Generalized Montgomery-Hooley formula; A survey

Robert C. Vaughan

February 1, 2023

- The progenitor of this talk is the following

Theorem 1 (Montgomery,[1970])

Define $V(x,Q)=\sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\psi(x;q,a)-\frac{x}{\phi(q)}\right|^2,$

$$\psi(x;q,a)=\sum_{\substack{n\le x\\ n\equiv a \pmod q}}\Lambda(n).$$

Let $A>0$ *and suppose that* $x>x_0(A)$. *When* $Q\le x$,

$$V(x,Q)=Qx\log x+O\big(Qx\log\tfrac{2x}{Q}\big)+O\big(x^2(\log x)^{-A}\big).$$

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- Some authors prefer ϑ or π. For consistency I will stay with ψ.

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$$V(x,Q)=Qx\log Q-cQX+O\big(Q^{\frac54}x^{\frac34}+x^2(\log x)^{-A}\big).$$

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Let $A>0$ *and suppose that* $x>x_0(A)$. *When* $Q\le x$,

$$V(x,Q)=Qx\log Q-cQX+O\big(Q^{\frac{5}{4}}x^{\frac{3}{4}}+x^2(\log x)^{-A}\big).$$

- One immediate observation. The conclusions become less precise as Q gets close to x.
- There is a good reason for this. When $q\approx x$ the number of residue classes is greater than the number of primes, so $\frac{x}{\phi(q)}$ will be a bad approximation.

- Earlier, Barban [1964] had established that if $B = B(A)$, $x > x_0(A)$, $Q \le x(\log x)^{-B}$, then

$$V(x, Q) \ll x^2(\log x)^{-A}$$

and this was refined by Davenport and Halberstam [1966] with $B = A + 5$ and Gallagher [1967] with $B = A + 1$.

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- Also Barban had apparently stated that

$$V(x,x) = x^2 \log x - cx^2 + O\big(x^2(\log x)^{-A}\big).$$

I have not seen this paper (Dokl. Akad. Nauk UzSSR). It is only three pages so presumably contains no details.

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- The results are not very surprising. After all if you average over enough things one should be able to establish a precise conclusion.
- Still, one can try to understand what ingredients are necessary for success and the extent to which they can be applied.
- Gallagher's proof is interesting because it reveals some of those ingredients.
- By using Dirichlet characters to pick out the residue classes, applying orthogonality and using the prime number theorem to deal with χ_0 one reduces to

$$\sum_{q\le Q}\frac{1}{\phi(q)}\sum_{\substack{\chi \pmod{q}\\ \chi\ne\chi_0}}|\psi(x;\chi)|^2.$$

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one replaces each character by the primitive character χ^* of conductor r which induces it to obtain essentially

$$\sum_{m\le Q}\frac{1}{\phi(m)}\sum_{1<r\le Q/m}\frac{1}{\phi(r)}\sum_{\chi^* \pmod r}|\psi(x;\chi^*)|^2.$$

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- Partial summation and the large sieve gives

$$\sum_{L<r\le Q/m}\frac{1}{\phi(r)}\sum_{\chi^* \pmod r}|\psi(x;\chi^*)|^2 \ll \left(\frac{x}{L}+\frac{Q}{m}\right)x\log x$$

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where $L=(\log x)^B$, say.
- The final ingredient is the Siegel-Walfisz theorem to cover the $r\le L$.

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- The standard initial step is to square and separate out the $p|q$ to obtain

$$V(x,Q) = S_0 + 2S_1 - 2S_2 + S_3 + O\big((Q+x)\log^4 x\big)$$

where

$$S_0 = Q\sum_{n\le x}\Lambda(n)^2,$$

$$S_1 = \sum_{q\le Q}\sum_{\substack{m<n\le x\\ m\equiv n \pmod q}}\Lambda(m)\Lambda(n),$$

$$S_2 = \sum_{q\le Q}\psi(x)\frac{x}{\phi(q)}, \quad S_3 = \sum_{q\le Q}\frac{x^2}{\phi(q)}.$$

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- All the sums here are easy to deal with except S_1. Hopefully one can obtain asymptotics for all and the main term for V(x,Q) will drop out from lower order terms.

- Hugh deals with

$$S_1 = \sum_{q \le Q} \sum_{\substack{m<n\le x \\ m \equiv n \pmod q}} \Lambda(m)\Lambda(n),$$

by writing this as

$$\sum_{h \le x} d_Q(h) R(x;h)$$

where

$$d_Q(h) = \sum_{\substack{q|h \\ q \le Q}} 1, \quad R(x;h) = \sum_{\substack{m,n \le x \\ n-m=h}} \Lambda(m)\Lambda(n).$$

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- One then appeals to Vinogradov's method in additive prime number theory to replace $R(x;h)$ by $\mathfrak{S}(h)(x-h)$ where $\mathfrak{S}$ is the appropriate singular series. The relevant theorem here actually is due to Lavrik [1960].

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- Although superseded by Hooley's idea I will return to this later.

- Hooley's idea is as follows. Let the large sieve deal with $q \le Q_0 = x(\log x)^{-B}$ and suppose $Q_0 < Q \le x$.

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$$= \sum_{m<n\le x} \sum_{\substack{r|n-m\\ r<\frac{n-m}{Q}}} \Lambda(m)\Lambda(n)$$

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$$\sum_{m<n\le x} \sum_{\substack{qr=n-m\\ Q<q\le x}} \Lambda(m)\Lambda(n) = \sum_{m<n\le x} \sum_{\substack{r|n-m\\ r<\frac{n-m}{Q}}} \Lambda(m)\Lambda(n) = \sum_{r<\frac{x}{Q}} \sum_{m<x-rQ} \Lambda(m) \sum_{\substack{m+rQ<n\le x\\ r|n-m}} \Lambda(n).$$

- Now the process can be completed *via* Siegel-Walfisz.

- In this talk I am not so concerned with refining these results, or speculation about sums like

$$\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\psi(x;q,a)-\frac{x}{\phi(q)}\right|^2$$

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- My main concern is the extent to which these ideas can be applied to functions significantly different from $\psi(x;q,a)$.
- Hooley wrote at least 20 papers in some of which these ideas are extended to a wide class of functions.
- The primes have the advantage that they are uniformly distributed into the reduced residue classes. Most sequences of number theoretic interest are not so well behaved. Even the square free numbers are deficient in this regard.

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- He requires an analogue of the Siegel-Walfisz theorem which, for some unfathomable reason, is labelled **Criterion U**. Let $\mathcal{S} \subset \mathbb{N}$ and

$$S(x;q,a) = \sum_{\substack{s\in\mathcal{S}, s\le x \\ s\equiv a \pmod q}} 1$$

and suppose that for $x > x_0(A)$ we have

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- The dependence of the main term on (q,a) rather than a is satisfactory for many applications, such as the squarefree numbers, but nevertheless signals a dependence on the large sieve.
- The final conclusion is that, when $Q \le x$,

$$\sum_{q\le Q}\sum_{a=1}^{q} \big|S(x;q,a) - f\big(q,(q,a)\big)x\big|^2 \ll Qx + x^2(\log x)^{-A}.$$

- By the way, in this regard the squarefree numbers have a substantial history. There are papers by Warlimont [1969], Orr [1969], [1971], Croft [1975], Warlimont [1972], [1980] and RCV [2005], and a very recent paper by Parry [2021] on squarefree k-tuples.

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- Let μ_k be the characteristic function of the k-free numbers,

$$Q_k(x;q,a) = \sum_{\substack{n\le x\\ n\equiv a \pmod q}} \mu_k(n),$$

$$f(q,a) = \sum_{\substack{m=1\\ (m^k,q)|a}}^{\infty} \frac{\mu(m)(m^k,q)}{m^k q},$$

$$V(x,Q) = \sum_{q\le Q}\sum_{a=1}^{q} |Q_k(x;q,a) - xf(q,a)|^2$$

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- In RCV [2005] it is shown that $V(x,Q) = A_k x^{\frac{1}{k}} Q^{2-\frac{1}{k}} +$

$$O\Big(x^{\frac{1}{2k}} Q^{2-\frac{1}{2k}} \exp\big(-F_1(\log(2x/Q))\big) + x^{1+\frac{1}{k}} \exp\big(-F_2(\log x)\big)\Big)$$

where the F_j are related to the zero free region of ζ.

$$f(q,a)=\sum_{\substack{m=1\\(m^k,q)|a}}^{\infty}\frac{\mu(m)(m^k,q)}{m^kq},$$

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  where the F_j are related to the zero free region of ζ.

- The method used are those described so far, pushed to the limit. Note the lack of uniformity as $Q\to x$ even though the k-frees have positive density.

- A question which seems to be significantly harder concerns the distribution of smooth numbers in arithmetic progressions. Let $\mathcal{A}(x,y)=\{n\le x: p|n\Rightarrow p\le y\}$

$$\Psi(x,y;q,a)=\sum_{\substack{n\in\mathcal{A}(x,y)\\ n\equiv a \pmod q}} .$$

and $\Psi_q(x,y)=\sum_{\substack{n\in\mathcal{A}(x,y)\\ (n,q)=1}} 1$ and $\Psi(x,y)=\Psi_1(x,y)$.

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- Harper [2012] has shown, for example, that when $\log\log x \ll \log y \le \log x$ and $1 \le Q \le \Psi(x,y)$,

$$V(x,y,Q) = \sum_{q\le Q} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \left|\Psi(x,y;q,a) - \frac{\Psi_q(x,y)}{\phi(q)}\right|^2,$$

$$\ll_A Q\Psi(x,y) + \Psi(x,y)^2\left((\log x)^{-A} + y^{-\delta}\right)$$

but as far as I am aware no asymptotic formula has been proved.

- At this point it is useful to introduce another perspective on the methods so far discussed.

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- A significant proportion of the work in the area had been consequent on the assumption of the generalised Riemann Hypothesis, and in Goldston and Vaughan [1996] an idea was introduced which, whilst facilitating the use of that hypothesis, might be thought of as being a backwards step. However it transpires that it plays a signicant rôle in some later work.

- Recall that a key ingredient to the original result is an estimate for

$$S_1 = \sum_{m<n\le x} \sum_{\substack{qr=n-m \\ q\le Q}} \Lambda(m)\Lambda(n)$$

and that Hugh's original method was based on Vinogradov's method.

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- Instead one can write directly

$$S_1 = \int_0^1 F(\alpha)|G(\alpha)|^2 d\alpha,$$

$$F(\alpha) = \sum_{q\le Q} \sum_{r\le x/q} e(\alpha qr), G(\alpha) = \sum_{n\le x} \Lambda(n)e(\alpha n).$$

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- The sum F is essentially trivial to estimate on minor arcs, so one can avoid Vinogradovs's method. Whilst not as simple as Hooley's, it has some advantage of flexibility and avoids the large sieve. Thus it opens up the possibility of dealing with sequences which are not so well distributed.

- This was exploited in RCV [1998a], [1998b]. Suppose a_n is a real sequence satisfying $\sum_{n\le x} a_n^2 \ll x$ and a Siegel-Walfisz condition $A(x;q,a) =$

$$\sum_{\substack{n\le x\\ n\equiv a \pmod q}} a_n = xf\big(q,(q,a)\big) + O\big(x/\Psi(x)\big).$$

Then $V(x,Q) = \sum_{q\le Q}\sum_{a=1}^{q} \left|A(x;q,a) - xf\big(q,(q,a)\big)\right|^2$

satisfies $V(x,Q) \sim Q\sum_{n\le x} a_n^2 - Qx\sum_{q=1}^{\infty} g(q)$ where

$$g(q) = \phi(q)\Bigg(\sum_{r|q} f(q,r)\mu(q/r)\Bigg)^2.$$

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- The quality of the result depends on the extent to which the second expression is an approximation to the first.

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and one would expect that

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is an approximation to the major arcs. Thus the estimate for V should relate to the minor arcs.

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- Although the hypothesis is false for $a_n = \Lambda(n)$ the conclusion can be adjusted, and one can recall the well known phenomenon that the minor arcs are the same size as the major arcs.
- For the k-frees the minor arcs are of smaller order so the two main terms almost cancel, which fits in with the previously stated result.

- Let me now advert to another Hooley paper, Hooley VIII [1998a]. Here he deals with a problem which is not directly of the kind which is our central interest, namely a third moment

$$\sum_{q\le Q}\phi(q)\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\psi(x;q,a)-\frac{x}{\phi(q)}\right)^3$$

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- Note the weight $\phi(q)$. This is necessary, since one is expecting that the sum over a is behaving roughly like

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- The $\phi(q)$ seems rather unnatural compared with a smooth weight, but it is there to alleviate some of the not inconsiderable difficulties that Hooley runs into.

- The method is to follow the pattern established for the second moment.

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- The hardest, coming from the product of three von Mangoldt functions, can be dealt with by Vinogradov's method.
- However the really big problem is to show that the main terms sum to 0. This is a major achievement and takes many pages. It also results in the paper being littered with quotations from Dante's Inferno.

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$$|\alpha - \tfrac{b}{r}| \le \frac{(\log x)^B}{rx},\ 1 \le b \le r \le R = (\log x)^B,\ (r,b) = 1,$$

$$G(\alpha) \sim G^*(\alpha) = \sum_{r \le R} \sum_{\substack{b=1 \\ (r,b)=1}}^{r} \frac{\mu(r)}{\phi(r)} \sum_{n \le x} e\big((\alpha - b/r)n\big).$$

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- If we rewrite the RHS as $\sum_{n \le x} e(\alpha n) \sum_{r \le R} \frac{\mu(r)}{\phi(r)} c_r(n)$ it makes sense to replace the approximation $x/\phi(q)$ by

$$\sum_{\substack{n \le x\\ n \equiv a \pmod q}} \sum_{r \le R} \frac{\mu(r)}{\phi(r)} c_r(n).$$

- Let $\Xi_R(n) = \sum_{r\le R} \frac{\mu(r)}{\phi(r)} c_r(n)$, $\rho(x;q,a) = \sum_{\substack{n\le x\\ n\equiv a \pmod q}} \Xi_R(n)$.

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- RCV [2003a]: if $x > x_0(A)$, $Q \le x$ and $R \le (\log x)^A$, then

$$\sum_{q\le Q} \sum_{a=1}^{q} \big(\psi(x;q,a) - \rho(x;q,a)\big)^2$$

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- In each case the result is uniform as $Q \to x$.
- Note also better than square root cancellation in the second result.

- Justification

$$\sum_{q\le Q} q \sum_{a=1}^{q} \big(\psi(x;q,a)-\rho(x;q,a)\big)^3$$
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$$\approx \phi(q)^{-1/2}x^{3/2} \dashrightarrow Q^{3/2}x^{3/2}.$$

- The simplest proof of $\sum_{q\leq Q}\sum_{a=1}^{q}\big(\psi(x;q,a)-\rho(x;q,a)\big)^2$

$$= Qx(\log x/R) - cQx + O\big(QxR^{-1/2} + x^2(\log x)^2R^{-1}\big).$$

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- However one can write

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$$+ 2\int_0^1 F(\alpha)|G(\alpha)-G^*(\alpha)|^2 d\alpha.$$
- The HL Method applies directly to the integral and shows that it is small compared with the main term.

- The treatment of $\sum_{q\le Q} q \sum_{a=1}^{q} \big(\psi(x;q,a)-\rho(x;q,a)\big)^3$ is to write $\Delta(n)=\Lambda(n)-\Xi_R(n)$

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and cube it out.
- Then the core part is

$$\sum_{q\le Q}\sum_{r,s}\sum_{\substack{l<m<n\le x\\ m-l=qr,\, n-m=qs}} \Delta(l)\Delta(m)\Delta(n)$$

- Let $E(\theta)=G(\theta)-G^*(\theta)$. Then this can be written as

$$\int_0^1\int_0^1 F(\alpha,\beta)E(\alpha)E(\beta-\alpha)E(-\beta)d\alpha d\beta.$$

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- Again the HL method is amenable.

- There are a number of other generalizations of these techniques.

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- For example Smith [2010] has established a version of the Montgomery-Hooley theorem when $\psi(x; q, a)$ is replaced by

$$\theta_K(x; q, a) = \sum_{\substack{\mathrm{N}\mathfrak{p} \le x \\ \mathrm{N}\mathfrak{p} \equiv a \pmod{q}}} \log \mathrm{N}\mathfrak{p}$$

on the assumption that K is a Galois extension of $\mathbb{Q}$.

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- There they establish an analogue of Montgomery-Hooley for function fields.

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- Dancs [2002] pushed the envelope by considering

$$V(x,Q)=\sum_{q\le Q}\sum_{a=1}^{q}\big(A(x;q,a)-\pi x f(q,a)\big)^2$$

with $a(n)=r(n)$, $r(n)=\text{card}\{x,y\in\mathbb{Z}^2 : x^2+y^2=n\}$, $f(q,a)=q^{-2}\,\text{card}\{x^2+y^2\equiv a \pmod q\}=$

$$\frac{1}{q^3}\sum_{b=1}^{q}S(q,b)^2e(-ab/q),\quad S(q,b)=\sum_{x=1}^{q}e(bx^2/q)$$

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$$\frac{1}{q^3}\sum_{b=1}^{q}S(q,b)^2e(-ab/q),\quad S(q,b)=\sum_{x=1}^{q}e(bx^2/q)$$

- For $Q\le x$ he obtains $V(x,Q)=8Qx\big(\log(x/Q)+C_1\big)$

$$+4Q^2\log Q+C_2Q^2+O(x^{5/3+\varepsilon}).$$

- A trickier example which was looked at by Motohashi [1973] in the special case $Q = x$ and in the general case by Pongsriiam [2012] (see also Pongsriiam and Vaughan [2018], [2021]).

$$A(x; q, a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} d(n).$$

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$$V(x,Q)=\sum_{q\le Q}\sum_{a=1}^{q}\big(A(x;q,a)-M(x;q,a)\big)^2.$$

- When $Q\le x$

$$V(x,Q)=\frac{Qx}{\pi^2}\left(\log\frac{Q^2}{x}\right)^3+$$

$$QxP(\log x,\log Q)+O\left(x^{\frac{7}{4}}(\log x)^3+Q^2(x/Q)^{\varepsilon}(\log x)^2\right)$$

where P is a polynomial of degree 2.

- Another curious example has been studied by Penyong Ding [2021]. He considers

$$A(x;q,a)=\sum_{\substack{n\le x\\ n\equiv a \pmod q}} r_3(n)$$

where $r_3(n)=\mathrm{card}\{l_1,l_2,l_3\in\mathbb{N}^3 : l_1^3+l_2^3+l_3^3\}$ and uses the approximation

$$\Gamma(4/3)^3 x\rho(q,a)q^{-3}$$

where $\rho(q,a)$ is the number of solutions of the congruence

$$l_1^3+l_2^3+l_3^3\equiv a \pmod q.$$

- The function $r_3(n)$ is somewhat mysterious since we don't know how
$$\sum_{n\le x} r_3(n)^2$$
behaves. The best that we know is that
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- We might expect that it is $\sim cx$, but Hooley has shown [1986] that if true the value of c is not obvious.
- For $r_3(n)$ it is natural to approach the question by using a variant of the HL method.

- Let $V(x, Q) =$

$$\sum_{q \le Q} \sum_{a=1}^{q} \left| \sum_{\substack{n \le x \\ n \equiv a \pmod{q}}} r_3(n) - \Gamma(4/3)^3 x \rho(q, a) q^{-3} \right|^2$$

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- Then the conclusion is

$$V(x,Q) = Q\sum_{n\le x} r_3(n)^2 - A_1 Qx + A_2 Q^{\frac{5}{3}}x^{\frac{1}{3}} + E$$

where

$$E \ll x^{\frac{10}{9}+\varepsilon}\left(\sum_{n\le x} r_3(n)^2\right)^{\frac{2}{3}} + Q^2(x/Q)^{\varepsilon}.$$

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$$E \ll x^{\frac{10}{9}+\varepsilon}\left(\sum_{n\le x} r_3(n)^2\right)^{\frac{2}{3}} + Q^2(x/Q)^{\varepsilon}.$$

- Here

$$A_1 = \Gamma(4/3)^6 \sum_{q=1}^{\infty} q^{-6} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |S_3(q,a)|^6$$

as expected.

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$$r_2(n) = card\{u, v \in \mathbb{N}^2 : u^3 + v^3 = n\}.$$

- Now, by the usual lattice point arguments we have

$$\sum_{n\le x} r_2(n) = Cx^{\frac{2}{3}} + O(x^{\frac{1}{3}}), \quad C = \frac{\Gamma(4/3)^2}{\Gamma(5/3)}$$

and by a celebrated theorem of Hooley [1963]

$$\sum_{n\le x} r_2(n)^2 \sim 2Cx^{\frac{2}{3}}.$$

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- We consider

$$V(x,Q)=\sum_{q\le Q}\sum_{a=1}^{q}\Bigg|\sum_{\substack{n\le x\\ n\equiv a \pmod q}} r_2(n)-\frac{\rho(q,a)}{q^2}Cx^{2/3}\Bigg|^2$$

- and show that when $x^{3/5+\varepsilon}<Q\le x$

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- This is consistent with our overall philosophy since the major arcs are $\ll Qx^{1/3}\log x$.

- It is noteworthy that the conclusion is deduced from a prior estimate for

$$\sum_{q\le Q}\sum_{a=1}^{q}\left|\sum_{\substack{n\le x\\ n\equiv a \pmod q}}\left(r(n)-\frac{2}{3}Cn^{-1/3}\mathfrak{S}(n;R)\right)\right|^2$$

where $\mathfrak{S}(n;R)$ is the truncated singular series

$$\mathfrak{S}(n;R)=\sum_{r\le R}\sum_{\substack{b=1\\ (b,r)=1}}^{r} r^{-2}S_3(r,b)^2e(-bn/r).$$

- In Brüdern and RCV [2022] we can also treat the case

$$r(n) = \text{card}\{u, v \in \mathbb{N}^2 : u^k + v^l = n\}$$

for various choices of $k < l$, namely

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- Let $\theta = \frac{1}{k} + \frac{1}{l}$,

$$C = \frac{\Gamma(1+1/k)\Gamma(1+1/l)}{\Gamma(1+1/k+1/l)},$$

$\rho(q, a)$ denote the number of solutions of $u^k + v^l \equiv a \pmod q$ and

$$V(x, Q) = \sum_{q \leq Q} \sum_{a=1}^{q} \left| \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} r(n) - \frac{\rho(q, a)}{q^2} C x^\theta \right|^2 .$$

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- Then $V(x, Q) \sim 2CQx^\theta$ for $x^{\theta-\eta} < Q \leq x^\theta$ for some $\eta > 0$ depending only on k, l.

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where $\lambda > 1/2$. I have taken the second moment here so that one can include examples such as $a(n) = \mu(n)$.
- By the way, I don't recall seeing it in the literature, but it is surely well known, and certainly easy to prove, that when $x(\log x)^{-A} \le Q = o(x)$ we have

$$\sum_{q\le Q}\sum_{a=1}^{q}\Bigg|\sum_{\substack{n\le x\\ n\equiv a \pmod q}} \mu(n)\Bigg|^2 \sim Q\sum_{n\le x}\mu(n)^2.$$

- **Question 1.** Is there a change of nature when

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- Indeed, are there any examples!

- Hooley [1974] has conjectured that

$$W(x,q)=\sum_{a=1}^{q}\left|\psi(x;q,a)-\frac{x}{\phi(q)}\right|^2\sim x\log q$$

and shown it p.p. q with $\frac{Q}{2}<q\le Q$, $\frac{x}{(\log x)^A}<Q\le x$.

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- Fiorilli and Martin [2020] have shown that this fails when q is small w.r.t. x. But surely Hooley would have been aware of this possibility and intended that q is large, perhaps $q>x^\delta$.
- Both Hooley III [1975], and Friedlander and Goldston [1996] have extended the range for Q on GRH.

- Also it can be shown RCV [2001] that if

$$W(x,q)=\sum_{a=1}^{q}\left|\psi(x;q,a)-\frac{x}{\phi(q)}\right|^2$$

and

$$U(x,q)=x\log q-x\Big(\gamma+\log 2\pi+\sum_{p|q}\frac{\log p}{p-1}\Big),$$

then there is an $F(y)$ a bit smaller than $y^{-1/2}$ so that.

$$M_k(x,Q)=\sum_{Q/2<q\le Q}|W(x,q)-U(x,q)|^k$$
$$\ll Qx^kF(x/Q)^k+Qx^k(\log x)^{-A}.$$

- **Question 2.** What can be said about

$$W(x,q)=\sum_{a=1}^{q}\left|A(x;q,a)-\rho(q,a)\Psi(x)\right|^2$$

when we know a Montgomery-Hooley estimate for

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- There are sequences a_n known for which there is an asymptotic formula for $W(x,q)$ when, say, $x^\theta < q \le x$. See examples by Lau and Zhao [2012], Nunes [2014].

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- There are sequences a_n known for which there is an asymptotic formula for $W(x,q)$ when, say, $x^\theta<q\le x$. See examples by Lau and Zhao [2012], Nunes [2014].
- Is there a much more general principle lurking here?
- Presumably there should be an estimate for almost all q. I have not checked, but Hooley may have explored this to some extent in some of his many papers on Barban-Davenport-Halbestam.

- During a previous talk Shaprlinski asked about restricting q to being prime in $V(x, Q)$. Certainly the technique using the Hardy-Littlewood method could deal with this.

- During a previous talk Shaprlinski asked about restricting q to being prime in $V(x,Q)$. Certainly the technique using the Hardy-Littlewood method could deal with this.
- Indeed Brüdern and Wooley [2011] have dealt with more general and thinner sequences including the squares.

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