\infty} a(x) = a$, then $\lim_{n\to\infty} b_n = a$.
7. Let r_m be a sequence of positive real numbers with $r_m \to 1^-$ as $m \to \infty$. For $m \ge 1, n \ge 1$, put $t_{mn} = n r_m^{n-1} (1 - r_m)^2$.
   (a) Show that $[t_{mn}]$ is regular.
   (b) Show that if $a_n = \sum_{k=0}^{n-1} c_k (1 - k/n)$ and b_m is defined by (5.32), then $b_m = \sum_{k=0}^{\infty} c_k r_m^k$.
   (c) Show that if $\sum c_n = c$ (C, 1), then $\sum c_n = c$ (A).
8. Suppose that $T = [t_{mn}]$ is given by
$$t_{mn} = \begin{cases} 0 & \text{if } n = 0, \\ \dfrac{m!n}{m^{n+1}(m-n)!} & \text{if } m \ge n > 0, \\ 0 & \text{if } m < n. \end{cases}$$

(a) Show that

$$\sum_{n=k}^{m} t_{mn} = \frac{m!}{m^k(m-k)!}$$

for $1 \le k \le m$.

(b) Verify that T is regular.

(c) Show that if $a_n = \sum_{k=0}^{n} x^k/k!$ for $n \ge 0$, then $b_m = (1 + x/m)^m$ for $m \ge 1$.

9. (Mercer's theorem) Suppose that

$$b_m = \frac{1}{2}a_m + \frac{1}{2} \cdot \frac{a_1 + a_2 + \cdots + a_m}{m}$$

for $m \ge 1$. Show that

$$a_n = \frac{2n}{n+1}b_n - \frac{2}{n(n+1)}\sum_{m=1}^{n-1} mb_m.$$

Conclude that $\lim_{n\to\infty} a_n = a$ if and only if $\lim_{m\to\infty} b_m = a$.

10. For a non-negative integer k we say that $\sum a_n = a$ (C, k) if

$$\lim_{x\to\infty} \sum_{n\le x} a_n \left(1 - \frac{n}{x}\right)^k = a.$$

This is *Cesàro summability of order k*.

(a) Show that if $\sum a_n = a$ (C, j), then $\sum a_n = a$ (C, k) for all $k \ge j$.

(b) Show that if $\sum a_n = a$ (C, k) for some k, then $\sum a_n = a$ (A).

11. Show that if $\sum a_n = a$ (A), then $\lim_{s\to 0^+} \sum a_n n^{-s} = a$. (See Wintner 1943 for Tauberian converses.)

12. For a non-negative integer k we say that $\sum a_n = a$ (R, k) if

$$\lim_{x\to\infty} \sum_{n\le x} a_n \left(1 - \frac{\log n}{\log x}\right)^k = a.$$

This is *Riesz summability of order k*.

(a) Show that if $\sum a_n = a$ (R, j), then $\sum a_n = a$ (R, k) for all $k \ge j$.

(b) Show that if $\sum a_n = a$ (R, k) for some k, then $\sum_{s\to 0^+} \alpha(s) = a$.

13. Put $t_{mn} = 0$ for $n > m$, set

$$t_{mm} = \frac{m+1}{\log(m+1)}(\log(m+1) - \log m),$$

while for $1 \le n < m$ put

$$t_{mn} = \frac{n+1}{\log(m+1)}(-\log n + 2\log(n+1) - \log(n+2)).$$

(a) Show that if

$$a_n = \sum_{k=1}^{n} c_k \left(1 - \frac{k}{n+1}\right)$$

for $n \ge 1$, then the b_m given in (5.32) satisfies

$$b_m = \sum_{k=1}^{m} c_k \left(1 - \frac{\log k}{\log(n+1)}\right).$$

(b) Show that $t_{mn} \ge 0$ for all m, n.

(c) Show that

$$\sum_{n=1}^{\infty} t_{mn} = 1 + \frac{\log 2}{\log(m+1)}.$$

(d) Show that $\lim_{m\to\infty} t_{mn} = 0$.

(e) Conclude that if $\sum c_k = c$ (C, 1), then $\sum c_k = c$ (R, 1).

14. Let $A(x) = \sum_{0<n\le x} a_n$.

(a) Show that

$$\sum_{n=1}^{N} a_n \left(1 - \frac{n}{N}\right) = \frac{1}{N}\int_0^N A(x)\,dx.$$

(b) Show that

$$\sum_{n=1}^{N} a_n \left(1 - \frac{\log n}{\log N}\right) = \frac{1}{\log N}\int_1^N \frac{A(x)}{x}\,dx.$$

(c) Suppose that t is a fixed non-zero real number. By Corollary 1.15, or otherwise, show that

$$\sum_{n=1}^{N} n^{-1-it}\left(1 - \frac{n}{N}\right) = \frac{N^{-it}}{(1-it)^2} + \zeta(1+it) + O\left(\frac{\log N}{N}\right).$$

(d) Similarly, show that

$$\sum_{n=1}^{N} n^{-1-it}\left(1 - \frac{\log n}{\log N}\right) = \zeta(1+it) + O\left(\frac{1}{\log N}\right).$$

(e) Conclude that $\sum_{n=1}^{\infty} n^{-1-it}$ is not summable (C, 1), but that it is summable (R, 1) to $\zeta(1+it)$.

15. We say that a series is *Lambert summable*, and write $\sum a_n = a$ (L), if

$$\lim_{r\to 1^-} (1-r)\sum_{n=1}^{\infty} \frac{n a_n r^n}{1-r^n} = a.$$

(a) Show that if $\sum a_n = a$, then $\sum a_n = a$ (L).

(b) Show that if a_n is a bounded sequence and $|z| < 1$, then

$$\sum_{n=1}^{\infty} \frac{na_n z^n}{1 - z^n} = \sum_{n=1}^{\infty} \Big(\sum_{d|n} da_d \Big) z^n.$$

(c) Show that $\sum_{n=1}^{\infty} \mu(n)/n = 0$ (L).

(d) Deduce that if $\sum_{n=1}^{\infty} \mu(n)/n$ converges, then its value is 0. (See (6.18) and (8.6).)

(e) Show that $\sum_{n=1}^{\infty} (\Lambda(n) - 1)/n = -2C_0$ (L).

(f) Deduce that if $\sum_{n \le x} \Lambda(n)/n = \log x + c + o(1)$ then $c = -C_0$. (See Exercise 8.1.1.)

16. (Bohr 1909; Riesz 1909; Phragmén (cf. Landau 1909, pp. 762, 904)) Let $\alpha(s) = \sum a_n n^{-s}$, $\beta(s) = \sum b_n n^{-s}$, and $\gamma(s) = \alpha(s)\beta(s) = \sum c_n n^{-s}$ where $c_n = \sum_{d|n} a_d b_{n/d}$. Further, put $A(x) = \sum_{n \le x} a_n$ and $B(x) = \sum_{n \le x} b_n$.

(a) Show that

$$\int_1^x A(y)B(x/y) \frac{dy}{y} = \sum_{n \le x} c_n \log x/n.$$

(b) Show that if $\sum a_n$ converges and $\sum b_n$ converges, then $\sum c_n = \alpha(0)\beta(0)$ (R, 1).

(c) (Landau 1907) By taking $j = 0$ in Exercise 12(a), or otherwise, show that if the three series $\sum a_n$, $\sum b_n$, $\sum c_n$ all converge, then $\sum c_n = \big(\sum a_n\big)\big(\sum b_n\big)$.

17. Suppose that $f(n) \nearrow \infty$. Construct a_n so that $|a_n| \le f(n)/n$ for all n,

$$\limsup_{N \to \infty} \sum_{n=1}^{N} a_n = 1, \qquad \liminf_{N \to \infty} \sum_{n=1}^{N} a_n = -1,$$

but

$$\lim_{N \to \infty} \sum_{n=1}^{N} a_n (1 - n/N) = 0.$$

18. (Landau 1908) Show that if $f(x) \sim x$ as $x \to \infty$ and $xf'(x)$ is increasing, then $\lim_{x \to \infty} f'(x) = 1$.

19. (Landau (1913); cf. Littlewood (1986, p. 54–55); Schoenberg 1973) Show that if $f(x) \to 0$ as $x \to \infty$, and if $f''(x) = O(1)$, then $f'(x) \to 0$ as $x \to \infty$.

20. (Tauber's 'second theorem') Suppose that $P(\delta) = \sum_{n=0}^{\infty} a_n e^{-n\delta}$ for $\delta > 0$, and put $s_N = \sum_{n=0}^{N} a_n$.

(a) Show that if $a_n = O(1/n)$, then $s_N = P(1/N) + O(1)$.

(b) Show that if $a_n = o(1/n)$, then $s_N = P(1/N) + o(1)$.

(c) Let $B(N) = \sum_{n=1}^{N} na_n$. Show that if $\sum a_n$ converges, then $B(N) = o(N)$ as $N \to \infty$.

(d) Show that if $P(\delta)$ converges for $\delta > 0$, then

$$s_N - P(1/N) = \frac{B(N)}{N} + \int_1^N B(u)\left(\frac{1}{u^2} - \frac{e^{-u/N}}{u^2} - \frac{e^{-u/N}}{uN}\right) du + \int_N^\infty B(u)e^{-u/N}\left(\frac{u}{N} - 1\right)\frac{du}{u^2}.$$

(e) Show that if $B(N) = o(N)$, then $s_N - P(1/N) = o(1)$.

(f) Show that if $\sum a_n = a$ (A), then $\sum a_n = a$ if and only if $B(N) = o(N)$.

21. (a) Using Ramanujan's identity $\sum_{n=1}^{\infty} d(n)^2 n^{-s} = \zeta(s)^4/\zeta(2s)$ and Theorem 5.11, show that $\sum_{n\le x} d(n)^2/n \sim (4\pi^2)^{-1}(\log x)^4$.

(b) Show that if $\sum_{n\le x} d(n)^2 \sim cx(\log x)^3$ as $x \to \infty$, then $c = 1/\pi^2$.

22. Show that $\sum_{n=1}^{\infty} 1/(d(n)n^s) \sim c(s-1)^{-1/2}$ as $s \to 1^+$ where

$$c = \prod_p \left((p^2 - p)^{1/2} \log\left(\frac{p}{p-1}\right)\right).$$

Deduce that

$$\sum_{n\le x} \frac{1}{nd(n)} \sim \frac{2c}{\sqrt{\pi}}(\log x)^{1/2}$$

as $x \to \infty$.

23. Show that if $\sum_{n\le N} a_n/n = O(1)$ and $\lim_{s\to 1^+} \sum_{n=1}^{\infty} a_n n^{-s} = a$, then

$$\lim_{x\to\infty} \sum_{n\le x} \frac{a_n}{n}\left(1 - \frac{\log n}{\log x}\right) = a.$$

24. Show that

$$\int_0^\infty \frac{\sin x}{x} e^{-sx}\, dx = \arctan 1/s$$

for $s > 0$. Using Theorem 5.7, deduce that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

25. Suppose that $f(u) \ge 0$, that $\int_0^\infty f(u)\, du < \infty$, and that $\int_0^\infty (1 - e^{-\delta u})\, du \sim \delta^{1/2}$ as $\delta \to 0^+$. Show that $\int_U^\infty f(u)\, du \sim (\pi U)^{-1/2}$ as $U \to \infty$.

26. Show that $\sum_{n=1}^{\infty} a_n = a$ if and only if

$$\lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n r^{2^n} = a.$$

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27. Suppose that for every $\varepsilon > 0$ there is an $\eta > 0$ such that $\sum_{N<n\le(1+\eta)N} |a_n| < \varepsilon$ whenever $N > 1/\eta$. Show that if $\sum a_n = a$ (A), then $\sum a_n = a$.
28. Show that if $\sum a_n = a$ (C, 1) and if $a_{n+1} - a_n = O(|a_n|/n)$, then $\sum a_n = a$.
29. (Hardy & Littlewood 1913, Theorem 27) Show that if $\sum a_n = a$ (A) and if $a_{n+1} - a_n = O(|a_n|/n)$, then $\sum a_n = a$.
30. (Hardy 1907) Show that

$$\lim_{x\to 1^-} \sum_{k=0}^{\infty} (-1)^k x^{2^k}$$

does not exist.

5.3 Notes

Section 5.1. Theorem 5.1 and the more general (5.22) were first proved rigorously by Perron (1908). Although the Mellin transform had been used by Riemann and Cahen, it was Mellin (1902) who first described a general class of functions for which the inversion succeeds. Hjalmar Mellin was Finnish, but his family name is of Swedish origin, so it is properly pronounced mĕ · lēn′. However, in English-speaking countries the uncultured pronunciation mĕl′· ĭn is universal.

In connection with Theorem 5.4, it should be noted that Plancherel's formula $\|f\|_2 = \|\widehat{f}\|_2$ holds not just for all $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ but actually for all $f \in L^2(\mathbb{R})$. However, in this wider setting one must adopt a new definition for $\widehat{f}$, since the definition we have taken is valid only for $f \in L^1(\mathbb{R})$. See Goldberg (1961, pp. 46–47) for a resolution of this issue.

For further material concerning properties of Dirichlet series, one should consult Hardy & Riesz (1915), Titchmarsh (1939, Chapter 9), or Widder (1971, Chapter 2). Beyond the theory developed in these sources, we call attention to two further topics of importance in number theory. Wiener (1932, p. 91) proved that if the Fourier series of $f \in L^1(\mathbb{T})$ is absolutely convergent and is never zero, then the Fourier series of $1/f$ is also absolutely convergent. Wiener's proof was rather difficult, but Gel'fand (1941) devised a simpler proof depending on his theory of normed rings. Lévy (1934) proved more generally that the Fourier series of $F(f)$ is absolutely convergent provided that F is analytic at all points in the range of f. Elementary proofs of these theorems have been given by Zygmund (1968, pp. 245–246) and Newman (1975). These theorems were generalized to absolutely convergent Dirichlet series by Hewitt & Williamson (1957), who showed that if $\alpha(s) = \sum a_n n^{-s}$ is absolutely convergent for $\sigma \ge \sigma_0$, then $1/\alpha(s)$ is represented by an absolutely convergent Dirichlet series

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in the same half-plane, if and only if the values taken by $\alpha(s)$ in this half-plane are bounded away from 0. Ingham (1962) noted a fallacy in Zygmund's account of Lévy's theorem, corrected it, and gave an elementary proof of the generalization to absolutely convergent Dirichlet series. See also Goodman & Newman (1984). Secondly, Bohr (1919) developed a theory concerning the values taken on by an absolutely convergent Dirichlet series. This is described by Titchmarsh (1986, Chapter 11), and in greater detail by Apostol (1976, Chapter 8). For a small footnote to this theory, see Montgomery & Schinzel (1977).

Section 5.2. That conditions (5.29)–(5.31) are necessary and sufficient for the transformation T to preserve limits was proved by Toeplitz (1911) for upper triangular matrices, and by Steinhaus (1911) in general. See also Kojima (1917) and Schur (1921). For more on the Toeplitz matrix theorem and various aspects of Tauberian theorems, see Peyerimhoff (1969).

Theorem 5.6 under the hypothesis (a) is trivial by dominated convergence. Theorem 5.6(b) is a special case of a theorem of Hardy (1910), who considered the more general (C,k) convergence, and Theorem 5.6(c) is similarly a special case of a theorem of Landau (1910, pp. 103–113).

Tauber (1897) proved two theorems, the second of which is found in Exercise 5.2.18. Littlewood (1911) derived his strengthening of Tauber's first theorem by using high-order derivatives. Subsequently Hardy & Littlewood (1913, 1914a, b, 1926, 1930) used the same technique to obtain Theorem 5.8 and its corollaries. Karamata (1930, 1931a, b) introduced the use of Weierstrass's approximation theorem. Karamata also considered a more general situation, in which the right-hand sides of (5.35) and (5.36) are multiplied by a slowly oscillating function $L(1/\delta)$, and the right-hand side of (5.37) is multiplied by $L(U)$. Our exposition employs a further simplification due to Wielandt (1952). Other proofs of Littlewood's theorem have been given by Delange (1952) and by Eggleston (1951). Ingham (1965) observed that a peak function similar to Littlewood's can be constructed by using high-order differencing instead of differentiation. Since many proofs of the Weierstrass theorem involve constructing a peak function, the two methods are not materially different. Sharp quantitative Tauberian theorems have been given by Postnikov (1951), Korevaar (1951, 1953, 1954a–d), Freud (1952, 1953, 1954), Ingham (1965), and Ganelius (1971).

For other accounts of the Hardy–Littlewood theorem, see Hardy (1949) or Widder (1946, 1971). For a brief survey of applications of summability to classical analysis, see Rubel (1989).

Wiener (1932, 1933) invented a general Tauberian theory that contains the Hardy–Littlewood theorems for power series (Theorem 5.8 and its corollaries) as a special case. Wiener's theory is discussed by Hardy (1949), Pitt (1958), and Widder (1946). Among the longer expositions of Tauberian theory, the recent accounts of Korevaar (2002, 2004) are especially recommended.

5.4 References

Apostol, T. (1976). *Modular Functions and Dirichlet Series in Number Theory*, Graduate Texts Math. 41. New York: Springer-Verlag.

Bohr, H. (1909). Über die Summabilität Dirichletscher Reihen, *Nachr. König. Gesell. Wiss. Göttingen Math.-Phys. Kl.*, 247–262; *Collected Mathematical Works*, Vol. I. København: Dansk Mat. Forening, 1952, A2.

(1919). Zur Theorie algemeinen Dirichletschen Reihen, *Math. Ann.* **79**, 136–156; *Collected Mathematical Works*, Vol. I. København: Dansk Mat. Forening, 1952, A13.

Delange, H. (1952). Encore une nouvelle démonstration du théorème taubérien de Littlewood, *Bull. Sci. Math.* (2) **76**, 179–189.

Edwards, D. A. (1957). On absolutely convergent Dirichlet series, *Proc. Amer. Math. Soc*. **8**, 1067–1074.

Eggleston, H. G. (1951). A Tauberian lemma, *Proc. London Math. Soc.* (3) **1**, 28–45.

Freud, G. (1952). Restglied eines Tauberschen Satzes, I, *Acta Math. Acad. Sci. Hungar*. **2**, 299–308.

(1953). Restglied eines Tauberschen Satzes, II, *Acta Math. Acad. Sci. Hungar*. **3**, 299–307.

(1954). Restglied eines Tauberschen Satzes, III, *Acta Math. Acad. Sci. Hungar*. **5**, 275–289.

Ganelius, T. (1971). *Tauberian Remainder Theorems*, Lecture Notes Math. 232. Berlin: Springer-Verlag.

Gel'fand, I. M. (1941). Über absolut konvergente trigonometrische Reihen und Integrale, *Mat. Sb. N. S*. **9**, 51–66.

Goldberg R. R. (1961). *Fourier Transforms*, Cambridge Tract 52. Cambridge: Cambridge University Press.

Goodman, A. & Newman, D. J. (1984). A Wiener type theorem for Dirichlet series, *Proc. Amer. Math. Soc*. **92**, 521–527.

Hardy, G. H. (1907). On certain oscillating series, *Quart. J. Math*. **38**, 269–288; *Collected Papers*, Vol. 6. Oxford: Clarendon Press, 1974, pp. 146–167.

(1910). Theorems relating to the summability and convergence of slowly oscillating series, *Proc. London Math. Soc*. (2) **8**, 301–320; *Collected Papers*, Vol. 6. Oxford: Clarendon Press, 1974, pp. 291–310.

(1949). *Divergent Series*, Oxford: Oxford University Press.

Hardy, G. H. & Littlewood, J. E. (1913). Contributions to the arithmetic theory of series, *Proc. London Math. Soc*. (2) **11**, 411–478; *Collected Papers*, Vol. 6. Oxford: Clarendon Press, 1974, pp. 428–495.

(1914a). Tauberian theorems concerning power series and Dirichlet series whose coefficients are positive, *Proc. London Math. Soc*. (2) **13**, 174–191; *Collected Papers*, Vol. 6. Oxford: Clarendon Press, 1974, pp. 510–527.

(1914b). Some theorems concerning Dirichlet's series, *Messenger Math.* **43**, 134–147; *Collected Papers*, Vol. 6. Oxford: Clarendon Press, 1974, pp. 542–555.

(1926). A further note on the converse of Abel's theorem, *Proc. London Math. Soc*. (2) **25**, 219–236; *Collected Papers*, Vol. 6. Oxford: Clarendon Press, 1974, pp. 699–716.

(1930). Notes on the theory of series XI: On Tauberian theorems, *Proc. London Math. Soc*. (2) **30**, 23–37; *Collected Papers*, Vol. 6. Oxford: Clarendon Press, 1974, pp. 745–759.

Hardy, G. H. & Riesz, M. (1915). *The General Theory of Dirichlet's Series*, Cambridge Tract No. 18. Cambridge: Cambridge University Press. Reprint: Stechert–Hafner (1964).

Hewitt, E. & Williamson, H. (1957). Note on absolutely convergent Dirichlet series, *Proc. Amer. Math. Soc*. **8**, 863–868.

Ingham, A. E. (1962). *On absolutely convergent Dirichlet series*. Studies in Mathematical Analysis and Related Topics. Stanford: Stanford University Press, pp. 156–164.

(1965). On tauberian theorems, *Proc. London Math. Soc*. (3) **14A**, 157–173.

Karamata, J. (1930). Über die Hardy–Littlewoodschen Umkehrungen des Abelschen Stetigkeitssatzes, *Math. Z.* **32**, 319–320.

(1931a). Neuer Beweis und Verallgemeinerung einiger Tauberian-Sätze, *Math. Z.* **33**, 294–300.

(1931b). Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stieltjessche Transformation betreffen, *J. Reine Angew. Math.* **164**, 27–40.

Kojima, T. (1917). On generalized Toeplitz's theorems on limit and their application, *Tôhoku Math. J*. **12**, 291–326.

Korevaar, J. (1951). An estimate of the error in Tauberian theorems for power series, *Duke Math. J*. **18**, 723–734.

(1953). Best L_1 approximation and the remainder in Littlewood's theorem, *Proc. Nederl. Akad. Wetensch. Ser. A* **56** (= Indagationes Math. **15**), 281–293.

(1954a). A very general form of Littlewood's theorem, *Proc. Nederl. Akad. Wetensch. Ser. A* **57** (= Indagationes Math. **16**), 36–45.

(1954b). Another numerical Tauberian theorem for power series, *Proc. Nederl. Akad. Wetensch. Ser. A* **57** (= Indagationes Math. **16**), 46–56.

(1954c). Numerical Tauberian theorems for Dirichlet and Lambert series, *Proc. Nederl. Akad. Wetensch. Ser*. A **57** (= Indagationes Math. **16**), 152–160.

(1954d). Numerical Tauberian theorems for power series and Dirichlet series, I, II, *Proc. Nederl. Akad. Wetensch. Ser*. A **57** (= Indagationes Math. **16**), 432–443, 444–455.

(2001). Tauberian theory, approximation, and lacunary series of powers, *Trends in approximation theory* (Nashville, 2000), Innov. Appl. Math. Nashville: Vanderbilt University Press, pp. 169–189.

(2002). A century of complex Tauberian theory, *Bull. Amer. Math. Soc.* (N.S.) **39**, 475–531.

(2004). Tauberian Theory. A Century of Developments. *Grundl. Math. Wiss*. 329. Berlin: Springer-Verlag.

Landau, E. (1907). Über die Multiplikation Dirichletscher Reihen, *Rend. Circ. Mat. Palermo* **24**, 81–160.

(1908). Zwei neue Herleitungen für die asymptotische Anzahl der Primzahlen unter einer gegebenen Grenze, *Sitzungsberichte Akad. Wiss*. Berlin 746–764; *Collected Works*, Vol.4. Essen: Thales Verlag, 1986, pp. 21–39.

(1909). *Handbuch der Lehre von der Verteilung der Primzahlen*, Leipzig: Teubner. Reprint: Chelsea (New York), 1953.

(1910). Über die Bedeutung einiger neuerer Grenzwertsätze der Herren Hardy und Axer, *Prace mat.-fiz.* (Warsaw) **21**, 97–177; *Collected Works*, Vol. 4. Essen: Thales Verlag, 1986, pp. 267–347.

(1913). Einige Ungleichungen für zweimal differentiierbare Funktionen, *Proc. London Math. Soc*. (2) **13**, 43–49; *Collected Works*, Vol. 6. Essen: Thales Verlag, 1986, pp. 49–55.

Lévy, P. (1934). Sur la convergence absolue des séries de Fourier, *Compositio Math*. **1**, 1–14.

Littlewood, J. E. (1911). The converse of Abel's theorem on power series, *Proc. London Math. Soc*. (2) **9**, 434–448; *Collected Papers*, Vol. 1. Oxford: Oxford University Press, 1982, pp. 757–773.

(1986). *Littlewood's Miscellany*, Bollobas, B. Ed., Cambridge: Cambridge University Press.

van de Lune, J. (1986). *An Introduction to Tauberian Theory: From Tauber to Wiener*. CWI Syllabus 12. Amsterdam: Mathematisch Centrum.

Mellin, H. (1902). Über den Zusammenhang zwischen den linearen Differential- und Differenzengleichungen, *Acta Math*. **25**, 139–164.

Montgomery, H. L. & Schinzel, A. (1977). Some arithmetic properties of polynomials in several variables. *Transcendence Theory: Advances and Applications* (Cambridge, 1976). London: Academic Press, pp. 195–203.

Newman, D. J. (1975). A simple proof of Wiener's $1/f$ theorem, *Proc. Amer. Math. Soc*. **48**, 264–265.

Perron, O. (1908). Zur Theorie der Dirichletschen Reihen, *J. Reine Angew. Math*. **134**, 95–143.

Peyerimhoff, A. (1969). *Lectures on summability*, Lecture Notes Math. 107. Berlin: Springer-Verlag.

Pitt, H. R. (1958). *Tauberian Theorems*. Tata Monographs. London: Oxford University Press.

Postnikov, A. G. (1951). The remainder term in the Tauberian theorem of Hardy and Littlewood, *Dokl. Akad. Nauk SSSR N. S*. **77**, 193–196.

Riesz, M. (1909). Sur la sommation des séries de Dirichlet, *C. R. Acad. Sci.* Paris **149**, 18–21.

Rubel, L. (1989). Summability theory: a neglected tool of analysis, *Amer. Math. Monthly* **96**, 421–423.

Schoenberg, I. J. (1973). The elementary cases of Landau's problem of inequalities between derivatives, *Amer. Math. Monthly* **80**, 121–158.

Schur, I. (1921). Über lineare Transformationen in der Theorie der unendlichen Reihen, *J. Reine Angew. Math.* **151**, 79–111.

Steinhaus, H. (1911). *Kilka słów o uogólnieniu pojęcia granicy*, Warsaw: Prace mat-fiz **22**, 121–134.

Tauber, A. (1897). Ein Satz aus der Theorie der unendlichen Reihen, *Monat. Math*. **8**, 273–277.

Titchmarsh, E. C. (1939). *The Theory of Functions*, Second Edition. Oxford: Oxford University Press.

(1986). *The Theory of the Riemann Zeta-function*, Second Edition. Oxford: Oxford University Press.

Toeplitz, O. (1911). *Über algemeine lineare Mittelbildungen*, Warsaw: Prace mat–fiz **22**, 113–119.

Widder, D. V. (1946). *The Laplace transform*, Princeton: Princeton University Press.

(1971). *An Introduction to Transform Theory*. New York: Academic Press.

Wielandt, H. (1952). Zur Umkehrung des Abelschen Stetigkeitssatzes, *Math Z.* **56**, 206–207.

Wiener, N. (1932). Tauberian theorems, *Ann. of Math*. (2) **33**, 1–100.

(1933). *The Fourier Integral, and Certain of its Applications*. Cambridge: Cambridge University Press.

Wintner, A. (1943). *Eratosthenian averages*. Baltimore: Waverly Press.

Zygmund, A. (1968). *Trigonometric series*, Vol. 1, Second Edition. Cambridge: Cambridge University Press.