# HEATH-BROWN JLMS1979

## 1. Introduction and Notation

This is a write-up of my notes on Heath-Brown's paper in the special case of the Riemann zeta function. Let $\mathcal{G}$ be a set of real numbers $g \in [-T, T]$. When $\mathcal{G}$ has the property that for each pair $g, g' \in \mathcal{G}$ with $g \neq g'$ we have $|g - g'| \geq 1$ we call $\mathcal{G}$ "1-separated". Also let $R = \operatorname{card}(\mathcal{G})$. Clearly if $\mathcal{G}$ is 1-separated, then $R \leq 2T + 1$. The main theorems or consequences of the paper are essentially as follows.

**Theorem 1.1.** *There is a positive constant $C$ such that when $\mathcal{G} \subset [-T, T]$ is 1-separated we have*

$$\sum_{\mathbf{g} \in \mathcal{G}^2} \left| \sum_{n=1}^{N} n^{i(g_1 - g_2)} \right|^2 \ll RN^2 + (R^{5/4} N T^{1/2} + NR^2) \exp(C \log T / \log\log T)$$

This is clearly best possible when $R = 1$. I expect that in many applications we have

$$N \gg \left(R^{1/4} T^{1/2} + R\right) \exp(C \log T / \log\log T)$$

so that the first term dominates. For large $R$ I do wonder if the term $RN^2$ ought to be replaced with something smaller. It leads to the term $R^{1/2}N$ in the theorem below, and I suspect that there the $R^{1/2}N$ ought to be $N$. The factor $\exp(C \log T / \log\log T)$ occurs solely because of the $D(P)^2$ in Lemma 2.4 and maybe there is a way of replacing it with a power of $\log T$. Note that it has nothing to do with the spacing of the elements of $\mathcal{G}$.

The superficially more general bound

$$\sum_{\mathbf{g} \in \mathcal{G}^2} \left| \sum_{n=1}^{N} a_n n^{i(g_1 - g_2)} \right|^2 \ll \left(RN^2 + \left((R^{5/4} N T^{1/2} + NR^2) \exp(C \log T / \log\log T)\right)\right) \max_n |a_n|^2$$

which holds when the $a_n$ are arbitrary complex numbers is a trivial consequence of Theorem 1.1 and Lemma 2.2.

**Theorem 1.2.** *There is a positive constant $C$ such that when $\mathcal{G} \subset [-T, T]$ is 1-separated and the $a_n$ are arbitrary complex numbers we have*

$$\sum_{g\in\mathcal{G}} \left| \sum_{n=1}^{N} a_n n^{ig} \right|^2 \ll$$
$$\left( R^{1/2}N + \left(R^{5/8}N^{1/2}T^{1/4} + RN^{1/2}\right) \exp\left( \frac{C\log T}{\log\log T} \right) \right) \sum_{n=1}^{N} |a_n|^2.$$

## 2. Lemmas

The following lemma is central in that the trace of a matrix is both the sum of the diagonal terms of a square matrix and the sum of its eigenvalues. Also the eigenvalues of a Hermitian matrix are non-negative and the eigenvalues of $\mathcal{H}^k$ are the $k$-the powers of the eigenvalues of $\mathcal{H}$.

**Lemma 2.1.** *Suppose that $\mathcal{H}$ is a $K \times K$ Hermitian matrix, $\tau_k$ is the trace of $\mathcal{H}^k$ and $\lambda$ is an eigenvalue of $\mathcal{H}$. Then, for $k \in \mathbb{N}$,*

$$\lambda \ll \frac{\tau_1}{K} + \left( \tau_k - \frac{\tau_1^k}{K^{k-1}} \right)^{1/k}.$$

This is just Hölder's inequality in disguise and really does not have anything to do with eigenvalues. Note that it is best possible when the eigenvalues are the same, or when there is one dominant eigenvalue and $K$ is large.

*Proof.* Let $\lambda_1, \ldots, \lambda_K$ be $K$ non-negative real numbers. Then

$$(\lambda_2 + \cdots \lambda_K)^k \le (K-1)^{k-1}(\lambda_2^k + \cdots + \lambda_K^k).$$

If we suppose that $\tau_k = \lambda_1^k + \cdots + \lambda_K^k$, then

$$(\tau_1 - \lambda_1)^k \le (K-1)^{k-1}(\tau_k - \lambda_1^k),$$

and we also have

$$\tau_1^k \le K^{k-1}\tau_k.$$

Thus if $\lambda = \lambda_1$ and $\lambda_1 \ll \frac{\tau_1}{K}$, then we are done. Thus we may suppose that

$$\tau_1 \le \delta K \lambda_1$$

where $\delta$ is chosen so that

$$1 - (1 - 1/K)^{1-1/k} < \frac{1}{\delta K} < 1.$$

From above we have

$$(K-1)^{1-k}(\tau_1 - \lambda_1)^k + \lambda_1^k - K^{1-k}\tau_1^k \le \tau_k - \tau_1^k K^{1-k}.$$

Viewing the left hand side as a function of $\tau_1$ but keeping $\lambda_1$ fixed we see that it is decreasing for $\tau_1 \le K\lambda_1$. Hence

$$(K-1)^{1-k}(\tau_1-\lambda_1)^k + \lambda_1^k - K^{1-k}\tau_1^k < 0$$

□

The following lemma is Lemma 2 of Jutila [1977].

**Lemma 2.2.** *Suppose that* $|a_n| \le Bb_n$ $(1 \le n \le N)$ *and* $\mathcal{G}$ *is a finite subset of* $\mathbb{R}$. *Then*

$$\sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{n=1}^{N} a_n n^{ig_1-ig_2} \right|^2 \le B^2 \sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{n=1}^{N} b_n n^{ig_1-ig_2} \right|^2.$$

*Proof.* The left hand side is

$$\sum_{\mathbf{g}\in\mathcal{G}^2} \sum_{m=1}^{N} \sum_{n=1}^{N} \overline{a_m} a_n (n/m)^{ig_1-ig_2} = \sum_{m=1}^{N} \sum_{n=1}^{N} \overline{a_m} a_n \left| \sum_{g\in\mathcal{G}} (n/m)^{ig} \right|^2.$$

We then apply the stated inequality for $a_n$ and reverse the process. □

The following lemma is essentially Lemma 3 of Heath-Brown [1979]. The proof is simpler, and also we avoid some of the complications arising when characters are included.

**Lemma 2.3.** *Suppose that* $J, K, M, N, \sigma \in \mathbb{R}$ *with* $K \ge J \ge 0$, $N > M \ge 0$, *and suppose further that* $\mathcal{G}$ *is a finite subset of* $\mathbb{R}$. *Then*

$$\sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{M<n\le N} a_n n^{-\sigma-i(g_1-g_2)} \right|^2$$
$$\le \frac{1}{\sum_{J\le p\le K} p^{-2\sigma}} \sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{JM<l\le KN} A(l) l^{-\sigma-i(g_1-g_2)} \right|^2$$

*where*

$$A(l) = \sum_{\substack{pn=l \\ J<p\le K \\ M<n\le N}} |a_n|.$$

Note for future reference that

$$\sum_{J\le p\le K} p^{-1} = \log\frac{\log K}{\log J} + O\big((\log J)^{-2}\big) \gg \frac{\log(K/J)}{\log J} \qquad (2.1)$$

when $K \ge 2J$ and $J$ is sufficiently large, and

$$A(l) \le \frac{\log KN}{\log K} \tag{2.2}$$

when $|a_n| \le 1$ $(M < n \le N)$.

*Proof.* We evaluate

$$\int_0^1 \sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{M<n\le N} a_n n^{-\sigma-i(g_1-g_2)} \right|^2 \left| \sum_{J\le p\le K} e(\alpha p) p^{-\sigma-i(g_1-g_2)} \right|^2 d\alpha$$

in two different ways.

First of all by Parseval's identity it is

$$\sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{M<n\le N} a_n n^{-\sigma-i(g_1-g_2)} \right|^2 \sum_{J\le p\le K} p^{-2\sigma}.$$

On the other hand the integrand is

$$\sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{JM<l\le KN} \Bigg( \sum_{\substack{pn=l\\ J<p\le K\\ M<n\le N}} a_n e(\alpha p) \Bigg) l^{-\sigma-i(g_1-g_2)} \right|^2.$$

The coefficient of $l^{-\sigma-i(g_1-g_2)}$ here is bounded by $A(l)$. Hence By Lemma 2.2 the above is at most

$$\sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{JM<l\le KN} A(l) l^{-\sigma-i(g_1-g_2)} \right|^2$$

□

We now have a version of Lemma 4 of Heath-Brown [1979]. Let

$$S_\sigma(N) = \sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{N<n\le 2N} n^{-\sigma-i(g_1-g_2)} \right|^2 \tag{2.3}$$

and in particular

$$S(N) = S_{\frac{1}{2}}(N) \tag{2.4}$$

**Lemma 2.4.** *Suppose that $N, P, \sigma \in \mathbb{R}$, $N \ge 0$, $P \ge 4N^2$ and $\mathcal{G}$ is a finite subset of $\mathbb{R}$ with* $\mathrm{card}(\mathcal{G}) = R$. *Then*

$$S_\sigma(N)^2 \ll R^2 D(P)^2 (P/N^2)^{2\sigma-1} S_\sigma(P)$$

*where*

$$D(P) = \max_{n\le P} d(n).$$

*Proof.* By Cauchy's inequality

$$S_\sigma(N)^2 \le R^2 \sum_{\mathbf{g}\in\mathcal{G}^2} \Big| \sum_{N<n\le 2N} n^{-\sigma-i(g_1-g_2)} \Big|^4$$

$$= R^2 \sum_{\mathbf{g}\in\mathcal{G}^2} \Big| \sum_{N^2<n\le 4N^2} a_n n^{-\sigma-i(g_1-g_2)} \Big|^2$$

where

$$a_n = \sum_{\substack{lm=n\\ N<l\le 2N\\ N<m\le 2N}} 1.$$

By Lemma 2.2 this is

$$\ll R^2 D(4N^2) \sum_{\mathbf{g}\in\mathcal{G}^2} \Big| \sum_{N^2<n\le 4N^2} n^{-\sigma-i(g_1-g_2)} \Big|^2.$$

Let

$$\theta = 2^{2/3}.$$

Then we split the sum over $n$ into three intervals

$$(N^2, \theta N^2], \quad (\theta N^2, \theta^2 N^2], \quad [\theta^2 N^2, 4N^2]$$

By Cauchy's inequality once more

$$\Big| \sum_{N^2<n\le 4N^2} n^{-\sigma-i(g_1-g_2)} \Big|^2 \le 3 \sum_{j=1}^{3} \Big| \sum_{\theta^{j-1}N^2<n\le\theta^j N^2} n^{-\sigma-i(g_1-g_2)} \Big|^2.$$

We now apply Lemma 2.3 to each of these sums, with $M, N$ replaced by $\theta^{j-1}N^2, \theta^j N^2$ and $J = \theta^{1-j}PN^{-2}$, $K = 2\theta^{-j}PN^{-2}$. Note that in each case

$$\sum_{J<n\le K} n^{-2\sigma} \gg (2-\theta)\theta^{-j}(P/N^2)^{1-2\sigma}.$$

□

Our next Lemma is a refinement of an argument of Jutila [1977]. The reason for taking $\sigma = \frac{1}{2}$ here and subsequently is because then the "critical line" for $\zeta(\frac{1}{2} + i(g_1 - g_2) + s)$ below is the 0-line.

**Lemma 2.5.** *Suppose that $\mathcal{G}$ is a 1-separated subset of $[0, T]$, $H \in \mathbb{N}$, $H \ge C \log T$ for a suitable positive constant $C$, and $(MN)^{1-\frac{1}{H}} \gg H(H+T)T^{\frac{2}{H}}$. Then*

$$S(N) \ll RN + (\log H)^2(\log(2M))^4 S(M)$$

*and in the special case* $N^{1-\frac{1}{H}} \gg H(H+T)T^{\frac{2}{H}}$ *we have*

$$S(N) \ll RN.$$

We remark that the condition $H \geq C\log T$ and the observation that in practice we may assume that $N \leq T^{C_1}$ for ssome positive constnat $C_1$ means that we really only need $MN \gg T\log T$.

*Proof.* Let

$$f : [0,\infty) \to [0,\infty) : f(x) = \exp\left(-\left(\frac{x}{2N}\right)^H\right) - \exp\left(-\left(\frac{x}{N}\right)^H\right).$$

Then $f$ has single maximum, $f(0) = 0$, $\lim_{x\to\infty} f(x) = 0$,

$$f(N) = \exp(-2^{-H}) - e^{-1} \geq e^{-1/2} - e^{-1} \gg 1$$

and

$$f(2N) = e^{-1} - exp(-2^H) \geq e^{-1} - e^{-2} \gg 1$$

so that $f(x) \gg 1$ $(N \leq x \leq 2N)$. Moreover

$$f(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \big((2N/x)^s - (N/x)^s\big)H^{-1}\Gamma(s/H)ds.$$

Thus, by Lemma 2.2

$$S(N) \ll \sum_{\mathbf{g}\in\mathcal{G}^2}\left|\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\zeta\big(\tfrac{1}{2}+i(g_1-g_2)+s\big)\big((2N)^s - N^s\big)\Gamma(\tfrac{s}{H})\tfrac{ds}{H}\right|^2.$$

We now move the line of integration to $\operatorname{Re} s = -H/2$, picking up a contribution from the reside at the simple pole of the integrand at $s = \frac{1}{2} - i(g_1 - g_2)$. Note that the integrand has a removable singularity at $s = 0$. We then apply the functional equation for the zeta function. Thus the integral above becomes

$$\mathcal{R} + \mathcal{I}_1$$

where

$$\mathcal{R} = \big((2N)^{\frac{1}{2}-i(g_1-g_2)} - N^{\frac{1}{2}-i(g_1-g_2)}\big)H^{-1}\Gamma\left(\frac{\frac{1}{2}-i(g_1-g_2)}{H}\right)$$
$$\ll N^{\frac{1}{2}}H^{-1}\exp\left(-\frac{\pi|g_1-g_2|}{2H}\right),$$

$\mathcal{I}_1 =$

$$\frac{1}{2\pi i}\int_{-\frac{H}{2}-i\infty}^{-\frac{H}{2}+i\infty} G\big(\tfrac{1}{2}+i(g_1-g_2)+s\big)\zeta\big(\tfrac{1}{2}-i(g_1-g_2)-s\big)(2^s-1)N^s\Gamma(\tfrac{s}{H})\tfrac{ds}{H}$$

and

$$G(z) = \pi^{z-\frac{1}{2}}\frac{\Gamma\big((1-z)/2\big)}{\Gamma(z/2)} = \pi^{-\frac{1}{2}}2^z\Gamma(1-z)\sin\frac{\pi z}{2}.$$

Unless $N^{1-\frac{1}{H}} \gg H(H+T)T^{\frac{2}{H}}$ we split the series

$$\zeta\big(\tfrac{1}{2}-i(g_1-g_2)-s\big) = \sum_{n=1}^{\infty} n^{-\frac{H+1}{2}+i(g_1-g_2-t)}$$

at $M$ and move the portion $n \le M$ to the 0-line. In the case $N^{1-\frac{1}{H}} \gg H(H+T)T^{\frac{2}{H}}$ we simply take $M = 1$ in the following argument. A careful analysis of the gamma factors shows that the portion on the $-\frac{H}{2}$-line contributes

$$\begin{aligned}&\ll \int_{-\infty}^{\infty}(H+|g_1-g_2+t|)^{\frac{H}{2}}e^{-\frac{H}{2}}M^{\frac{1-H}{2}}N^{-\frac{H}{2}}\exp\left(-\frac{\pi|t|}{2H}\right)\frac{dt}{H}\\ &= \int_{-\infty}^{\infty}(H+|g_1-g_2+Hu|)^{\frac{H}{2}}e^{-\frac{H}{2}}M^{\frac{1}{2}}(MN)^{-\frac{H}{2}}\exp\left(-\frac{\pi|u|}{2}\right)du\\ &\ll N^{-\frac{1}{2}}T^{-1}.\end{aligned}$$

The portion now on the 0-line contributes

$$\ll \int_{-\infty}^{\infty}\left|\sum_{n\le M} n^{-\frac{1}{2}+i(g_1-g_2)+it}\right| |(2^{it}-1)\Gamma(\tfrac{it}{H})|\tfrac{dt}{H}.$$

The parts of this with $|t| \ge H^2$ contribute (for the bound for the *Gamma* function see (C.19) on page 523 of [2006]

$$\ll M^{\frac{1}{2}}\int_H^{\infty} u^{-\frac{1}{2}}e^{-\pi u/2}du \ll M^{1/2}H^{-\frac{1}{2}}e^{-\pi H/2} \ll T^{-2}.$$

The remaining part is

$$\begin{aligned}&\ll \int_{-H^2}^{H^2}\left|\sum_{n\le M} n^{-\frac{1}{2}+i(g_1-g_2)+it}\right| |(2^{it}-1)\Gamma(\tfrac{it}{H})|\tfrac{dt}{H}\\ &= \int_{-H}^{H}\left|\sum_{n\le M} n^{-\frac{1}{2}+i(g_1-g_2)+iHu}\right| |(2^{iHu}-1)\Gamma(iu)|du\\ &\ll \int_{-H}^{H}\left|\sum_{n\le M} n^{-\frac{1}{2}+i(g_1-g_2)+iHu}\right| \varphi(u)du\end{aligned}$$

where

$$\varphi(u)=\begin{cases} H & 0\le |u|\le 1/H,\\ \frac{1}{|u|} & 1/H\le |u|\le 1,\\ e^{-\pi|u|/2} & 1<|u|.\end{cases}$$

Collecting estimates together we have

$$S(N)\ll \sum_{\mathbf{g}\in\mathcal{G}^2} NH^{-2}\exp\left(-\frac{\pi|g_1-g_2|}{H}\right)+N^{-1}T^{-2}+T^{-4}$$
$$+\sum_{\mathbf{g}\in\mathcal{G}^2}\left(\int_{-H}^{H}\left|\sum_{n\le M} n^{-\frac{1}{2}+i(g_1-g_2)+iHu}\right|\varphi(u)du\right)^2$$

Hence, by Schwarz' inequality,

$$S(N)$$
$$\ll NR+\sum_{\mathbf{g}\in\mathcal{G}^2}\int_{-H}^{H}\varphi(u)\left|\sum_{n\le M} n^{-\frac{1}{2}+i(g_1-g_2)+iHu}\right|^2 du\int_{-H}^{H}\varphi(v)dv$$
$$\ll NR+(\log H)\int_{-H}^{H}\varphi(u)\sum_{\mathbf{g}\in\mathcal{G}^2}\left|\sum_{n\le M} n^{-\frac{1}{2}+i(g_1-g_2)+iHu}\right|^2 du$$

Let $\lambda=\sqrt{2}$. Then by Lemma 2.2 this is

$$\ll NR+(\log H)^2\sum_{\mathbf{g}\in\mathcal{G}^2}\left|\sum_{n\le M} n^{-\frac{1}{2}+i(g_1-g_2)}\right|^2 du$$

and the multiple sum here is

$$\ll (\log M)\sum_{\substack{j\ge 1\\ \lambda^j\le M}}\sum_{\mathbf{g}\in\mathcal{G}^2}\left|\sum_{M\lambda^{-j}<n\le M\lambda^{1-j}} n^{-\frac{1}{2}+i(g_1-g_2)}\right|^2.$$

By Lemma 2.3 with $J=\lambda^j$ and $K=2\lambda^{j-1}$, (2.1) and (2.2), and Lemma 2.2 we have

$$\sum_{\mathbf{g}\in\mathcal{G}^2}\left|\sum_{M\lambda^{-j}<n\le M\lambda^{1-j}} n^{-\frac{1}{2}+i(g_1-g_2)}\right|^2\ll(\log M)^3S(M).$$

Hence

$$S(N)\ll NR+(\log H)^2(\log M)^4S(M)$$

as required. $\square$

We now consolidate the above results as follows. We define

$$D(T)=\max_{N\le T^3}\max_n \operatorname{card}\{l,m : lm=n, N<l,m\le 2N\}. \tag{2.5}$$

**Lemma 2.6.** *Suppose $T\ge 1$ and that $\mathcal{G}$ is a 1-separated subset of $[0,T]$. Then there are positive constants $C_1$ and $C_2$ such that*

*(i) if $C_1T\log T\le N\le T^{C_2}$, then*

$$S(N)\ll RN, \tag{2.6}$$

*(ii) if $C_2(T\log T)^{2/3}\le N\le C_1T\log T$, then*

$$S(N)\ll RN+(\log\log T)^4(\log T)^{14}R^2D(T)^2, \tag{2.7}$$

*(iii) if $(\log\log T)(\log T)^{5/2}R^{1/4}T^{1/2}D(T)^{1/2}\le N\le C_2(T\log T)^{2/3}$, then*

$$S(N)\ll RN+(\log\log T)^4(\log T)^{11}R^2D(T)^2, \tag{2.8}$$

*and*

*(iv) if $1\le N\le(\log\log T)(\log T)^{5/2}R^{1/4}T^{1/2}D(T)^{1/2}$, then*

$$\begin{aligned}S(N)\ll{}&(\log\log T)(\log T)^{11/2}R^{5/4}T^{1/2}D(T)^{1/2}\\&+(\log\log T)^4(\log T)^{14}R^2D(T)^2.\end{aligned} \tag{2.9}$$

*Proof.* The bound (2.6) is immediate by the second part of Lemma 2.5. Now suppose that $C_2(T\log T)^{2/3}\le N\le C_1T\log T$. Then by the first part of Lemma 2.5 with $M=C_1(\log T)TN^{-1}$ we have

$$S(N)\ll RN+(\log\log T)^2(\log T)^4S\big(C_1(\log T)TN^{-1}\big), \tag{2.10}$$

and for future reference we observe by Lemma 2.5 this holds generally even when $N>C_1T\log T$.

The condition $C_2(T\log T)^{2/3}\le N$ ensures that

$$4\big(C_1(\log T)TN^{-1}\big)^2\le N.$$

Hence, by Lemma 2.4,

$$S(M)^2\ll R^2D(T)^2S(4M^2).$$

We split the sum over $(4M^2,8M^2]$ into two sums over $(4M^2,4\sqrt{2}M^2]$ and $(4\sqrt{2}M^2,8M^2]$, and then apply Lemma 2.3 with $JN/(4M^2)$ or $N/(4\sqrt{2})M^2$ and $K=j\sqrt{2}$ and obtain, *via* and (2.1) and (2.2),

$$S(M)^2\ll(\log T)^3R^2D(T)^2S(N).$$

Hence

$$S(N)\ll RN+(\log\log T)^2(\log T)^7RD(T)S(N)^{1/2}$$

and so

$$S(N)\ll RN+(\log\log T)^4(\log T)^{14}R^2D(T)^2$$

as required.

Now suppose that $N \le C_2(T\log T)^{2/3}$ and choose

$$Q = C_1 T N^{-1}\log T.$$

Then for $C_2$ suitably chosen we have $4Q^2 \ge C_2(T\log T)^{2/3}$ and so by parts (i) and (ii)

$$S(4Q^2) \ll RQ^2 + (\log\log T)^4(\log T)^{14}R^2D(T)^2. \tag{2.11}$$

Moreover, as noted in the proof of (ii), (2.10) continues to hold, so that

$$S(N) \ll RN + (\log\log T)^2(\log T)^4 S(Q).$$

Also by Lemmas 2.4 and 2.3, and (2.1) and (2.2), by splitting into sub-sums once more, $S(Q)^2 \ll R^2D(T)^2S(4Q^2)$, whence by (2.11)

$$S(Q)^2 \ll R^3Q^2D(T)^2 + (\log\log T)^4(\log T)^{14}R^4D(T)^4.$$

Therefore

$$\begin{aligned} S(N) \ll{}& RN + (\log\log T)^2(\log T)^4R^{3/2}QD(T)\\ &+ (\log\log T)^4(\log T)^{11}R^2D(T)^2\\ \ll{}& RN + (\log\log T)^2(\log T)^5R^{3/2}TN^{-1}D(T)\\ &+ (\log\log T)^4(\log T)^{11}R^2D(T)^2. \end{aligned}$$

The condition $(\log\log T)(\log T)^{5/2}R^{1/4}T^{1/2}D(T)^{1/2} \le N$ in case (iii) shows that then the first term dominates the second and gives the desired conclusion. Finally let

$$P = (\log\log T)(\log T)^{5/2}R^{1/4}T^{1/2}D(T)^{1/2}$$

and suppose $N \le P$. By a now familiar argument based on splitting the sum and Lemma 2.2 we have

$$\begin{aligned} S(N) \ll&(\log T)^3S(P)\\ \ll&(\log\log T)(\log T)^{11/2}R^{5/4}T^{1/2}D(T)^{1/2}\\ &+ (\log\log T)^4(\log T)^{14}R^2D(T)^2. \end{aligned}$$

□

## 3. Proofs of Theorems 1.1 and 1.2

Let

$$S^*(N) = \sum_{\mathbf{g}\in\mathcal{G}^2}\left|\sum_{n=1}^{N} n^{i(g_1-g_2)}\right|^2. \tag{3.1}$$

Then by (2.4)

$$S^*(N) = \sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{\substack{j\ge 1 \\ 2^j\le N}} \sum_{N2^{-j}\le n\le N2^{1-j}} n^{i(g_1-g_2)} \right|^2$$
$$\ll \left(\sum_{2^j\le N} j^{-2}\right) \sum_{2^j\le N} j^2 \sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{N2^{-j}\le n\le N2^{1-j}} n^{i(g_1-g_2)} \right|^2.$$

Multiplying out

$$\sum_{\mathbf{g}\in\mathcal{G}^2} \left| \sum_{N2^{-j}<n\le N2^{1-j}} n^{i(g_1-g_2)} \right|^2 = \sum_{N2^{-j}<m,n\le N2^{1-j}} \left| \sum_{g\in\mathcal{G}} (m/n)^{ig} \right|^2$$
$$\ll \frac{N}{2^j} \sum_{N2^{-j}<m,n\le N2^{1-j}} \frac{1}{\sqrt{mn}} \left| \sum_{g\in\mathcal{G}} \left(\frac{m}{n}\right)^{ig} \right|^2$$
$$= N2^{-j} S(N2^{-j}).$$

Thus

$$S^*(N) \ll \sum_{2^j\le N} \frac{j^2 N}{2^j} S(N2^{-j})$$

Hence, by Lemma 2.6,

$$S^*(N) \ll \sum_{2^j\le N} \frac{j^2N}{2^j} \left( \frac{RN}{2^j} + (R^{\frac{5}{4}}T^{\frac{1}{2}} + R^2)\exp\left(\frac{C\log T}{\log\log T}\right)\right)$$
$$\ll RN^2 + \left(R^{5/4}NT^{1/2} + NR^2\right)\exp(C\log T/\log\log T)\big)$$

for some positive constant $C$ (probably $2\log 2$), and by (3.1) this gives Theorem 1.1.

We observe that $S^*(N)$ is the trace of the $N\times N$ Hermitian matrix

$$(\mathcal{M}^*\mathcal{M})^2$$

where $\mathcal{M}$ is the $R\times N$ matrix with general entry $n^{ig}$ $(g\in\mathcal{G})$, $(1\le n\le N)$. Thus by Lemma 2.1 it gives a bound for the largest eigenvalue $\lambda$ of $\mathcal{M}^*\mathcal{M}$ (of course the trace of $\mathcal{M}^*\mathcal{M}$ is $NR$), and hence shows that

$$\sum_{g\in\mathcal{G}} \left| \sum_{n=1}^N a_n n^{ig} \right|^2 \ll$$
$$\left( NR^{1/2} + (RN^{1/2} + R^{5/8}N^{1/2}T^{1/4})\exp\left(\frac{C\log T}{\log\log T}\right)\right) \sum_{n=1}^N |a_n|^2.$$

Alternatively by Cauchy's inequality

$$\left(\sum_{g\in\mathcal{G}}\left|\sum_{n=1}^{N}a_n n^{ig}\right|^2\right)^2 = \left(\sum_{m,n\leq N}a_m\overline{a_n}\sum_{g\in\mathcal{G}}(m/n)^{ig}\right)^2$$
$$\leq \left(\sum_{m,n\leq N}|a_m a_n|^2\right)S^*(N).$$

Either way we have Theorem 1.2.

## References


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