Background to Guth-Maynard

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Motivation

- The question is, how do we deal with

$$\pi(x+h)-\pi(x),$$

or equivalently

$$\psi(x+h)-\psi(x)$$

when

$$h = x^\theta \text{ with } \theta < 1.$$

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- The classical prime number theorem is not good enough, so try the explicit formula
$$\psi_0(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1-x^{-2})$$

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- Unfortuantely here there is a form of the Heisenberg uncertainty principle - to obtain finer detail of a function we need a wider range of information from the Fourier transform.

- More precisely to control this we need to truncate at height T.

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- and so when $h \ll x$ we have

$$\psi(x+h) - \psi(x) = h - \sum_{\substack{\rho \\ |\gamma| \le T}} \frac{(x+h)^\rho - x^\rho}{\rho} + O\big(\log x + xT^{-1}\log^2(xT)\big)$$

- Clearly we will need to take T just a bit larger than $x(\log x)^2/h$. This emphasizes that the shorter h is longer T has to be.

- The crucial part is

$$\sum_{\substack{\rho \\ |\gamma|\leq T}} \frac{(x+h)^\rho - x^\rho}{\rho} = \sum_{\substack{\rho \\ |\gamma|\leq T}} \int_x^{x+h} u^{\rho-1} du.$$

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- On RH the integral is $\ll hx^{-1/2}$, so we certainly need to understand $N(T)$, the number of zeros ρ with $0 < \gamma \le T$. Note that by symmetry and the functional equation ρ is a zero of ζ iff any one of $1-\rho$, $\overline{\rho}$ and $1-\overline{\rho}$ is.

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- The $x^{1/2}$ we can treat as on RH. That leaves

$$hx^{-1}\sum_{0<\gamma\le T}\int_{1/2}^{\beta} x^u\log x du = hx^{-1}\int_{1/2}^{1} N(u,T)x^u\log x du$$

where $N(u,T)=\text{card}\{\rho:\beta\ge u, 0<\gamma\le T\}$.

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$$\ll (x/h)^{A-1}(\log x)^{B'}\int_{1/2}^{1}(h^A x^{1-A})^u du$$

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- Obviously we have to put in information about zeros near the 1-line.

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- which is still not good enough since $h^A x^{1-A}$ and T are both some positive power of x.
- Until about 1938 the best zero-free region that was known was due to Littlewood, namely that all zeros of zeta satisfy

$$\beta \le 1 - C\log\log(3+|\gamma|)/\log(3+|\gamma|).$$

- The extra loglog gives a bound

$$(x/h)^{A-1}(h^A x^{1-A})^{1-C(\log\log T)/\log T}(\log x)^{B''}$$
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- $\log T$ should be of order $\log x$. Since B'' is rather large and C is rather small, in order to get a result

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will have to be largish. Substituting $h = x^{1-\delta}$ and $T = x/h$ gives $(1-\delta A)/\delta$ so δ has to be small.

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- Thus Hoheisel needed to take $h = x^\theta$ with

$$\theta > 1 - \frac{1}{33000}.$$

[The other constraint $\theta > 1 - 1/A$ is far less demanding.]

- By working very hard Heilbronn reduced this to

$$\theta > 1 - \frac{1}{250}.$$

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- To summarise. The fundamental ingredients are the explicit formula, an improved zero-free region and a zero density estimate of the form (where $A \ge 2$)

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- Note that we could replace this by $T\log T$ when $u \le 1 - 1/A$.

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- In particular the ξ function is an entire function of order 1 and so
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- It is also fundamental to the zero-free region.
- Thus it is important to have an estimate for the size of $\zeta(\sigma + it)$ when σ is close to 1 and t is large.

- To get the Chudakove style zero free region we need a bound of the kind

$$\zeta(\sigma + it) \ll \big(\log(2+|t|)\big)^{\theta}(2+|t|)^{(1-\sigma)^{1/\theta}}$$

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- and by partial summation it suffices to treat sums

$$\sum_{n=N+1}^{N+M} n^{it} = \sum_{n=1}^{M} e^{it\log(N+n)} = N^{it}\sum_{n=1}^{M}\exp\big(it\log(1+n/N)\big)$$

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- A standard way of dealing with such is to perturb it

$$\begin{aligned}\sum_{n=1}^{M} e\left(\sum_{k=1}^{K} \alpha_k n^k\right) &= \sum_{n=r+1}^{M+r} e\left(\sum_{k=1}^{K} \alpha_k n^k\right) + O(r) \\ &= \sum_{n=1}^{M} e\left(\sum_{k=1}^{K} \alpha_k (n+r)^k\right) + O(r)\end{aligned}$$

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is a polynomial

$$\sum_{k=0}^{K} \alpha_k^*(r) n^k$$

in n and the coefficients α_k^* vary as r varies.

- $\displaystyle\sum_{n=1}^{M} e\left(\sum_{k=1}^{K} \alpha_k (n+r)^k\right) + O(r).$
- Now
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in n and the coefficients α_k^* vary as r varies.
- Then one can average over the r and look at
$$\frac{1}{R}\sum_{r=1}^{R}\left|\sum_{n=1}^{M} e\left(\sum_{k=1}^{K}\alpha_k^*(r)n^k\right)\right|^{2b}.$$
- Now we can hope to relate this to
$$\int_{[0,1]^k}\left|\sum_{n=1}^{M} e\left(\sum_{k=1}^{K} x_k n^k\right)\right|^{2b} dx_1 \dots dx_K.$$

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- By the orthogonality relationship

$$\int_0^1 e(xh)dx = \begin{cases} 1 & h = 0, \\ 0 & h \in \mathbb{Z} \setminus \{0\}, \end{cases}$$

this is the number of solutions of

$$m_1 + \cdots + m_b = n_1 + \cdots + n_b$$
$$\vdots$$
$$m_1^k + \cdots + m_b^k = n_1^k + \cdots + n_b^k$$
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$$m_1^K + \cdots + m_b^K = n_1^K + \cdots + n_b^K.$$
- Any non-trivial estimate for this is known as the "Vinogradov Mean Value Theorem" and this is the core part of Chapter 24.

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- Moreover they collectively give

$$N(u, T) \ll T^{\frac{8(1-u)}{5}} (\log T)^{14}$$

and so we get the desired estimate when $\theta > \frac{5}{8}$.

- In 1972 Huxley refined Montgomery's method to give

$$N(u, T) \ll T^{\frac{(5u-3)(1-u)}{u^2+u-1}} (\log T)^9$$

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- **Important Remark**. These exponents all give an $A < \frac{12}{5}$ when u is not close to $\frac{3}{4}$. So you only have to improve the zero density estimates in the neighbourhood of $u = \frac{3}{4}$. This is what Guth and Maynard do. Thus to get the final result you also need to know the earlier work!

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- Then for $\sigma > 1$

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- In that half-plane $M(s)\zeta(s)$ is close to 1 if K is quite large and we might hope that persists when $1/2 < \operatorname{Re} s = \sigma < 1$. Indeed this can be shown to be true on RH.
- On the other hand $M(s)\zeta(s)$ will be 0 when $\zeta(s) = 0$.

- One way this can be realised is through the transformation

$$\sum_{n=1}^{\infty} a(n)n^{-s}e^{-n/Y}$$
$$= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} M(s+w)\zeta(s+w)Y^w\Gamma(w)dw.$$

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- In view of the bounds given by Corollaries 10.5 and 10.10, Lemma 10.15 and (C.19) of Theorem C.1 we are able to move the path of integration to the line $\operatorname{Re} w = \frac{1}{2} - \sigma$ and pick up the residues at 0 and $w = 1 - s$.

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- Thus

$$\sum_{n=1}^{\infty} a(n)n^{-s}e^{-n/Y} =$$

$$M(1)Y^{1-s}\Gamma(1-s) + M(s)\zeta(s) +$$

$$\int_{-\infty}^{\infty} M(\tfrac{1}{2}+it+iv)\zeta(\tfrac{1}{2}+it+iv)Y^{\frac{1}{2}-\sigma+iv}\Gamma(\tfrac{1}{2}-\sigma+iv)\tfrac{dv}{2\pi}.$$

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- $$\sum_{n=1}^{\infty} a(n)n^{-s}e^{-n/Y} = M(1)Y^{1-s}\Gamma(1-s) + M(s)\zeta(s) + \int_{-\infty}^{\infty} M(\tfrac12+it+iv)\zeta(\tfrac12+it+iv)Y^{\frac12-\sigma+iv}\Gamma(\tfrac12-\sigma+iv)\frac{dv}{2\pi}.$$
- If $\zeta(\rho)=0$ we have

$$e^{-\frac{1}{Y}} + \sum_{n=K+1}^{\infty} a(n)n^{-\rho}e^{-\frac{n}{Y}} = M(1)Y^{1-\rho}\Gamma(1-\rho) + \int_{-\infty}^{\infty} M(\tfrac12+i\gamma+iv)\zeta(\tfrac12+i\gamma+iv)Y^{\frac12-\beta+iv}\Gamma(\tfrac12-\beta+iv)\frac{dv}{2\pi}.$$

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  $$\int_{-\infty}^{\infty} M(\tfrac12+i\gamma+iv)\zeta(\tfrac12+i\gamma+iv)Y^{\frac12-\beta+iv}\Gamma(\tfrac12-\beta+iv)\tfrac{dv}{2\pi}.$$
- We have $\Gamma(1-\rho) \ll e^{-|\gamma|}$, so for $K=T\le Y\ll T^2$, $Z=Y(\log T)^2$ and $(\log T)^2\le\gamma\le T$ have
  $$e^{-1/Y} + \sum_{K<n\le Z} a(n)n^{-\rho}e^{-n/Y} = O(T^{-1}) +$$
  $$\int_{-\infty}^{\infty} M(\tfrac12+i\gamma+iv)\zeta(\tfrac12+i\gamma+iv)Y^{\frac12-\beta+iv}\Gamma(\tfrac12-\beta+iv)\tfrac{dv}{2\pi}.$$

- $$e^{-1/Y} + \sum_{K<n\le Z} a(n)n^{-\rho}e^{-n/Y} = O(T^{-1}) + \int_{-\infty}^{\infty} M(\tfrac12 + i\gamma + iv)\zeta(\tfrac12 + i\gamma + iv)Y^{\frac12-\beta+iv}\Gamma(\tfrac12 - \beta + iv)\frac{dv}{2\pi}.$$

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$$1 \ll \left|\sum_{K<n\le Z} a(n)n^{-\rho}e^{-n/Y}\right| +$$
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- We can partition the set $\mathcal{R}$ of zeros counted by $N(u,T)$ into two classes $\mathcal{R}_1$ and $\mathcal{R}_2$, in the first of which the first sum is larger and the second contains the remainder, and let their counts be $N_1(u,T)$ and $N_2(u,T)$. This gives tremendous flexibility.

- $1 \ll \Big| \sum_{K<n\le Z} a(n) n^{-\rho} e^{-n/Y} \Big| +$

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- In the proof of Ingham's bound we use

$$N_1(u,T) \ll \sum_{\rho\in\mathcal{R}} \Big| \sum_{K<n\le Z} a(n) n^{-\rho} e^{-n/Y} \Big|^2$$

and $N_2(u,T) \ll$

$$Y^{\frac{2}{3}-\frac{4u}{3}} \int_{-\infty}^{\infty} \sum_{\rho\in\mathcal{R}} \Big| M\big(\tfrac{1}{2}+i\gamma+iv\big)\zeta\big(\tfrac{1}{2}+i\gamma+iv\big)\Big|^{\frac{4}{3}} e^{-|v|} dv.$$

This makes best use of the various mean value theorems we have for Dirichlet polynomials.

- We already saw in the outline of the Chudakov-Korobov-Vinogradov zero free region that one needs to relate discrete mean values to continuous mean values and we will see more of this here.

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- Those who have seen the large sieve, say in Math 571 or Math 572 will be aware of the inequality

$$\sum_{r=1}^{R} |S(x_r)|^2 \ll (N+\delta^{-1}) \int_0^1 |S(x)|^2 dx$$

where

$$S(x) = \sum_{m=1M+1}^{M+N} c_n e(nx)$$

and

$$\|x_q - x_r\| := \min_{n\in\mathbb{Z}} |x_q - x_r - n| \ge \delta.$$

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- A method of Gallagher which will give this can be adapted to many situations, such as sums over zeros ρ.

- $f(x) - f(y) = \int_y^x f'(v)dv$

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- This can be rearranged to give

$$2\delta f(x) =$$
$$\int_{x-\delta}^{x+\delta} f(y)dy + \int_{x-\delta}^{x+\delta} f'(v)sgn(x-v)(\delta - |x-v|)dv$$

- $f(x) - f(y) = \int_y^x f'(v)dv$
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- $|f(x)| \leq \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} |f(y)|dy + \frac{1}{2} \int_{x-\delta}^{x+\delta} |f'(v)|dv$

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- Say something about spacing and two dimensions.

- To return to Ingham.

$$N_1(u,T) \ll \sum_{\rho\in\mathcal{R}} \Big| \sum_{K<n\le Z} a(n) n^{-\rho} e^{-n/Y} \Big|^2$$

and $N_2(u,T) \ll$

$$Y^{\frac{2}{3}-\frac{4u}{3}} \int_{-\infty}^{\infty} \sum_{\rho\in\mathcal{R}} \big| M(\tfrac12 + i\gamma + iv)\zeta(\tfrac12 + i\gamma + iv)\big|^{\frac43} e^{-|v|} dv.$$

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- The first inequality will give $N_1(u,T) \ll Y^{2(1-u)}(\log T)^C$, which looks promising, especially if we could take $Y \approx T$.
- The second inequality is trickier. If we knew the mean-value version of the Lindelöf hypothesis $\zeta(1/2+it) \ll_\varepsilon (2+|t|)^\varepsilon$, namely

$$\int_0^T |\zeta(1/2+it)|^k dt \ll_{\varepsilon,k} T^{1+\varepsilon},$$

then we could also deduce $N_2(u,T) \ll T^{2(1-u)+\varepsilon}$. Thus Lindelöf $\Rightarrow$ MVLindelöf $\Rightarrow$ Density Hypothesis $\Rightarrow \theta > 1/2$.

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- This motivates the $\frac{4}{3}$ in $N_2(u,T) \ll$

$$Y^{\frac{2}{3}-\frac{4u}{3}} \int_{-\infty}^{\infty} \sum_{\rho\in\mathcal{R}} \left|M\left(\tfrac{1}{2}+i\gamma+iv\right)\zeta\left(\tfrac{1}{2}+i\gamma+iv\right)\right|^{\frac{4}{3}} e^{-|v|} dv.$$

since we can use Hölder's inequality to give

$$\int_0^T \left|M\left(\tfrac{1}{2}+it\right)\zeta\left(\tfrac{1}{2}+it\right)\right|^{\frac{4}{3}} dt \le$$

$$\left(\int_0^T \left|M\left(\tfrac{1}{2}+it\right)\right|^2 dt\right)^{\frac{2}{3}} \left(\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 dt\right)^{\frac{1}{3}}.$$

- The Huxley proof is somewhat more sophisticated and involves in addition a version of mean value theorems called a large values theorem. This is where the breakthrough occurs in the Guth-Maynard paper.