A NOTE ON ŠNIREL'MAN'S APPROACH TO GOLDBACH'S PROBLEM

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1. *Introduction and statement of theorem*

The object of this note is to demonstrate that Šnirel'man's method [38], [39] (see also Landau [22], [23]) as modified by Shapiro and Warga [36] will give the following theorem.

THEOREM. *Every sufficiently large odd number can be written as the sum of five odd prime numbers, and thus every sufficiently large number is the sum of at most six prime numbers.*

It is apparent from the proof that the method only just misses giving four for even numbers, and that is the best that has been obtained for even numbers by the much more powerful Hardy–Littlewood–Vinogradov method (see I. M. Vinogradov [42], [43], [44]).

In [38] Šnirel'man has a large positive constant in place of the six, and this has been reduced successively to 2208 by Romanov [35], to 71 by Heilbronn, Landau and Scherk [17], 67 by Ricci [32], [33], 20 by Shapiro and Warga [36], 18 by Yin [45], 12 by Klimov and Kondakova [20] and 10 by Čečuro and Kuzjašev [5] and Siebert [37] independently. Čečuro and Kuzjašev, and Siebert use the Bombieri–A. I. Vinogradov theorem [4], [41], and in view of the work of Linnik [25], [26], Čudakov [8] and Montgomery [29, Chapter 16] we know that this is of about the same "depth" as the Hardy–Littlewood–Vinogradov method.

The argument presented here does not use the Bombieri–Vinogradov theorem, but depends instead on a theorem of Davenport and Halberstam [10] (see also Barban [1], [2], [3, Theorem 3.2] and Gallagher [14, Theorem 3]) which is a simple consequence of the Siegel–Val'fiš theorem and the large sieve. In principle, the new idea is that counting all the numbers in an interval which are representable as the sum of two prime numbers enables one to average over all the residue classes of all the sifting moduli. One of the fascinating aspects of this subject is the interrelation between the various methods. Montgomery's later sharpening [28], [29, Chapter 17] of the Davenport–Halberstam theorem involved the Hardy–Littlewood–Vinogradov method, but Hooley has recently shown [18] that estimates for exponential sums involving primes, of the Vinogradov type, can be avoided and indeed obtains still sharper results.

Although largely superseded by I. M. Vinogradov's work, the Šnirel'man method has the advantage that it is relatively easy to compute an s such that *every* even number is the sum of at most s prime numbers. Klimov [19] has given $s = 6\times 10^9$ and more recently, with Pil'tjaĭ and Septickaja [20a] has reduced this to $s = 115$.†

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† Note added in proof: This can be reduced further to 27. The proof is to appear in *J. Reine Angew. Math.*

Lavrik has observed [24] that the original Šnirel'man device of using Cauchy's inequality

$$\Big(\sum_{n\leqslant x} r(n)\Big)^2 \leqslant \sum_{\substack{m\leqslant x\\ r(m)>0}} \sum_{n\leqslant x} r(n)^2$$

where $r(n)$ is the number of representations of n as the sum of two prime numbers, will never enable one to do better than the above theorem with six replaced by eight or so. Instead we follow in the spirit of Shapiro and Warga [36], who applied the Selberg sieve directly to a sum of the kind $\sum_{n\leqslant x} r(n)a_n$, where the a_n are suitable weights which smooth out the irregularities of $r(n)$. For some brief comments on the method see Čudakov and Klimov [9].

2. *Proof of the theorem*

Let x be a large real number,

$$R(n) = \sum_{3\leqslant p_1\leqslant x} \sum_{\substack{p_2\geqslant 3\\ p_1+p_2=n}} \log p_1, \tag{1}$$

$$f_n(p) = \begin{cases} 0 & p\mid n \\ \dfrac{1}{p-2} & p\nmid n \end{cases} (2\mid n), \quad f_n(q) = \mu(q)^2 \prod_{p\mid q} f_n(p), \tag{2}$$

$$\vartheta(x) = \sum_{3\leqslant p\leqslant x} \log p, \quad \vartheta(x,d,n) = \sum_{\substack{3\leqslant p\leqslant x\\ p\equiv n (\mathrm{mod}\ d)}} \log p, \tag{3}$$

and

$$\mathscr{L} = \log x, \quad y = x\mathscr{L}^{-20}, \quad \xi = y^{\frac{1}{2}}, \tag{4}$$

where μ is the Möbius function. Then it is a straightforward consequence of Selberg's upper bound sieve method (see, for example, Theorem 3.2 of [15]. Note that the weights $\log p$ in (3) make no essential difference to the argument.) that

$$R(n) \leqslant \frac{\vartheta(x)}{\sum_{q\leqslant\xi} f_n(q)} + y^{\frac{1}{2}} \log x + E(n) \quad (n > x, 2\mid n) \tag{5}$$

where

$$E(n) = \sum_{\substack{d\leqslant y\\ (d,n)=1}} 3^{\omega(d)} \left| \vartheta(x,d,n) - \frac{\vartheta(x)}{\phi(d)} \right|, \tag{6}$$

$\omega(d)$ denotes the number of different prime divisors of d, and ϕ is Euler's function.

By (6),

$$\sum_{\substack{x<n\leqslant 2x\\ 2\mid n}} E(n) \ll x \sum_{d\leqslant y} 3^{\omega(d)} d^{-1} \sum_{\substack{n=1\\ (n,d)=1}}^{d} \left| \vartheta(x,d,n) - \frac{\vartheta(x)}{\phi(d)} \right|$$

so that by Cauchy's inequality

$$\left(\sum_{\substack{x<n\leqslant 2x\\ 2|n}} E(n)\right)^2 \ll x^2\left(\prod_{p\leqslant y}\left(1+\frac{9}{p-1}\right)\right)\sum_{d\leqslant y}\sum_{\substack{n=1\\ (n,d)=1}}^{d}\left|\vartheta(x,d,n)-\frac{\vartheta(x)}{\phi(d)}\right|^2. \tag{7}$$

By elementary prime number theory

$$\prod_{p\leqslant y}\left(1+\frac{9}{p-1}\right) \ll \mathscr{L}^9$$

and

$$\sum_{d\leqslant y}\sum_{\substack{n=1\\ (n,d)=1}}^{d}\left|\vartheta(x,d,n)-\psi(x,d,n)-\frac{\vartheta(x)-\psi(x)}{\phi(d)}\right|^2 \ll yx^{\frac{1}{2}}\mathscr{L}$$

where as usual

$$\psi(x)=\sum_{m\leqslant x}\Lambda(m), \quad \psi(x,d,n)=\sum_{\substack{m\leqslant x\\ m\equiv n(\mathrm{mod}\ d)}}\Lambda(m)$$

with Λ von Mangoldt's function. Hence, by (7) and Theorem 3 of Gallagher **[14]** (by slightly altering the value of y one could equally well use the Davenport–Halberstam theorem **[10]**),

$$\sum_{\substack{x<n\leqslant 2x\\ 2|n}} E(n) \ll x^2\mathscr{L}^{-5}. \tag{8}$$

By (2) and (4),

$$\sum_{q\leqslant \xi} f_n(q) \ll \mathscr{L}^2$$

and, by (1), $R(n)=0$ if $2\nmid n$. Hence, by (4), (5) and (8),

$$\sum_{x<n\leqslant 2x} R(n)\sum_{q\leqslant\xi} f_n(q) \leqslant \sum_{\substack{x<n\leqslant 2x\\ R(n)>0}} \vartheta(x)+O(x^2\mathscr{L}^{-3}). \tag{9}$$

The next step is to estimate the left of (9) from below. By (2),

$$\sum_{q\leqslant\xi} f_n(q) \geqslant \left(\prod_{\substack{p|n\\ p>2}}\frac{p-2}{p-1}\right)\sum_{\substack{q\leqslant\xi\\ q\ \mathrm{odd}}}\frac{\mu(q)^2}{\phi_1(q)} \tag{10}$$

where

$$\phi_1(q)=\prod_{p|q}(p-2). \tag{11}$$

Thus, by (1),

$$\sum_{x<n\leqslant 2x} R(n)\sum_{q\leqslant\xi} f_n(q) \geqslant \left(\sum_{\substack{q\leqslant\xi\\ q\text{ odd}}}\frac{\mu(q)^2}{\phi_1(q)}\right)\sum_{\substack{r\leqslant 2x\\ r\text{ odd}}}\frac{\mu(r)}{\phi(r)}\sum_{\substack{p_1\\ 3\leqslant p_1\leqslant x}}\sum_{\substack{p_2\geqslant 3\\ r\mid p_1+p_2\\ x-p_1<p_2\leqslant 2x-p_1}}\log p_1. \tag{12}$$

When $r > \mathscr{L}^{10}$ one uses the crude estimate

$$\sum_{\substack{p_2\geqslant 3\\ r\mid p_1+p_2\\ x-p_1<p_2\leqslant 2x-p_1}} 1 \ll 1+\frac{x}{r}$$

and when $r \leqslant \mathscr{L}^{10}$ one observes that the number of different prime divisors of r is $\ll\mathscr{L}$. Hence, by (4) and the Siegel–Val'fiš theorem (c.f. §22 of **[11]**),

$$\sum_{\substack{r\leqslant 2x\\ r\text{ odd}}}\frac{\mu(r)}{\phi(r)}\sum_{\substack{p_1\\ 3\leqslant p_1\leqslant x}}\sum_{\substack{p_2\geqslant 3\\ r\mid p_1+p_2\\ x-p_1<p_2\leqslant 2x-p_1}}\log p_1$$

$$= \left(\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\right)\int_3^{2x}\frac{\min(u, 2x-u)}{\log u}\,du+O(x^2\mathscr{L}^{-4}). \tag{13}$$

By (4) and (11),

$$\sum_{\substack{q\leqslant\xi\\ q\text{ odd}}}\frac{\mu(q)^2}{\phi_1(q)} = \left(\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\right)\frac{\log y}{4}+O(1).$$

Hence, by (4), (9), (12) and (13),

$$\sum_{\substack{x<n\leqslant 2x\\ R(n)>0}} 1 \geqslant \frac{\log y}{4\vartheta(x)}\int_3^{2x}\frac{\min(u, 2x-u)}{\log u}\,du+O(x\mathscr{L}^{-1}). \tag{14}$$

The final stage of the proof consists of appealing to a number of density theorems concerning the addition of sets. Whenever a script letter denotes a set, the corresponding italic letter denotes the counting function of the natural numbers contained in the set. Let

$$\mathscr{A} = \{a : 2a = p_1+p_2, p_1 \geqslant 3, p_2 \geqslant 3\}.$$

Then, by (4) and (14),

$$\varliminf_{n\to\infty}\frac{A(n)}{n} \geqslant \tfrac{1}{2}. \tag{15}$$

Let $\mathscr{B} = \{b : b = a_1+a_2, a_1\in\mathscr{A}, a_2\in\mathscr{A}\}$. Then since $3, 4, 5\in\mathscr{A}$, a theorem of Ostmann **[31]** (Theorem 4.1 of **[27]**) implies that

$$\varliminf_{n\to\infty}\frac{B(n)}{n} \geqslant \tfrac{3}{5}. \tag{16}$$

If, instead of Ostmann's theorem one uses a deeper theorem of Kneser [21] (see Theorem 18 of [16, Chapter 1] or Theorem 4.2 of [27]) combined with the identity

$$\sum_{\substack{r=1 \\ (r(2n-r),\,2g)=1}}^{2g} 1 = 2g\left(\prod_{\substack{p|2g \\ p|2n}}\left(1-\frac{1}{p}\right)\right)\prod_{\substack{p|2g \\ p\nmid 2n}}\left(1-\frac{2}{p}\right)$$

and the prime number theorem for arithmetic progressions, then one finds that

$$\lim_{n\to\infty}\frac{B(n)}{n} = 1.$$

Thus almost every even number is the sum of four primes, which compares with two by the Hardy–Littlewood–Vinogradov method, due to Van der Corput [6], Čudakov [7] and Estermann [13] independently (see also [30], [40]).

By (15), (16) and Dirichlet's box principle, every sufficiently large integer can be written in the form $a+b$ with $a\in\mathscr{A}$, $b\in\mathscr{B}$. Thus the set $\mathscr{C} = \{c : c = \tfrac{1}{2}(p-1), p \geqslant 3\}$ forms an asymptotic basis. Let $\mathscr{D} = \{d : d = a+c, a\in\mathscr{A}, c\in\mathscr{C}\}$. Then, by (15) and a theorem of Rohrbach [34] (an analogue of a theorem of Erdős [12], see Theorem 13 of [16, Chapter 1]),

$$\varliminf_{n\to\infty}\frac{D(n)}{n} > \tfrac{1}{2}. \tag{17}$$

The theorem now follows from (15) and (17) by Dirichlet's box principle.

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