# 1

# Dirichlet series: I

## 1.1 Generating functions and asymptotics

The general rationale of analytic number theory is to derive statistical information about a sequence $\{a_n\}$ from the analytic behaviour of an appropriate generating function, such as a power series $\sum a_n z^n$ or a Dirichlet series $\sum a_n n^{-s}$. The type of generating function employed depends on the problem being investigated. There are no rigid rules governing the kind of generating function that is appropriate – the success of a method justifies its use – but we usually deal with additive questions by means of power series or trigonometric sums, and with multiplicative questions by Dirichlet series. For example, if

$$f(z) = \sum_{n=1}^{\infty} z^{n^k}$$

for $|z| < 1$, then the $n^{\text{th}}$ power series coefficient of $f(z)^s$ is the number $r_{k,s}(n)$ of representations of $n$ as a sum of $s$ positive $k^{\text{th}}$ powers,

$$n = m_1^k + m_2^k + \cdots + m_s^k.$$

We can recover $r_{k,s}(n)$ from $f(z)^s$ by means of Cauchy's coefficient formula:

$$r_{k,s}(n) = \frac{1}{2\pi i} \oint \frac{f(z)^s}{z^{n+1}}\, dz.$$

By choosing an appropriate contour, and estimating the integrand, we can determine the asymptotic size of $r_{k,s}(n)$ as $n \to \infty$, provided that $s$ is sufficiently large, say $s > s_0(k)$. This is the germ of the Hardy–Littlewood circle method, but considerable effort is required to construct the required estimates.

To appreciate why power series are useful in dealing with additive problems, note that if $A(z) = \sum a_k z^k$ and $B(z) = \sum b_m z^m$ then the power series




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coefficients of $C(z) = A(z)B(z)$ are given by the formula

$$c_n = \sum_{k+m=n} a_k b_m. \tag{1.1}$$

The terms are grouped according to the sum of the indices, because $z^k z^m = z^{k+m}$.

A *Dirichlet series* is a series of the form $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ where $s$ is a complex variable. If $\beta(s) = \sum_{m=1}^{\infty} b_m m^{-s}$ is a second Dirichlet series and $\gamma(s) = \alpha(s)\beta(s)$, then (ignoring questions relating to the rearrangement of terms of infinite series)

$$\gamma(s) = \sum_{k=1}^{\infty} a_k k^{-s} \sum_{m=1}^{\infty} b_m m^{-s} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_k b_m (km)^{-s} = \sum_{n=1}^{\infty}\Bigg(\sum_{km=n} a_k b_m\Bigg) n^{-s}. \tag{1.2}$$

That is, we expect that $\gamma(s)$ is a Dirichlet series, $\gamma(s) = \sum_{n=1}^{\infty} c_n n^{-s}$, whose coefficients are

$$c_n = \sum_{km=n} a_k b_m. \tag{1.3}$$

This corresponds to (1.1), but the terms are now grouped according to the product of the indices, since $k^{-s} m^{-s} = (km)^{-s}$.

Since we shall employ the complex variable $s$ extensively, it is useful to have names for its real and complex parts. In this regard we follow the rather peculiar notation that has become traditional: $s = \sigma + it$.

Among the Dirichlet series we shall consider is the *Riemann zeta function*, which for $\sigma > 1$ is defined by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1.4}$$

As a first application of (1.3), we note that if $\alpha(s) = \beta(s) = \zeta(s)$ then the manipulations in (1.3) are justified by absolute convergence, and hence we see that

$$\sum_{n=1}^{\infty} d(n) n^{-s} = \zeta(s)^2 \tag{1.5}$$

for $\sigma > 1$. Here $d(n)$ is the *divisor function*, $d(n) = \sum_{d|n} 1$.

From the rate of growth or analytic behaviour of generating functions we glean information concerning the sequence of coefficients. In expressing our findings we employ a special system of notation. For example, we say, '$f(x)$ is asymptotic to $g(x)$' as $x$ tends to some limiting value (say $x \to \infty$), and write


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$f(x) \sim g(x)\ \ (x \to \infty)$, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

An instance of this arises in the formulation of the Prime Number Theorem (PNT), which concerns the asymptotic size of the number $\pi(x)$ of prime numbers not exceeding $x$; $\pi(x) = \sum_{p\le x} 1$. Conjectured by Legendre in 1798, and finally proved in 1896 independently by Hadamard and de la Vallée Poussin, the Prime Number Theorem asserts that

$$\pi(x) \sim \frac{x}{\log x}.$$

Alternatively, we could say that

$$\pi(x) = (1 + o(1))\frac{x}{\log x},$$

which is to say that $\pi(x)$ is $x/\log x$ plus an error term that is in the limit negligible compared with $x/\log x$. More generally, we say, '$f(x)$ is small oh of $g(x)$', and write $f(x) = o(g(x))$, if $f(x)/g(x) \to 0$ as $x$ tends to its limit.

The Prime Number Theorem can be put in a quantitative form,

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right). \tag{1.6}$$

Here the last term denotes an implicitly defined function (the difference between the other members of the equation); the assertion is that this function has absolute value not exceeding $Cx(\log x)^{-2}$. That is, the above is equivalent to asserting that there is a constant $C > 0$ such that the inequality

$$\left|\pi(x) - \frac{x}{\log x}\right| \le \frac{Cx}{(\log x)^2}$$

holds for all $x \ge 2$. In general, we say that $f(x)$ is 'big oh of $g(x)$', and write $f(x) = O(g(x))$ if there is a constant $C > 0$ such that $|f(x)| \le Cg(x)$ for all $x$ in the appropriate domain. The function $f$ may be complex-valued, but $g$ is necessarily non-negative. The constant $C$ is called the *implicit constant*; it is an absolute constant unless the contrary is indicated. For example, if $C$ is liable to depend on a parameter $\alpha$, we might say, 'For any fixed value of $\alpha$, $f(x) = O(g(x))$'. Alternatively, we might say, '$f(x) = O(g(x))$ where the implicit constant may depend on $\alpha$', or more briefly, $f(x) = O_\alpha(g(x))$.

When there is no main term, instead of writing $f(x) = O(g(x))$ we save a pair of parentheses by writing instead $f(x) \ll g(x)$. This is read, '$f(x)$ is less-than-less-than $g(x)$', and we write $f(x) \ll_\alpha g(x)$ if the implicit constant may depend on $\alpha$. To provide an example of this notation, we recall that Chebyshev

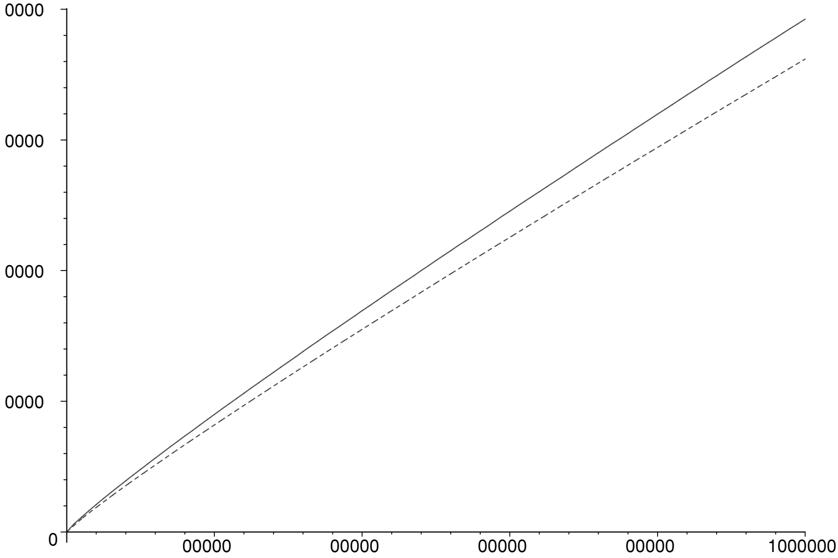


**Figure 1.1** Graph of $\pi(x)$ (solid) and $x/\log x$ (dotted) for $2 \le x \le 10^6$.

proved that $\pi(x) \ll x/\log x$. This is of course weaker than the Prime Number Theorem, but it was derived much earlier, in 1852. Chebyshev also showed that $\pi(x) \gg x/\log x$. In general, we say that $f(x) \gg g(x)$ if there is a positive constant $c$ such that $f(x) \ge cg(x)$ and $g$ is non-negative. In this situation both $f$ and $g$ take only positive values. If both $f \ll g$ and $f \gg g$ then we say that $f$ and $g$ have the same order of magnitude, and write $f \asymp g$. Thus Chebyshev's estimates can be expressed as a single relation,

$$\pi(x) \asymp \frac{x}{\log x}.$$

The estimate (1.6) is best possible to the extent that the error term is not $o(x(\log x)^{-2})$. We have also a special notation to express this:

$$\pi(x) - \frac{x}{\log x} = \Omega\left(\frac{x}{(\log x)^2}\right).$$

In general, if $\limsup_{x\to\infty} |f(x)|/g(x) > 0$ then we say that $f(x)$ is 'Omega of $g(x)$', and write $f(x) = \Omega(g(x))$. This is precisely the negation of the statement '$f(x) = o(g(x))$'. When studying numerical values, as in Figure 1.1, we find that the fit of $x/\log x$ to $\pi(x)$ is not very compelling. This is because the error term in the approximation is only one logarithm smaller than the main term. This error term is not oscillatory – rather there is a second main term of this size:

$$\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right).$$

This is also best possible, but the main term can be made still more elaborate to give a smaller error term. Gauss was the first to propose a better approximation to $\pi(x)$. Numerical studies led him to observe that the density of prime numbers in the neighbourhood of $x$ is approximately $1/\log x$. This suggests that the number of primes not exceeding $x$ might be approximately equal to the *logarithmic integral*,

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log u}\,du.$$

(Orally, 'li' rhymes with 'pi'.) By repeated integration by parts we can show that

$$\mathrm{li}(x) = x \sum_{k=1}^{K-1} \frac{(k-1)!}{(\log x)^k} + O_K\left(\frac{x}{(\log x)^K}\right)$$

for any positive integer $K$; thus the secondary main terms of the approximation to $\pi(x)$ are contained in li(x).

In Chapter 6 we shall prove the Prime Number Theorem in the sharper quantitative form

$$\pi(x) = \mathrm{li}(x) + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right)$$

for some suitable positive constant $c$. Note that $\exp(c\sqrt{\log x})$ tends to infinity faster than any power of $\log x$. The error term above seems to fall far from what seems to be the truth. Numerical evidence, such as that in Table 1.1, suggests that the error term in the Prime Number Theorem is closer to $\sqrt{x}$ in size. Gauss noted the good fit, and also that $\pi(x) < \mathrm{li}(x)$ for all $x$ in the range of his extensive computations. He proposed that this might continue indefinitely, but the numerical evidence is misleading, for in 1914 Littlewood showed that

$$\pi(x) - \mathrm{li}(x) = \Omega_{\pm}\left(\frac{x^{1/2}\log\log\log x}{\log x}\right).$$

Here the subscript $\pm$ indicates that the error term achieves the stated order of magnitude infinitely often, and in both signs. In particular, the difference $\pi - \mathrm{li}$ has infinitely many sign changes. More generally, we write $f(x) = \Omega_+(g(x))$ if $\limsup_{x\to\infty} f(x)/g(x) > 0$, we write $f(x) = \Omega_-(g(x))$ if $\liminf_{x\to\infty} f(x)/g(x) < 0$, and we write $f(x) = \Omega_{\pm}(g(x))$ if both these relations hold.

Table 1.1 Values of $\pi(x)$, li(x), $x/\log x$ *for* $x = 10^k$, $1 \le k \le 22$.

| $x$ | $\pi(x)$ | $\mathrm{li}(x)$ | $x/\log x$ |
|---|---|---|---|
| 10 | 4 | 5.12 | 4.34 |
| $10^2$ | 25 | 29.08 | 21.71 |
| $10^3$ | 168 | 176.56 | 144.76 |
| $10^4$ | 1229 | 1245.09 | 1085.74 |
| $10^5$ | 9592 | 9628.76 | 8685.89 |
| $10^6$ | 78498 | 78626.50 | 72382.41 |
| $10^7$ | 664579 | 664917.36 | 620420.69 |
| $10^8$ | 5761455 | 5762208.33 | 5428681.02 |
| $10^9$ | 50847534 | 50849233.90 | 48254942.43 |
| $10^{10}$ | 455052511 | 455055613.54 | 434294481.90 |
| $10^{11}$ | 4118054813 | 4118066399.58 | 3948131653.67 |
| $10^{12}$ | 37607912018 | 37607950279.76 | 36191206825.27 |
| $10^{13}$ | 346065536839 | 346065458090.05 | 334072678387.12 |
| $10^{14}$ | 3204941750802 | 3204942065690.91 | 3102103442166.08 |
| $10^{15}$ | 29844570422669 | 29844571475286.54 | 28952965460216.79 |
| $10^{16}$ | 279238341033925 | 279238344248555.75 | 271434051189532.39 |
| $10^{17}$ | 2623557157654233 | 2623557165610820.07 | 2554673422960304.87 |
| $10^{18}$ | 24739954287740860 | 24739954309690413.98 | 24127471216847323.76 |
| $10^{19}$ | 234057667276344607 | 234057667376222382.22 | 228576043106974646.13 |
| $10^{20}$ | 2220819602560918840 | 2220819602783663483.55 | 2171472409516259138.26 |
| $10^{21}$ | 21127269486018731928 | 21127269486616126182.33 | 20680689614440563221.48 |
| $10^{22}$ | 201467286689315906290 | 201467286691248261498.15 | 197406582683296285295.97 |

In the exercises below we give several examples of the use of generating functions, mostly power series, to establish relations between various counting functions.

### 1.1.1 Exercises

1. Let $r(n)$ be the number of ways that $n$ cents of postage can be made, using only 1 cent, 2 cent, and 3 cent stamps. That is, $r(n)$ is the number of ordered triples $(x_1, x_2, x_3)$ of non-negative integers such that $x_1 + 2x_2 + 3x_3 = n$.
   (a) Show that
$$\sum_{n=0}^{\infty} r(n)z^n = \frac{1}{(1-z)(1-z^2)(1-z^3)}$$
   for $|z| < 1$.
   (b) Determine the partial fraction expansion of the rational function above.

That is, find constants $a, b, \ldots, f$ so that the above is

$$\frac{a}{(z-1)^3}+\frac{b}{(z-1)^2}+\frac{c}{z-1}+\frac{d}{z+1}+\frac{e}{z-\omega}+\frac{f}{z-\overline{\omega}}$$

where $\omega = e^{2\pi i/3}$ and $\overline{\omega} = e^{-2\pi i/3}$ are the primitive cube roots of unity.

(c) Show that $r(n)$ is the integer nearest $(n+3)^2/12$.

(d) Show that $r(n)$ is the number of ways of writing $n = y_1 + y_2 + y_3$ with $y_1 \ge y_2 \ge y_3 \ge 0$.

2. Explain why

$$\prod_{k=0}^{\infty}\left(1+z^{2^k}\right) = 1 + z + z^2 + \cdots$$

for $|z| < 1$.

3. (L. Mirsky & D. J. Newman) Suppose that $0 \le a_k < m_k$ for $1 \le k \le K$, and that $m_1 < m_2 < \cdots < m_K$. This is called a *family of covering congruences* if every integer $x$ satisfies at least one of the congruences $x \equiv a_k \pmod{m_k}$. A system of covering congruences is called *exact* if for every value of $x$ there is exactly one value of $k$ such that $x \equiv a_k \pmod{m_k}$. Show that if the system is exact then

$$\sum_{k=1}^{K}\frac{z^{a_k}}{1-z^{m_k}} = \frac{1}{1-z}$$

for $|z| < 1$. Show that the left-hand side above is

$$\sim \frac{e^{2\pi i a_K/m_K}}{m_K(1-r)}$$

when $z = re^{2\pi i/m_K}$ and $r \to 1^-$. On the other hand, the right-hand side is bounded for $z$ in a neighbourhood of $e^{2\pi i/m_K}$ if $m_K > 1$. Deduce that a family of covering congruences is not exact if $m_k > 1$.

4. Let $p(n;k)$ denote the number of partitions of $n$ into at most $k$ parts, that is, the number of ordered $k$-tuples $(x_1, x_2, \ldots, x_k)$ of non-negative integers such that $n = x_1 + x_2 + \cdots + x_k$ and $x_1 \ge x_2 \ge \cdots \ge x_k$. Let $p(n) = p(n;n)$ denote the total number of partitions of $n$. Also let $p_o(n)$ be the number of partitions of $n$ into an odd number of parts, $p_o(n) = \sum_{2\nmid k} p(n;k)$. Finally, let $p_d(n)$ denote the number of partitions of $n$ into distinct parts, so that $x_1 > x_2 > \cdots > x_k$. By convention, put $p(0) = p_o(0) = p_d(0) = 1$.

(a) Show that there are precisely $p(n;k)$ partitions of $n$ into parts not exceeding $k$.

(b) Show that

$$\sum_{n=0}^{\infty} p(n;k)z^n = \prod_{j=1}^{k}(1-z^j)^{-1}$$

for $|z| < 1$.

(c) Show that

$$\sum_{n=0}^{\infty} p(n)z^n = \prod_{k=1}^{\infty}(1-z^k)^{-1}$$

for $|z| < 1$.

(d) Show that

$$\sum_{n=0}^{\infty} p_{\mathrm{d}}(n)z^n = \prod_{k=1}^{\infty}(1+z^k)$$

for $|z| < 1$.

(e) Show that

$$\sum_{n=0}^{\infty} p_{\mathrm{o}}(n)z^n = \prod_{k=1}^{\infty}(1-z^{2k-1})^{-1}$$

for $|z| < 1$.

(f) By using the result of Exercise 2, or otherwise, show that the last two generating functions above are identically equal. Deduce that $p_{\mathrm{o}}(n) = p_{\mathrm{d}}(n)$ for all $n$.

5. Let $A(n)$ denote the number of ways of associating a product of $n$ terms; thus $A(1) = A(2) = 1$ and $A(3) = 2$. By convention, $A(0) = 0$.

(a) By considering the possible positionings of the outermost parentheses, show that

$$A(n) = \sum_{k=1}^{n-1} A(k)A(n-k)$$

for all $n \geq 2$.

(b) Let $P(z) = \sum_{n=0}^{\infty} A(n)z^n$. Show that

$$P(z)^2 = P(z) - z.$$

Deduce that

$$P(z) = \frac{1-\sqrt{1-4z}}{2} = \sum_{n=1}^{\infty}\binom{1/2}{n}2^{2n-1}(-1)^{n-1}z^n.$$

(c) Conclude that $A(n) = \binom{2n-2}{n-1}/n$ for all $n \geq 1$. These are called the *Catalan numbers*.

(d) What needs to be said concerning the convergence of the series used above?

6. (a) Let $n_k$ denote the total number of monic polynomials of degree $k$ in $\mathbb{F}_p[x]$. Show that $n_k = p^k$.

(b) Let $P_1, P_2, \ldots$ be the irreducible monic polynomials in $\mathbb{F}_p[x]$, listed in some (arbitrary) order. Show that

$$\prod_{r=1}^{\infty}(1 + z^{\deg P_r} + z^{2\deg P_r} + z^{3\deg P_r} + \cdots) = 1 + pz + p^2z^2 + p^3z^3 + \cdots$$

for $|z| < 1/p$.

(c) Let $g_k$ denote the number of irreducible monic polynomials of degree $k$ in $\mathbb{F}_p[x]$. Show that

$$\prod_{k=1}^{\infty}(1 - z^k)^{-g_k} = (1 - pz)^{-1} \qquad (|z| < 1/p).$$

(d) Take logarithmic derivatives to show that

$$\sum_{k=1}^{\infty} kg_k \frac{z^{k-1}}{1 - z^k} = \frac{p}{1 - pz} \qquad (|z| < 1/p).$$

(e) Show that

$$\sum_{k=1}^{\infty} kg_k \sum_{m=1}^{\infty} z^{mk} = \sum_{n=1}^{\infty} p^n z^n \qquad (|z| < 1/p).$$

(f) Deduce that

$$\sum_{k|n} kg_k = p^n$$

for all positive integers $n$.

(g) (Gauss) Use the Möbius inversion formula to show that

$$g_n = \frac{1}{n}\sum_{k|n} \mu(k)p^{n/k}$$

for all positive integers $n$.

(h) Use (f) (not (g)) to show that

$$\frac{p^n}{n} - \frac{2p^{n/2}}{n} \leq g_n \leq \frac{p^n}{n}.$$

(i) If a monic polynomial of degree $n$ is chosen at random from $\mathbb{F}_p[x]$, about how likely is it that it is irreducible? (Assume that $p$ and/or $n$ is large.)

(j) Show that $g_n > 0$ for all $p$ and all $n \ge 1$. (If $P \in \mathbb{F}_p[x]$ is irreducible and has degree $n$, then the quotient ring $\mathbb{F}_p[x]/(P)$ is a field of $p^n$ elements. Thus we have proved that there is such a field, for each prime $p$ and integer $n \ge 1$. It may be further shown that the order of a finite field is necessarily a prime power, and that any two finite fields of the same order are isomorphic. Hence the field of order $p^n$, whose existence we have proved, is essentially unique.)

7. (E. Berlekamp) Let $p$ be a prime number. We recall that polynomials in a single variable (mod $p$) factor uniquely into irreducible polynomials. Thus a monic polynomial $f(x)$ can be expressed uniquely (mod $p$) in the form $g(x)h(x)^2$ where $g(x)$ is square-free (mod $p$) and both $g$ and $h$ are monic. Let $s_n$ denote the number of monic square-free polynomials (mod $p$) of degree $n$. Show that

$$\Big(\sum_{k=0}^{\infty} s_k z^k\Big)\Big(\sum_{m=0}^{\infty} p^m z^{2m}\Big) = \sum_{n=0}^{\infty} p^n z^n$$

for $|z| < 1/p$. Deduce that

$$\sum_{k=0}^{\infty} s_k z^k = \frac{1 - pz^2}{1 - pz},$$

and hence that $s_0 = 1$, $s_1 = p$, and that $s_k = p^k(1 - 1/p)$ for all $k \ge 2$.

8. (cf Wagon 1987) (a) Let $\mathcal{I} = [a, b]$ be an interval. Show that $\int_{\mathcal{I}} e^{2\pi i x}\, dx = 0$ if and only if the length $b - a$ of $\mathcal{I}$ is an integer.

(b) Let $\mathcal{R} = [a, b] \times [c, d]$ be a rectangle. Show that $\iint_{\mathcal{R}} e^{2\pi i(x+y)}\, dx\, dy = 0$ if and only if at least one of the edge lengths of $\mathcal{R}$ is an integer.

(c) Let $\mathcal{R}$ be a rectangle that is a union of finitely many rectangles $\mathcal{R}_i$; the $\mathcal{R}_i$ are disjoint apart from their boundaries. Show that if all the $\mathcal{R}_i$ have the property that at least one of their side lengths is an integer, then $\mathcal{R}$ also has this property.

9. (L. Moser) If $\mathcal{A}$ is a set of non-negative integers, let $r_{\mathcal{A}}(n)$ denote the number of representations of $n$ as a sum of two distinct members of $\mathcal{A}$. That is, $r_{\mathcal{A}}(n)$ is the number of ordered pairs $(a_1, a_2)$ for which $a_1 \in \mathcal{A}$, $a_2 \in \mathcal{A}$, $a_1 + a_2 = n$, and $a_1 \ne a_2$. Let $A(z) = \sum_{a\in\mathcal{A}} z^a$.

(a) Show that $\sum_n r_{\mathcal{A}}(n) z^n = A(z)^2 - A(z^2)$ for $|z| < 1$.

(b) Suppose that the non-negative integers are partitioned into two sets $\mathcal{A}$ and $\mathcal{B}$ in such a way that $r_{\mathcal{A}}(n) = r_{\mathcal{B}}(n)$ for all non-negative integers $n$. Without loss of generality, $0 \in \mathcal{A}$. Show that $1 \in \mathcal{B}$, that $2 \in \mathcal{B}$, and that $3 \in \mathcal{A}$.

(c) With $\mathcal{A}$ and $\mathcal{B}$ as above, show that $A(z) + B(z) = 1/(1 - z)$ for $|z| < 1$.

(d) Show that $A(z) - B(z) = (1 - z)\big(A(z^2) - B(z^2)\big)$, and hence by induction that

$$A(z) - B(z) = \prod_{k=0}^{\infty} \left(1 - z^{2^k}\right)$$

for $|z| < 1$.

(e) Let the *binary weight* of $n$, denoted $w(n)$, be the number of 1's in the binary expansion of $n$. That is, if $n = 2^{k_1} + \cdots + 2^{k_r}$ with $k_1 > \cdots > k_r$, then $w(n) = r$. Show that $\mathcal{A}$ consists of those non-negative integers $n$ for which $w(n)$ is even, and that $\mathcal{B}$ is the set of those integers for which $w(n)$ is odd.

## 1.2 Analytic properties of Dirichlet series

Having provided some motivation for the use of Dirichlet series, we now turn to the task of establishing some of their basic analytic properties, corresponding to well-known facts concerning power series.

**Theorem 1.1** *Suppose that the Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges at the point $s = s_0$, and that $H > 0$ is an arbitrary constant. Then the series $\alpha(s)$ is uniformly convergent in the sector $\mathcal{S} = \{s : \sigma \ge \sigma_0, |t - t_0| \le H(\sigma - \sigma_0)\}$.*

By taking $H$ large, we see that the series $\alpha(s)$ converges for all $s$ in the half-plane $\sigma > \sigma_0$, and hence that the domain of convergence is a half-plane. More precisely, we have

**Corollary 1.2** *Any Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has an* abscissa of convergence *$\sigma_c$ with the property that $\alpha(s)$ converges for all $s$ with $\sigma > \sigma_c$, and for no $s$ with $\sigma < \sigma_c$. Moreover, if $s_0$ is a point with $\sigma_0 > \sigma_c$, then there is a neighbourhood of $s_0$ in which $\alpha(s)$ converges uniformly.*

In extreme cases a Dirichlet series may converge throughout the plane ($\sigma_c = -\infty$), or nowhere ($\sigma_c = +\infty$). When the abscissa of convergence is finite, the series may converge everywhere on the line $\sigma_c + it$, it may converge at some but not all points on this line, or nowhere on the line.

*Proof of Theorem 1.1* Let $R(u) = \sum_{n>u} a_n n^{-s_0}$ be the remainder term of the series $\alpha(s_0)$. First we show that for any $s$,

$$\sum_{n=M+1}^{N} a_n n^{-s} = R(M)M^{s_0-s} - R(N)N^{s_0-s} + (s_0 - s)\int_M^N R(u)u^{s_0-s-1}\,du. \tag{1.7}$$


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To see this we note that $a_n = (R(n-1) - R(n))\, n^{s_0}$, so that by partial summation

$$\sum_{n=M+1}^{N} a_n n^{-s} = \sum_{n=M+1}^{N} (R(n-1) - R(n)) n^{s_0-s}$$
$$= R(M)M^{s_0-s} - R(N)N^{s_0-s} - \sum_{n=M+1}^{N} R(n-1)((n-1)^{s_0-s} - n^{s_0-s}).$$

The second factor in this last sum can be expressed as an integral,

$$(n-1)^{s_0-s} - n^{s_0-s} = -(s_0 - s)\int_{n-1}^{n} u^{s_0-s-1}\, du,$$

and hence the sum is

$$(s - s_0) \sum_{n=M+1}^{N} R(n-1) \int_{n-1}^{n} u^{s_0-s-1}\, du = (s - s_0) \sum_{n=M+1}^{N} \int_{n-1}^{n} R(u) u^{s_0-s-1}\, du$$

since $R(u)$ is constant in the interval $[n-1, n)$. The integrals combine to give (1.7).

If $|R(u)| \le \varepsilon$ for all $u \ge M$ and if $\sigma > \sigma_0$, then from (1.7) we see that

$$\left| \sum_{n=M+1}^{N} a_n n^{-s} \right| \le 2\varepsilon + \varepsilon |s - s_0| \int_{M}^{\infty} u^{\sigma_0-\sigma-1}\, du \le \left(2 + \frac{|s - s_0|}{\sigma - \sigma_0}\right)\varepsilon.$$

For $s$ in the prescribed region we see that

$$|s - s_0| \le \sigma - \sigma_0 + |t - t_0| \le (H + 1)(\sigma - \sigma_0),$$

so that the sum $\sum_{M+1}^{N} a_n n^{-s}$ is uniformly small, and the result follows by the uniform version of Cauchy's principle. □

In deriving (1.7) we used partial summation, although it would have been more efficient to use the properties of the Riemann–Stieltjes integral (see Appendix A):

$$\sum_{n=M+1}^{N} a_n n^{-s} = -\int_{M}^{N} u^{s_0-s}\, dR(u) = -u^{s_0-s} R(u)\Big|_{M}^{N} + \int_{M}^{N} R(u)\, du^{s_0-s}$$

by Theorems A.1 and A.2. By Theorem A.3 this is

$$= M^{s_0-s} R(M) - N^{s_0-s} R(N) + (s_0 - s) \int_{M}^{N} R(u) u^{s_0-s-1}\, du.$$

In more complicated situations it is an advantage to use the Riemann–Stieltjes integral, and subsequently we shall do so without apology.

The series $\alpha(s) = \sum a_n n^{-s}$ is locally uniformly convergent for $\sigma > \sigma_c$, and each term is an analytic function, so it follows from a general principle of

Weierstrass that $\alpha(s)$ is analytic for $\sigma > \sigma_c$, and that the differentiated series is locally uniformly convergent to $\alpha'(s)$:

$$\alpha'(s) = -\sum_{n=1}^{\infty} a_n(\log n)n^{-s} \tag{1.8}$$

for $s$ in the half-plane $\sigma > \sigma_c$.

Suppose that $s_0$ is a point on the line of convergence (i.e., $\sigma_0 = \sigma_c$), and that the series $\alpha(s_0)$ converges. It can be shown by example that

$$\lim_{\substack{s\to s_0\\ \sigma>\sigma_c}} \alpha(s)$$

need not exist. However, $\alpha(s)$ is continuous in the sector $\mathcal{S}$ of Theorem 1.1, in view of the uniform convergence there. That is,

$$\lim_{\substack{s\to s_0\\ s\in\mathcal{S}}} \alpha(s) = \alpha(s_0), \tag{1.9}$$

which is analogous to Abel's theorem for power series.

We now express a convergent Dirichlet series as an absolutely convergent integral.

**Theorem 1.3** *Let $A(x) = \sum_{n\le x} a_n$. If $\sigma_c < 0$, then $A(x)$ is a bounded function, and*

$$\sum_{n=1}^{\infty} a_n n^{-s} = s\int_1^{\infty} A(x)x^{-s-1}\,dx \tag{1.10}$$

*for $\sigma > 0$. If $\sigma_c \ge 0$, then*

$$\limsup_{x\to\infty} \frac{\log|A(x)|}{\log x} = \sigma_c, \tag{1.11}$$

*and* (1.10) *holds for $\sigma > \sigma_c$.*

*Proof* We note that

$$\sum_{n=1}^{N} a_n n^{-s} = \int_{1^-}^{N} x^{-s}\,dA(x) = A(x)x^{-s}\Big|_{1^-}^{N} - \int_{1^-}^{N} A(x)\,dx^{-s}$$
$$= A(N)N^{-s} + s\int_1^N A(x)x^{-s-1}\,dx.$$

Let $\phi$ denote the left-hand side of (1.11). If $\theta > \phi$ then $A(x) \ll x^{\theta}$ where the implicit constant may depend on the $a_n$ and on $\theta$. Thus if $\sigma > \theta$, then the integral in (1.10) is absolutely convergent. Thus we obtain (1.10) by letting $N \to \infty$, since the first term above tends to 0 as $N \to \infty$.

Suppose that $\sigma_c < 0$. By Corollary 1.2 we know that $A(x)$ tends to a finite limit as $x \to \infty$, and hence $\phi \le 0$, so that (1.10) holds for all $\sigma > 0$.

Now suppose that $\sigma_c \geq 0$. By Corollary 1.2 we know that the series in (1.10) diverges when $\sigma < \sigma_c$. Hence $\phi \geq \sigma_c$. To complete the proof it suffices to show that $\phi \leq \sigma_c$. Choose $\sigma_0 > \sigma_c$. By (1.7) with $s = 0$ and $M = 0$ we see that

$$A(N) = -R(N)N^{\sigma_0} + \sigma_0 \int_0^N R(u)u^{\sigma_0 - 1}\,du.$$

Since $R(u)$ is a bounded function, it follows that $A(N) \ll N^{\sigma_0}$ where the implicit constant may depend on the $a_n$ and on $\sigma_0$. Hence $\phi \leq \sigma_0$. Since this holds for any $\sigma_0 > \sigma_c$, we conclude that $\phi \leq \sigma_c$. □

The terms of a power series are majorized by a geometric progression at points strictly inside the circle of convergence. Consequently power series converge very rapidly. In contrast, Dirichlet series are not so well behaved. For example, the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} \tag{1.12}$$

converges for $\sigma > 0$, but it is absolutely convergent only for $\sigma > 1$. In general we let $\sigma_a$ denote the infimum of those $\sigma$ for which $\sum_{n=1}^{\infty}|a_n|n^{-\sigma} < \infty$. Then $\sigma_a$, the *abscissa of absolute convergence*, is the abscissa of convergence of the series $\sum_{n=1}^{\infty}|a_n|n^{-s}$, and we see that $\sum a_n n^{-s}$ is absolutely convergent if $\sigma > \sigma_a$, but not if $\sigma < \sigma_a$. We now show that the strip $\sigma_c \leq \sigma \leq \sigma_a$ of conditional convergence is never wider than in the example (1.12).

**Theorem 1.4** *In the above notation,* $\sigma_c \leq \sigma_a \leq \sigma_c + 1$.

*Proof* The first inequality is obvious. To prove the second, suppose that $\varepsilon > 0$. Since the series $\sum a_n n^{-\sigma_c-\varepsilon}$ is convergent, the summands tend to 0, and hence $a_n \ll n^{\sigma_c+\varepsilon}$ where the implicit constant may depend on the $a_n$ and on $\varepsilon$. Hence the series $\sum a_n n^{-\sigma_c-1-2\varepsilon}$ is absolutely convergent by comparison with the series $\sum n^{-1-\varepsilon}$. □

Clearly a Dirichlet series $\alpha(s)$ is uniformly bounded in the half-plane $\sigma > \sigma_a + \varepsilon$, but this is not generally the case in the strip of conditional convergence. Nevertheless, we can limit the rate of growth of $\alpha(s)$ in this strip.

To aid in formulating our next result we introduce a notational convention that arises because many estimates relating to Dirichlet series are expressed in terms of the size of $|t|$. Our interest is in large values of this quantity, but in order that the statements be valid for small $|t|$ we sometimes write $|t| + 4$. Since this is cumbersome in complicated expressions, we introduce a shorthand: $\tau = |t| + 4$.

**Theorem 1.5** *Suppose that* $\alpha(s) = \sum a_n n^{-s}$ *has abscissa of convergence* $\sigma_c$. *If* $\delta$ *and* $\varepsilon$ *are fixed,* $0 < \varepsilon < \delta < 1$, *then*

$$\alpha(s) \ll \tau^{1-\delta+\varepsilon}$$

*uniformly for* $\sigma \geq \sigma_c + \delta$. *The implicit constant may depend on the coefficients* $a_n$, *on* $\delta$, *and on* $\varepsilon$.

By the example found in Exercise 8 at the end of this section, we see that the bound above is reasonably sharp.

*Proof* Let $s$ be a complex number with $\sigma \geq \sigma_c + \delta$. By (1.7) with $s_0 = \sigma_c + \varepsilon$ and $N \to \infty$, we see that

$$\alpha(s) = \sum_{n=1}^{M} a_n n^{-s} + R(M)M^{\sigma_c+\varepsilon-s} + (\sigma_c + \varepsilon - s)\int_M^\infty R(u)u^{\sigma_c+\varepsilon-s-1}\,du.$$

Since the series $\alpha(\sigma_c + \varepsilon)$ converges, we know that $a_n \ll n^{\sigma_c+\varepsilon}$, and also that $R(u) \ll 1$. Thus the above is

$$\ll \sum_{n=1}^{M} n^{-\delta+\varepsilon} + M^{-\delta+\varepsilon} + \frac{|\sigma_c + \varepsilon - s|}{\sigma - \sigma_c - \varepsilon} M^{\sigma_c+\varepsilon-\sigma}.$$

By the integral test the sum here is

$$< \int_0^M u^{-\delta+\varepsilon}\,du = \frac{M^{1-\delta+\varepsilon}}{1-\delta+\varepsilon} \ll M^{1-\delta+\varepsilon}.$$

Hence on taking $M = [\tau]$ we obtain the stated estimate. □

We know that the power series expansion of a function is unique; we now show that the same is true for Dirichlet series expansions.

**Theorem 1.6** *If* $\sum a_n n^{-s} = \sum b_n n^{-s}$ *for all* $s$ *with* $\sigma > \sigma_0$ *then* $a_n = b_n$ *for all positive integers* $n$.

*Proof* We put $c_n = a_n - b_n$, and consider $\sum c_n n^{-s}$. Suppose that $c_n = 0$ for all $n < N$. Since $\sum c_n n^{-\sigma} = 0$ for $\sigma > \sigma_0$ we may write

$$c_N = -\sum_{n>N} c_n (N/n)^\sigma.$$

By Theorem 1.4 this sum is absolutely convergent for $\sigma > \sigma_0 + 1$. Since each term tends to 0 as $\sigma \to \infty$, we see that the right-hand side tends to 0, by the principle of dominated convergence. Hence $c_N = 0$, and by induction we deduce that this holds for all $N$. □

Suppose that $f$ is analytic in a domain $\mathcal{D}$, and that $0 \in \mathcal{D}$. Then $f$ can be expressed as a power series $\sum_{n=0}^{\infty} a_n z^n$ in the disc $|z| < r$ where $r$ is the distance from 0 to the boundary $\partial\mathcal{D}$ of $\mathcal{D}$. Although Dirichlet series are analytic functions, the situation regarding Dirichlet series expansions is very different: The collection of functions that may be expressed as a Dirichlet series in some half-plane is a very special class. Moreover, the line $\sigma_c + it$ of convergence need not contain a singular point of $\alpha(s)$. For example, the Dirichlet series (1.12) has abscissa of convergence $\sigma_c = 0$, but it represents the entire function $(1 - 2^{1-s})\zeta(s)$. (The connection of (1.12) to the zeta function is easy to establish, since

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} = \sum_{n=1}^{\infty} n^{-s} - 2\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} n^{-s} = \zeta(s) - 2^{1-s}\zeta(s)$$

for $\sigma > 1$. That this is an entire function follows from Theorem 10.2.) Since a Dirichlet series does not in general have a singularity on its line of convergence, it is noteworthy that a Dirichlet series with non-negative coefficients not only has a singularity on the line $\sigma_c + it$, but actually at the point $\sigma_c$.

**Theorem 1.7** (Landau) *Let $\alpha(s) = \sum a_n n^{-s}$ be a Dirichlet series whose abscissa of convergence $\sigma_c$ is finite. If $a_n \ge 0$ for all $n$ then the point $\sigma_c$ is a singularity of the function $\alpha(s)$.*

It is enough to assume that $a_n \ge 0$ for all sufficiently large $n$, since any finite sum $\sum_{n=1}^{N} a_n n^{-s}$ is an entire function.

*Proof* By replacing $a_n$ by $a_n n^{-\sigma_c}$, we may assume that $\sigma_c = 0$. Suppose that $\alpha(s)$ is analytic at $s = 0$, so that $\alpha(s)$ is analytic in the domain $\mathcal{D} = \{s : \sigma > 0\} \cup \{|s| < \delta\}$ if $\delta > 0$ is sufficiently small. We expand $\alpha(s)$ as a power series at $s = 1$:

$$\alpha(s) = \sum_{k=0}^{\infty} c_k (s-1)^k. \tag{1.13}$$

The coefficients $c_k$ can be calculated by means of (1.8),

$$c_k = \frac{\alpha^{(k)}(1)}{k!} = \frac{1}{k!}\sum_{n=1}^{\infty} a_n(-\log n)^k n^{-1}.$$

The radius of convergence of the power series (1.13) is the distance from 1 to the nearest singularity of $\alpha(s)$. Since $\alpha(s)$ is analytic in $\mathcal{D}$, and since the nearest points not in $\mathcal{D}$ are $\pm i\delta$, we deduce that the radius of convergence is at least $\sqrt{1+\delta^2} = 1 + \delta'$, say. That is,

$$\alpha(s) = \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \sum_{n=1}^{\infty} a_n (\log n)^k n^{-1}$$

for $|s-1| < 1+\delta'$. If $s < 1$ then all terms above are non-negative. Since series of non-negative numbers may be arbitrarily rearranged, for $-\delta' < s < 1$ we may interchange the summations over $k$ and $n$ to see that

$$\begin{aligned}\alpha(s) &= \sum_{n=1}^{\infty} a_n n^{-1} \sum_{k=0}^{\infty} \frac{(1-s)^k (\log n)^k}{k!} \\ &= \sum_{n=1}^{\infty} a_n n^{-1} \exp\big((1-s)\log n\big) = \sum_{n=1}^{\infty} a_n n^{-s}.\end{aligned}$$

Hence this last series converges at $s = -\delta'/2$, contrary to the assumption that $\sigma_c = 0$. Thus $\alpha(s)$ is not analytic at $s = 0$. □

### 1.2.1 Exercises

1. Suppose that $\alpha(s)$ is a Dirichlet series, and that the series $\alpha(s_0)$ is boundedly oscillating. Show that $\sigma_c = \sigma_0$.
2. Suppose that $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is a Dirichlet series with abscissa of convergence $\sigma_c$. Suppose that $\alpha(0)$ converges, and put $R(x) = \sum_{n>x} a_n$. Show that $\sigma_c$ is the infimum of those numbers $\theta$ such that $R(x) \ll x^{\theta}$.
3. Let $A_k(x) = \sum_{n\le x} a_n (\log n)^k$.
   (a) Show that
   $$A_0(x) - \frac{A_1(x)}{\log x} = a_1 + \int_2^x \frac{A_1(u)}{u(\log u)^2}\,du.$$
   (b) Suppose that $A_1(x) \ll x^{\theta}$ where $\theta > 0$ and the implicit constant may depend on the sequence $\{a_n\}$. Show that
   $$A_0(x) = \frac{A_1(x)}{\log x} + O(x^{\theta}(\log x)^{-2}).$$
   (c) Let $\sigma_c$ denote the abscissa of convergence of $\sum a_n n^{-s}$, and $\sigma_c'$ the abscissa of convergence of $\sum a_n (\log n) n^{-s}$. Show that $\sigma_c' = \sigma_c$. (The remarks following the proof of Theorem 1.1 imply only that $\sigma_c' \le \sigma_c$.)
4. (Landau 1909b) Let $\alpha(s) = \sum a_n n^{-s}$ be a Dirichlet series with abscissa of convergence $\sigma_c$ and abscissa of absolute convergence $\sigma_a > \sigma_c$. Let $C(x) = \sum_{n\le x} a_n n^{-\sigma_c}$ and $A(x) = \sum_{n\le x} |a_n| n^{-\sigma_c}$.
   (a) By a suitable application of Theorem 1.3, or otherwise, show that $C(x) \ll x^{\varepsilon}$ and that $A(x) \ll x^{\sigma_a - \sigma_c + \varepsilon}$ for any $\varepsilon > 0$, where the implicit constants may depend on $\varepsilon$ and on the sequence $\{a_n\}$.
   (b) Show that if $\sigma > \sigma_c$ then
   $$\sum_{n>N} a_n n^{-s} = -C(N)N^{\sigma_c - s} + (s-\sigma_c)\int_N^{\infty} C(u) u^{\sigma_c - s - 1}\,du.$$

Deduce that the above is $\ll \tau N^{\sigma_c-\sigma+\varepsilon}$ uniformly for $s$ in the half-plane $\sigma \ge \sigma_c + \varepsilon$ where the implicit constant may depend on $\varepsilon$ and on the sequence $\{a_n\}$.

(c) Show that

$$\sum_{n=1}^{N} |a_n| n^{-\sigma} = A(N)N^{-\sigma+\sigma_c} + (\sigma - \sigma_c)\int_1^N A(u)u^{-\sigma+\sigma_c-1}\,du$$

for any $\sigma$. Deduce that the above is $\ll N^{\sigma_a-\sigma+\varepsilon}$ uniformly for $\sigma$ in the interval $\sigma_c \le \sigma \le \sigma_a$, for any given $\varepsilon > 0$. Here the implicit constant may depend on $\varepsilon$ and on the sequence $\{a_n\}$.

(d) Let $\theta(\sigma) = (\sigma_a - \sigma)/(\sigma_a - \sigma_c)$. By making a suitable choice of $N$, show that

$$\alpha(s) \ll \tau^{\theta(\sigma)+\varepsilon}$$

uniformly for $s$ in the strip $\sigma_c + \varepsilon \le \sigma \le \sigma_a$.

5. (a) Show that if $\alpha(s) = \sum a_n n^{-s}$ has abscissa of convergence $\sigma_c < \infty$, then

$$\lim_{\sigma\to\infty} \alpha(\sigma) = a_1.$$

(b) Show that $\zeta'(s) = -\sum_{n=1}^{\infty}(\log n)n^{-s}$ for $\sigma > 1$.

(c) Show that $\lim_{\sigma\to\infty}\zeta'(\sigma) = 0$.

(d) Show that there is no half-plane in which $1/\zeta'(s)$ can be written as a convergent Dirichlet series.

6. Let $\alpha(s) = \sum a_n n^{-s}$ be a Dirichlet series with $a_n \ge 0$ for all $n$. Show that $\sigma_c = \sigma_a$, and that

$$\sup_t |\alpha(s)| = \alpha(\sigma)$$

for any given $\sigma > \sigma_c$.

7. (Vivanti 1893; Pringsheim 1894) Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1 and that $a_n \ge 0$ for all $n$. Show that $z = 1$ is a singular point of $f$.

8. (Bohr 1910, p. 32) Let $t_1 = 4$, $t_{r+1} = 2^{t_r}$ for $r \ge 1$. Put $\alpha(s) = \sum a_n n^{-s}$ where $a_n = 0$ unless $n \in [t_r, 2t_r]$ for some $r$, in which case put

$$a_n = \begin{cases} t_r^{it_r} & (n = t_r), \\ n^{it_r} - (n-1)^{it_r} & (t_r < n < 2t_r), \\ -(2t_r - 1)^{it_r} & (n = 2t_r). \end{cases}$$

(a) Show that $\sum_{t_r}^{2t_r} a_n = 0$.

(b) Show that if $t_r \le x < 2t_r$ for some $r$, then $A(x) = [x]^{it_r}$ where $A(x) = \sum_{n \le x} a_n$.
(c) Show that $A(x) \ll 1$ uniformly for $x \ge 1$.
(d) Deduce that $\alpha(s)$ converges for $\sigma > 0$.
(e) Show that $\alpha(it)$ does not converge; conclude that $\sigma_c = 0$.
(f) Show that if $\sigma > 0$, then

$$\alpha(s) = \sum_{r=1}^{R} \sum_{n=t_r}^{2t_r} a_n n^{-s} + s \int_{t_{R+1}}^{\infty} A(x) x^{-s-1}\, dx\,.$$

(g) Suppose that $\sigma > 0$. Show that the above is

$$\sum_{n=t_R}^{2t_R} a_n n^{-s} + O\big(t_{R-1}\big) + O\left(\frac{|s|}{\sigma t_{R+1}^{\sigma}}\right).$$

(h) Show that if $\sigma > 0$, then

$$\sum_{n=t_R}^{2t_R} a_n n^{-s} = s \int_{t_R}^{2t_R} [x]^{it_R} x^{-s-1}\, dx\,.$$

(i) Show that if $n \le x < n+1$, then $\Re(n^{it_R} x^{-it_R}) \ge 1/2$. Deduce that

$$\left| \int_{t_R}^{2t_R} [x]^{it_R} x^{-\sigma - it_R - 1}\, dx \right| \gg t_R^{-\sigma}.$$

(j) Suppose that $\delta > 0$ is fixed. Conclude that if $R \ge R_0(\delta)$, then $|\alpha(\sigma + it_R)| \gg t_R^{1-\sigma}$ uniformly for $\delta \le \sigma \le 1 - \delta$.
(k) Show that $\sum |a_n| n^{-\sigma} < \infty$ when $\sigma > 1$. Deduce that $\sigma_a = 1$.

## 1.3 Euler products and the zeta function

The situation regarding products of Dirichlet series is somewhat complicated, but it is useful to note that the formal calculation in (2) is justified if the series are absolutely convergent.

**Theorem 1.8** *Let $\alpha(s) = \sum a_n n^{-s}$ and $\beta(s) = \sum b_n n^{-s}$ be two Dirichlet series, and put $\gamma(s) = \sum c_n n^{-s}$ where the $c_n$ are given by* (1.3). *If $s$ is a point at which the two series $\alpha(s)$ and $\beta(s)$ are both absolutely convergent, then $\gamma(s)$ is absolutely convergent and $\gamma(s) = \alpha(s)\beta(s)$.*

The mere convergence of $\alpha(s)$ and $\beta(s)$ is not sufficient to justify (1.2). Indeed, the square of the series (1.12) can be shown to have abscissa of convergence $\ge 1/4$.

A function is called an *arithmetic function* if its domain is the set $\mathbb{Z}$ of integers, or some subset of the integers such as the natural numbers. An arithmetic function $f(n)$ is said to be *multiplicative* if $f(1) = 1$ and if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. Also, an arithmetic function $f(n)$ is called *totally multiplicative* if $f(1) = 1$ and if $f(mn) = f(m)f(n)$ for all $m$ and $n$. If $f$ is multiplicative then the Dirichlet series $\sum f(n)n^{-s}$ factors into a product over primes. To see why this is so, we first argue formally (i.e., we ignore questions of convergence). When the product

$$\prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \cdots)$$

is expanded, the generic term is

$$\frac{f\big(p_1^{k_1}\big)f\big(p_2^{k_2}\big)\cdots f\big(p_r^{k_r}\big)}{\big(p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}\big)^s}.$$

Set $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$. Since $f$ is multiplicative, the above is $f(n)n^{-s}$. Moreover, this correspondence between products of prime powers and positive integers $n$ is one-to-one, in view of the fundamental theorem of arithmetic. Hence after rearranging the terms, we obtain the sum $\sum f(n)n^{-s}$. That is, we expect that

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots). \tag{1.14}$$

The product on the right-hand side is called the *Euler product* of the Dirichlet series. The mere convergence of the series on the left does not imply that the product converges; as in the case of the identity (1.2), we justify (1.14) only under the stronger assumption of absolute convergence.

**Theorem 1.9** *If $f$ is multiplicative and $\sum |f(n)|n^{-\sigma} < \infty$, then* (1.14) *holds.*

If $f$ is totally multiplicative, then the terms on the right-hand side in (1.14) form a geometric progression, in which case the identity may be written more concisely,

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p (1 - f(p)p^{-s})^{-1}. \tag{1.15}$$

*Proof* For any prime $p$,

$$\sum_{k=0}^{\infty} |f(p^k)|p^{-k\sigma} \le \sum_{n=1}^{\infty} |f(n)|n^{-\sigma} < \infty,$$

so each sum on the right-hand side of (1.14) is absolutely convergent. Let $y$ be a positive real number, and let $\mathcal{N}$ be the set of those positive integers composed entirely of primes not exceeding $y$, $\mathcal{N} = \{n : p|n \Rightarrow p \le y\}$. (Note that $1 \in \mathcal{N}$.) Since a product of finitely many absolutely convergent series may be arbitrarily rearranged, we see that

$$\Pi_y = \prod_{p \le y} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right) = \sum_{n \in \mathcal{N}} f(n)n^{-s}.$$

Hence

$$\left|\Pi_y - \sum_{n=1}^{\infty} f(n)n^{-s}\right| \le \sum_{n \notin \mathcal{N}} |f(n)|n^{-\sigma}.$$

If $n \le y$ then all prime factors of $n$ are $\le y$, and hence $n \in \mathcal{N}$. Consequently the sum on the right above is

$$\le \sum_{n > y} |f(n)|n^{-\sigma},$$

which is small if $y$ is large. Thus the partial products $\Pi_y$ tend to $\sum f(n)n^{-s}$ as $y \to \infty$. □

Let $\omega(n)$ denote the number of distinct primes dividing $n$, and let $\Omega(n)$ be the number of distinct prime powers dividing $n$. That is,

$$\omega(n) = \sum_{p|n} 1, \qquad \Omega(n) = \sum_{p^k|n} 1 = \sum_{p^k \| n} k. \tag{1.16}$$

It is easy to distinguish these functions, since $\omega(n) \le \Omega(n)$ for all $n$, with equality if and only if $n$ is square-free. These functions are examples of *additive functions* because they satisfy the functional relation $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$. Moreover, $\Omega(n)$ is *totally additive* because this functional relation holds for all pairs $m, n$. An exponential of an additive function is a multiplicative function. In particular, the *Liouville lambda function* is the totally multiplicative function $\lambda(n) = (-1)^{\Omega(n)}$. Closely related is the *Möbius mu function*, which is defined to be $\mu(n) = (-1)^{\omega(n)}$ if $n$ is square-free, $\mu(n) = 0$ otherwise. By the fundamental theorem of arithmetic we know that a multiplicative (or additive) function is uniquely determined by its values at prime powers, and similarly that a totally multiplicative (or totally additive) function is uniquely determined by its values at the primes. Thus $\mu(n)$ is the unique multiplicative function that takes the value $-1$ at every prime, and the value 0 at every higher power of a prime, while $\lambda(n)$ is the unique totally multiplicative function that takes the value $-1$ at every prime. By using Theorem 1.9 we can determine the Dirichlet series generating functions of $\lambda(n)$ and of $\mu(n)$ in terms of the Riemann zeta function.

**Corollary 1.10** *For $\sigma > 1$,*

$$\sum_{n=1}^{\infty} n^{-s} = \zeta(s) = \prod_p (1 - p^{-s})^{-1}, \tag{1.17}$$

$$\sum_{n=1}^{\infty} \mu(n) n^{-s} = \frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}), \tag{1.18}$$

*and*

$$\sum_{n=1}^{\infty} \lambda(n) n^{-s} = \frac{\zeta(2s)}{\zeta(s)} = \prod_p (1 + p^{-s})^{-1}. \tag{1.19}$$

*Proof* All three series are absolutely convergent, since $\sum n^{-\sigma} < \infty$ for $\sigma > 1$, by the integral test. Since the coefficients are multiplicative, the Euler product formulae follow by Theorem 1.9. In the first and third cases use the variant (1.15). On comparing the Euler products in (1.17) and (1.18), it is immediate that the second of these Dirichlet series is $1/\zeta(s)$. As for (1.19), from the identity $1 + z = (1 - z^2)/(1 - z)$ we deduce that

$$\prod_p (1 + p^{-s}) = \frac{\prod_p (1 - p^{-2s})}{\prod_p (1 - p^{-s})} = \frac{\zeta(s)}{\zeta(2s)}.$$

□

The manipulation of Euler products, as exemplified above, provides a powerful tool for relating one Dirichlet series to another.

In (1.17) we have expressed $\zeta(s)$ as an absolutely convergent product; hence in particular $\zeta(s) \neq 0$ for $\sigma > 1$. We have not yet defined the zeta function outside this half-plane, but we shall do so shortly, and later we shall find that the zeta function does have zeros in the half-plane $\sigma \le 1$. These zeros play an important role in determining the distribution of prime numbers.

Many important relations involving arithmetic functions can be expressed succinctly in terms of Dirichlet series. For example, the fundamental elementary identity

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{1.20}$$

is equivalent to the identity

$$\zeta(s) \cdot \frac{1}{\zeta(s)} = 1,$$

in view of (1.3), (1.17), (1.18), and Theorem 1.6. More generally, if

$$F(n) = \sum_{d|n} f(d) \tag{1.21}$$

for all $n$, then, apart from questions of convergence,

$$\sum F(n)n^{-s} = \zeta(s) \sum f(n)n^{-s}.$$

By Möbius inversion, the identity (1.21) is equivalent to the relation

$$f(n) = \sum_{d|n} \mu(d)F(n/d),$$

which is to say that

$$\sum f(n)n^{-s} = \frac{1}{\zeta(s)} \sum F(n)n^{-s}.$$

Such formal manipulations can be used to suggest (or establish) many useful elementary identities.

For $\sigma > 1$ the product (1.17) is absolutely convergent. Since $\log(1-z)^{-1} = \sum_{k=1}^{\infty} z^k/k$ for $|z| < 1$, it follows that

$$\log \zeta(s) = \sum_p \log(1-p^{-s})^{-1} = \sum_p \sum_{k=1}^{\infty} k^{-1} p^{-ks}.$$

On differentiating, we find also that

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \sum_{k=1}^{\infty} (\log p)p^{-ks}$$

for $\sigma > 1$. This is a Dirichlet series, whose $n^{\text{th}}$ coefficient is the von Mangoldt lambda function: $\Lambda(n) = \log p$ if $n$ is a power of $p$, $\Lambda(n) = 0$ otherwise.

**Corollary 1.11** *For $\sigma > 1$,*

$$\log \zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s}$$

*and*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s}.$$

The quotient $f'(s)/f(s)$, obtained by differentiating the logarithm of $f(s)$, is known as the *logarithmic derivative* of $f$. Subsequently we shall often write it more concisely as $\frac{f'}{f}(s)$.

The important elementary identity

$$\sum_{d|n} \Lambda(d) = \log n \tag{1.22}$$

is reflected in the relation

$$\zeta(s)\Big(-\frac{\zeta'}{\zeta}(s)\Big) = -\zeta'(s),$$

since

$$-\zeta'(s) = \sum_{n=1}^{\infty} (\log n) n^{-s}$$

for $\sigma > 1$.

We now continue the zeta function beyond the half-plane in which it was initially defined.

**Theorem 1.12** *Suppose that* $\sigma > 0$, $x > 0$, *and that* $s \neq 1$. *Then*

$$\zeta(s) = \sum_{n \le x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^\infty \{u\} u^{-s-1}\, du. \tag{1.23}$$

Here $\{u\}$ denotes the fractional part of $u$, so that $\{u\} = u - [u]$ where $[u]$ denotes the integral part of $u$.

*Proof of Theorem 1.12* For $\sigma > 1$ we have

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n \le x} n^{-s} + \sum_{n > x} n^{-s}.$$

This second sum we write as

$$\int_x^\infty u^{-s}\, d[u] = \int_x^\infty u^{-s}\, du - \int_x^\infty u^{-s}\, d\{u\}.$$

We evaluate the first integral on the right-hand side, and integrate the second one by parts. Thus the above is

$$= \frac{x^{1-s}}{s-1} + \{x\} x^{-s} + \int_x^\infty \{u\}\, du^{-s}.$$

Since $(u^{-s})' = -su^{-s-1}$, the desired formula now follows by Theorem A.3. The integral in (1.23) is convergent in the half-plane $\sigma > 0$, and uniformly so for $\sigma \ge \delta > 0$. Since the integrand is an analytic function of $s$, it follows that the integral is itself an analytic function for $\sigma > 0$. By the uniqueness of analytic continuation the formula (1.23) holds in this larger half-plane. □


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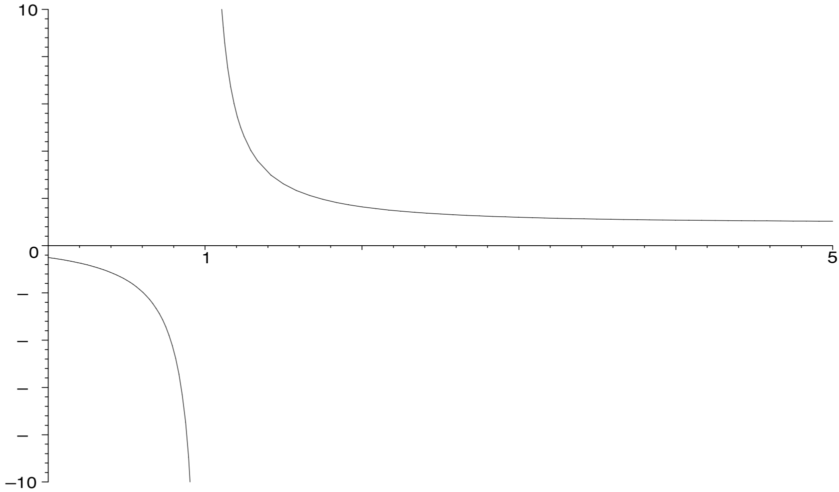


**Figure 1.2** The Riemann zeta function $\zeta(s)$ for $0 < s \le 5$.

By taking $x = 1$ in (1.23) we obtain in particular the identity

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{u\}u^{-s-1}\,du \tag{1.24}$$

for $\sigma > 0$. Hence we have

**Corollary 1.13** *The Riemann zeta function has a simple pole at $s = 1$ with residue 1, but is otherwise analytic in the half-plane $\sigma > 0$.*

A graph of $\zeta(s)$ that exhibits the pole at $s = 1$ is provided in Figure 1.2. By repeatedly integrating by parts we can continue $\zeta(s)$ into successively larger half-planes; this is systematized by using the Euler–Maclaurin summation formula (see Theorem B.5). In Chapter 10 we shall continue the zeta function by a different method. For the present we note that (1.24) yields useful inequalities for the zeta function on the real line.

**Corollary 1.14** *The inequalities*

$$\frac{1}{\sigma-1} < \zeta(\sigma) < \frac{\sigma}{\sigma-1}$$

*hold for all $\sigma > 0$. In particular, $\zeta(\sigma) < 0$ for $0 < \sigma < 1$.*

*Proof* From the inequalities $0 \le \{u\} < 1$ it follows that

$$0 \le \int_1^\infty \{u\}u^{-\sigma-1}\,du < \int_1^\infty u^{-\sigma-1}\,du = \frac{1}{\sigma}.$$

This suffices. □

We now put the parameter $x$ in (1.23) to good use.

**Corollary 1.15** *Let $\delta$ be fixed, $\delta > 0$. Then for $\sigma \ge \delta$, $s \ne 1$,*

$$\sum_{n\le x} n^{-s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(\tau x^{-\sigma}). \tag{1.25}$$

*In addition,*

$$\sum_{n\le x} \frac{1}{n} = \log x + C_0 + O(1/x) \tag{1.26}$$

*where $C_0$ is Euler's constant,*

$$C_0 = 1 - \int_1^\infty \{u\}u^{-2}\,du = 0.5772156649\ldots. \tag{1.27}$$

*Proof* The first estimate follows by crudely estimating the integral in (1.23):

$$\int_x^\infty \{u\}u^{-s-1}\,du \ll \int_x^\infty u^{-\sigma-1}\,du = \frac{x^{-\sigma}}{\sigma}.$$

As for the second estimate, we note that the sum is

$$\begin{aligned}\int_{1^-}^x u^{-1}\,d[u] &= \int_{1^-}^x u^{-1}\,du - \int_{1^-}^x u^{-1}\,d\{u\}\\ &= \log x + 1 - \{x\}/x - \int_1^x \{u\}u^{-2}\,du.\end{aligned}$$

The result now follows by writing $\int_1^x = \int_1^\infty - \int_x^\infty$, and noting that

$$\int_x^\infty \{u\}u^{-2}\,du \ll \int_x^\infty u^{-2}\,du = 1/x.$$

□

By letting $s \to 1$ in (1.25) and comparing the result with (1.26), or by letting $s \to 1$ in (1.24) and comparing the result with (1.27), we obtain

**Corollary 1.16** *Let*

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^\infty a_k(s-1)^k \tag{1.28}$$

*be the Laurent expansion of $\zeta(s)$ at $s = 1$. Then $a_0$ is Euler's constant, $a_0 = C_0$.*

Euler's constant also arises in the theory of the gamma function. (See Appendix C and Chapter 10.)

**Corollary 1.17** *Let $\delta > 0$ be fixed. Then*

$$\zeta(s) = \frac{1}{s-1} + O(1)$$


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*uniformly for s in the rectangle* $\delta \le \sigma \le 2$, $|t| \le 1$, *and*

$$\zeta(s) \ll (1 + \tau^{1-\sigma}) \min\left(\frac{1}{|\sigma - 1|}, \log \tau\right)$$

*uniformly for* $\delta \le \sigma \le 2$, $|t| \ge 1$.

*Proof* The first assertion is clear from (1.24). When $|t|$ is larger, we obtain a bound for $|\zeta(s)|$ by estimating the sum in (1.25). Assume that $x \ge 2$. We observe that

$$\sum_{n \le x} n^{-s} \ll \sum_{n \le x} n^{-\sigma} \ll 1 + \int_1^x u^{-\sigma}\, du$$

uniformly for $\sigma \ge 0$. If $0 \le \sigma \le 1 - 1/\log x$, then this integral is $(x^{1-\sigma} - 1)/(1 - \sigma) < x^{1-\sigma}/(1 - \sigma)$. If $|\sigma - 1| \le 1/\log x$, then $u^{-\sigma} \asymp u^{-1}$ uniformly for $1 \le u \le x$, and hence the integral is $\asymp \int_1^x u^{-1}\, du = \log x$. If $\sigma \ge 1 + 1/\log x$, then the integral is $< \int_1^\infty u^{-\sigma}\, du = 1/(\sigma - 1)$. Thus

$$\sum_{n \le x} n^{-s} \ll (1 + x^{1-\sigma}) \min\left(\frac{1}{|\sigma - 1|}, \log x\right) \tag{1.29}$$

uniformly for $0 \le \sigma \le 2$. The second assertion now follows by taking $x = \tau$ in (1.25). □

### 1.3.1 Exercises

1. Suppose that $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$, and that $f$ is not identically 0. Deduce that $f(1) = 1$, and hence that $f$ is multiplicative.
2. (Stieltjes 1887) Suppose that $\sum a_n$ converges, that $\sum |b_n| < \infty$, and that $c_n$ is given by (1.3). Show that $\sum c_n$ converges to $(\sum a_n)(\sum b_n)$. (Hint: Write $\sum_{n \le x} c_n = \sum_{n \le x} b_n A(x/n)$ where $A(y) = \sum_{n \le y} a_n$.)
3. Determine $\sum \varphi(n)n^{-s}$, $\sum \sigma(n)n^{-s}$, and $\sum |\mu(n)|n^{-s}$ in terms of the zeta function. Here $\varphi(n)$ is Euler's 'totient function', which is the number of a, $1 \le a \le n$, such that $(a, n) = 1$.
4. Let $q$ be a positive integer. Show that if $\sigma > 1$, then
$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty} n^{-s} = \zeta(s) \prod_{p|q} (1 - p^{-s}).$$
5. Show that if $\sigma > 1$, then
$$\sum_{n=1}^{\infty} d(n)^2 n^{-s} = \zeta(s)^4/\zeta(2s).$$

6. Let $\sigma_a(n) = \sum_{d|n} d^a$. Show that

$$\sum_{n=1}^{\infty} \sigma_a(n)\sigma_b(n)n^{-s} = \zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)/\zeta(2s-a-b)$$

when $\sigma > \max(1, 1+\Re a, 1+\Re b, 1+\Re(a+b))$.

7. Let $F(s) = \sum_p (\log p)p^{-s}$, $G(s) = \sum_p p^{-s}$ for $\sigma > 1$. Show that in this half-plane,

$$-\frac{\zeta'}{\zeta}(s) = \sum_{k=1}^{\infty} F(ks),$$

$$F(s) = -\sum_{d=1}^{\infty} \mu(d)\frac{\zeta'}{\zeta}(ds),$$

$$\log \zeta(s) = \sum_{k=1}^{\infty} G(ks)/k,$$

$$G(s) = \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \log \zeta(ds).$$

8. Let $F(s)$ and $G(s)$ be defined as in the preceding problem. Show that if $\sigma > 1$, then

$$\sum_{n=1}^{\infty} \omega(n)n^{-s} = \zeta(s)G(s) = \zeta(s)\sum_{d=1}^{\infty} \frac{\mu(d)}{d} \log \zeta(ds),$$

$$\sum_{n=1}^{\infty} \Omega(n)n^{-s} = \zeta(s)\sum_{k=1}^{\infty} G(ks) = \zeta(s)\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log \zeta(ks).$$

9. Let $t$ be a fixed real number, $t \neq 0$. Describe the limit points of the sequence of partial sums $\sum_{n \le x} n^{-1-it}$.

10. Show that $\sum_{n=1}^{N} n^{-1} > \log N + C_0$ for all positive integers $N$, and that $\sum_{n \le x} n^{-1} > \log x$ for all positive real numbers $x$.

11. (a) Show that if $a_n$ is totally multiplicative, and if $\alpha(s) = \sum a_n n^{-s}$ has abscissa of convergence $\sigma_c$, then

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n n^{-s} = (1 - 2a_2 2^{-s})\alpha(s)$$

for $\sigma > \sigma_c$.

(b) Show that

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = (1 - 2^{1-s})\zeta(s)$$

for $\sigma > 0$.

(c) (Shafer 1984) Show that

$$\sum_{n=1}^{\infty}(-1)^n(\log n)n^{-1} = C_0 \log 2 - \frac{1}{2}(\log 2)^2.$$

12. (Stieltjes 1885) Show that if $k$ is a positive integer, then

$$\sum_{n\le x}\frac{(\log n)^k}{n} = \frac{(\log x)^{k+1}}{k+1} + C_k + O_k\Big(\frac{(\log x)^k}{x}\Big)$$

for $x \ge 1$ where

$$C_k = \int_1^\infty \{u\}(\log u)^{k-1}(k - \log u)u^{-2}\,du.$$

Show that the numbers $a_k$ in (1.28) are given by $a_k = (-1)^k C_k/k!$.

13. Let $\mathcal{D}$ be the disc of radius 1 and centre 2. Suppose that the numbers $\varepsilon_k$ tend monotonically to 0, that the numbers $t_k$ tend monotonically to 0, and that the numbers $N_k$ tend monotonically to infinity. We consider the Dirichlet series $\alpha(s) = \sum_n a_n n^{-s}$ with coefficients $a_n = \varepsilon_k n^{it_k}$ for $N_{k-1} < n \le N_k$. For suitable choices of the $\varepsilon_k$, $t_k$, and $N_k$ we show that the series converges at $s = 1$ but that it is not uniformly convergent in $\mathcal{D}$.
   (a) Suppose that $\sigma_k = 2 - \sqrt{1 - t_k^2}$, so that $s_k = \sigma_k + it_k \in \mathcal{D}$. Show that if

$$N_k^{t_k^2} \ll 1, \tag{1.30}$$

   then

$$\Big|\sum_{N_{k-1}<n\le N_k} a_n n^{-s_k}\Big| \gg \varepsilon_k \log\frac{N_k}{N_{k-1}}.$$

   Thus if

$$\varepsilon_k \log\frac{N_k}{N_{k-1}} \gg 1 \tag{1.31}$$

   then the series is not uniformly convergent in $\mathcal{D}$.
   (b) By using Corollary 1.15, or otherwise, show that if $(a, b] \subseteq (N_{k-1}, N_k]$, then

$$\sum_{a<n\le b} a_n n^{-1} \ll \frac{\varepsilon_k}{t_k}.$$

   Hence if

$$\sum_{k=1}^{\infty}\frac{\varepsilon_k}{t_k} < \infty, \tag{1.32}$$

   then the series $\alpha(1)$ converges.

(c) Show that the parameters can be chosen so that (1.30)–(1.32) hold, say by taking $N_k = \exp(1/\varepsilon_k)$ and $t_k = \varepsilon_k^{1/2}$ with $\varepsilon_k$ tending rapidly to 0.

14. Let $t(n) = (-1)^{\Omega(n)-\omega(n)} \prod_{p|n}(p-1)^{-1}$, and put $T(s) = \sum_n t(n)n^{-s}$.
   (a) Show that for $\sigma > 0$, $T(s)$ has the absolutely convergent Euler product
$$T(s) = \prod_p \Big(1 + \frac{1}{(p-1)(p^s+1)}\Big).$$
   (b) Determine all zeros of the function $1 + 1/((p-1)(p^s+1))$.
   (c) Show that the line $\sigma = 0$ is a natural boundary of the function $T(s)$.

15. Suppose throughout that $0 < \alpha \le 1$. For $\sigma > 1$ we define the *Hurwitz zeta function* by the formula
$$\zeta(s,\alpha) = \sum_{n=0}^{\infty}(n+\alpha)^{-s}.$$
Thus $\zeta(s,1) = \zeta(s)$.
   (a) Show that $\zeta(s,1/2) = (2^s-1)\zeta(s)$.
   (b) Show that if $x \ge 0$ then
$$\zeta(s,\alpha) = \sum_{0\le n\le x}(n+\alpha)^{-s} + \frac{(x+\alpha)^{1-s}}{s-1} + \frac{\{x\}}{(x+\alpha)^s} - s\int_x^\infty \{u\}(u+\alpha)^{-s-1}\,du.$$
   (c) Deduce that $\zeta(s,\alpha)$ is an analytic function of $s$ for $\sigma > 0$ apart from a simple pole at $s = 1$ with residue 1.
   (d) Show that
$$\lim_{s\to 1}\Big(\zeta(s,\alpha) - \frac{1}{s-1}\Big) = 1/\alpha - \log\alpha - \int_0^\infty \frac{\{u\}}{(u+\alpha)^2}\,du.$$
   (e) Show that
$$\lim_{s\to 1}\Big(\zeta(s,\alpha) - \frac{1}{s-1}\Big) = \sum_{0\le n\le x}\frac{1}{n+\alpha} - \log(x+\alpha) + \frac{\{x\}}{x+\alpha} - \int_x^\infty \frac{\{u\}}{(u+\alpha)^2}\,du.$$
   (f) Let $x \to \infty$ in the above, and use (C.2), (C.10) to show that
$$\lim_{s\to 1}\Big(\zeta(s,\alpha) - \frac{1}{s-1}\Big) = -\frac{\Gamma'}{\Gamma}(\alpha).$$
   (This is consistent with Corollary 1.16, in view of (C.11).)

## 1.4 Notes

Section 1.1. For a brief introduction to the Hardy–Littlewood circle method, including its application to Waring's problem, see Davenport (2005). For a comprehensive account of the method, see Vaughan (1997). Other examples of the fruitful use of generating functions are found in many sources, such as Andrews (1976) and Wilf (1994).

Algorithms for the efficient computation of $\pi(x)$ have been developed by Meissel (Lehmer, 1959), Mapes (1963), Lagarias, Miller & Odlyzko (1985), Deléglise & Rivat (1996), and by X. Gourdon. For discussion of these methods, see Chapter 1 of Riesel (1994) and the web page of Gourdon & Sebah at http://numbers.computation.free.fr/Constants/Primes/countingPrimes.html.

The 'big oh' notation was introduced by Paul Bachmann (1894, p. 401). The 'little oh' was introduced by Edmund Landau (1909a, p. 61). The $\asymp$ notation was introduced by Hardy (1910, p. 2). Our notation $f \sim g$ also follows Hardy (1910). The Omega notation was introduced by G. H. Hardy and J. E. Littlewood (1914, p. 225). Ingham (1932) replaced the $\Omega_R$ and $\Omega_L$ of Hardy and Littlewood by $\Omega_+$ and $\Omega_-$. The $\ll$ notation is due to I. M. Vinogradov.

Section 1.2. The series $\sum a_n n^{-s}$ is called an *ordinary* Dirichlet series, to distinguish it from a *generalized* Dirichlet series, which is a sum of the form $\sum a_n e^{-\lambda_n s}$ where $0 < \lambda_1 < \lambda_2 < \cdots, \lambda_n \to \infty$. We see that generalized Dirichlet series include both ordinary Dirichlet series ($\lambda_n = \log n$) and power series ($\lambda_n = n$). Theorems 1.1, 1.3, 1.6, and 1.7 extend naturally to generalized Dirichlet series, and even to the more general class of functions $\int_0^\infty e^{-us}\,dA(u)$ where $A(u)$ is assumed to have finite variation on each finite interval $[0, U]$. The proof of the general form of Theorem 1.6 must be modified to depend on uniform, rather than absolute, convergence, since a generalized Dirichlet series may be never more than conditionally convergent (e.g., $\sum(-1)^n(\log n)^{-s}$). If we put $a = \limsup(\log n)/\lambda_n$, then the general form of Theorem 1.4 reads $\sigma_c \le \sigma_a \le \sigma_c + a$. Hardy & Riesz (1915) have given a detailed account of this subject, with historical attributions. See also Bohr & Cramér (1923).

Jensen (1884) showed that the domain of convergence of a generalized Dirichlet series is always a half-plane. The more precise information provided by Theorem 1.1 is due to Cahen (1894) who proved it not only for ordinary Dirichlet series but also for generalized Dirichlet series.

The construction in Exercise 1.2.8 would succeed with the simpler choice $a_n = n^{it_r}$ for $t_r \le n \le 2t_r$, $a_n = 0$ otherwise, but then to complete the argument one would need a further tool, such as the Kusmin–Landau inequality


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(cf. Mordell 1958). The square of the Dirichlet series in Exercise 1.2.8 has abscissa of convergence $1/2$; this bears on the result of Exercise 2.1.9. Information concerning the convergence of the product of two Dirichlet series is found in Exercises 1.3.2, 2.1.9, 5.2.16, and in Hardy & Riesz (1915).

Theorem 1.7 originates in Landau (1905). The analogue for power series had been proved earlier by Vivanti (1893) and Pringsheim (1894). Landau's proof extends to generalized Dirichlet series (including power series).

Section 1.3. The hypothesis $\sum |f(n)|n^{-\sigma} < \infty$ of Theorem 1.9 is equivalent to the assertion that

$$\prod_p (1 + |f(p)|p^{-\sigma} + |f(p^2)|p^{-2\sigma} + \cdots) < \infty,$$

which is slightly stronger than merely asserting that the Euler product converges absolutely. We recall that a product $\prod_n (1 + a_n)$ is said to be absolutely convergent if $\prod_n (1 + |a_n|) < \infty$. To see that the hypothesis $\prod_p (1 + |f(p)p^{-s} + \cdots|) < \infty$ is not sufficient, consider the following example due to Ingham: For every prime $p$ we take $f(p) = 1$, $f(p^2) = -1$, and $f(p^k) = 0$ for $k > 2$. Then the product is absolutely convergent at $s = 0$, but the terms $f(n)$ do not tend to 0, and hence the series $\sum f(n)$ diverges. Indeed, it can be shown that $\sum_{n \le x} f(n) \sim cx$ as $x \to \infty$ where $c = \prod_p \left(1 - 2p^{-2} + p^{-3}\right) > 0$.

Euler (1735) defined the constant $C_0$, which he denoted $C$. Mascheroni (1790) called the constant $\gamma$, which is in common use, but we wish to reserve this symbol for the imaginary part of a zero of the zeta function or an $L$-function. It is conjectured that Euler's constant $C_0$ is irrational. The early history of the determination of the initial digits of $C_0$ has been recounted by Nielsen (1906, pp. 8–9). More recently, Wrench (1952) computed 328 digits, Knuth (1963) computed 1,271 digits, Sweeney (1963) computed 3,566 digits, Beyer & Waterman (1974) computed 4,879 digits, Brent (1977) computed 20,700 digits, Brent & McMillan (1980) computed 30,100 digits. At this time, it seems that more than $10^8$ digits have been computed – see the web page of X. Gourdon & P. Sebah at http://numbers.computation.free.fr/Constants/Gamma/gamma.html. To 50 places, Euler's constant is

$$C_0 = 0.57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243\ 10421\ 59335\ 93992.$$

Statistical analysis of the continued fraction coefficients of $C_0$ suggest that it satisfies the Gauss–Kusmin law, which is to say that $C_0$ seems to be a typical irrational number.

Landau & Walfisz (1920) showed that the functions $F(s)$ and $G(s)$ of Exercise 1.3.7 have the imaginary axis $\sigma = 0$ as a natural boundary. For further work on Dirichlet series with natural boundaries see Estermann (1928a,b) and Kurokawa (1987).

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