Zero Density Estimates

Robert C. Vaughan

December 10, 2024

- So far I have given an outline of the proof of the main theorem

  Theorem 1.1 *Suppose* (b_n) *is a sequence of complex numbers with* $|b_n| \le 1$ *and* (t_r) *is a sequence of* 1*-separated points in* $[0, T]$ *such that*

$$\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$$

for all $r \le R$*. Then*

$$R \ll_\varepsilon T^\varepsilon \left(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4} \right).$$

- So far I have given an outline of the proof of the main theorem

  Theorem 1.1 *Suppose* (b_n) *is a sequence of complex numbers with* $|b_n| \le 1$ *and* (t_r) *is a sequence of* 1*-separated points in* $[0, T]$ *such that*

$$\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$$

  for all $r \le R$*. Then*

$$R \ll_\varepsilon T^\varepsilon \big(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4} \big).$$

- There are various things one can observe about this.

- So far I have given an outline of the proof of the main theorem
  Theorem 1.1 *Suppose* (b_n) *is a sequence of complex numbers with* $|b_n| \le 1$ *and* (t_r) *is a sequence of* 1*-separated points in* $[0, T]$ *such that*
  $$\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$$
  for all $r \le R$. *Then*
  $$R \ll_\varepsilon T^\varepsilon \big(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4}\big).$$
- There are various things one can observe about this.
- The core argument is for S_3 and gives a bound for the largest eigenvalue of the matrix $(\mathcal{M}\mathcal{M}^*)^3$.

- So far I have given an outline of the proof of the main theorem

  Theorem 1.1 *Suppose* (b_n) *is a sequence of complex numbers with* $|b_n| \le 1$ *and* (t_r) *is a sequence of* 1*-separated points in* $[0, T]$ *such that*

$$\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$$

  for all $r \le R$. *Then*

$$R \ll_\varepsilon T^\varepsilon \big(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4} \big).$$

- There are various things one can observe about this.
- The core argument is for S_3 and gives a bound for the largest eigenvalue of the matrix $(\mathcal{M}\mathcal{M}^*)^3$.
- Thus the theorem could be stated for b_n much more general provided that a factor N in each term on the right is replaced by

$$\sum_n |b_n|^2.$$

- **Theorem 1.1** *Suppose* (b_n) *is a sequence of complex numbers with* $|b_n| \le 1$ *and* (t_r) *is a sequence of* 1*-separated points in* $[0, T]$ *such that*

$$\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$$

for all $r \le R$. *Then*

$$R \ll_\varepsilon T^\varepsilon \left(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4} \right).$$

- **Theorem 1.1** *Suppose (b_n) is a sequence of complex numbers with $|b_n| \le 1$ and (t_r) is a sequence of 1-separated points in $[0, T]$ such that*

$$\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$$

for all $r \le R$. Then

$$R \ll_\varepsilon T^\varepsilon \left(N^2 V^{-2} + N^{18/5} V^{-4} + TN^{12/5} V^{-4} \right).$$

- Another observation is that if instead of $|b_n| \le 1$ one assumes only that $|b_n| \le B$, then the theorem still holds at the expense of an extra factor B^2 on the right since one can replace b_n by b_n/B and V by V/B.

- **Theorem 1.1** *Suppose* (b_n) *is a sequence of complex numbers with* $|b_n| \le 1$ *and* (t_r) *is a sequence of* 1*-separated points in* $[0, T]$ *such that*

$$\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$$

for all $r \le R$. *Then*

$$R \ll_\varepsilon T^\varepsilon \big(N^2 V^{-2} + N^{18/5} V^{-4} + TN^{12/5} V^{-4}\big).$$

- Another observation is that if instead of $|b_n| \le 1$ one assumes only that $|b_n| \le B$, then the theorem still holds at the expense of an extra factor B^2 on the right since one can replace b_n by b_n/B and V by V/B.
- This is important since in applications we may only know, for example, that $b_n \ll d_k(n)$ and so one would need to take

$$B = \max_{N \le n \le 2N} d_k(n).$$

- **Theorem 1.1** *Suppose* (b_n) *is a sequence of complex numbers with* $|b_n| \le 1$ *and* (t_r) *is a sequence of* 1*-separated points in* $[0, T]$ *such that*

$$\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$$

for all $r \le R$*. Then*

$$R \ll_\varepsilon T^\varepsilon \big(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4} \big).$$

- **Theorem 1.1** *Suppose (b_n) is a sequence of complex numbers with $|b_n| \le 1$ and (t_r) is a sequence of 1-separated points in $[0, T]$ such that*

$$\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$$

for all $r \le R$. Then

$$R \ll_\varepsilon T^\varepsilon \bigl(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4} \bigr).$$

- Anyway since $|D_n(it) = \sum_{n=1}^{N} b_n n^{it}| \ll N$ we have for some $D_0 > 0$

$$\sum_{r=1}^{R} |D_N(it_r)|^2 \le \sum_{\substack{r=1 \\ |D_N(it_r)| \le D_0}}^{R} D_0^2 + \sum_{\substack{r=1 \\ |D_N(it_r)| > D_0}}^{R} \left(D_0^2 + \int_{D_0}^{|D_N(it_r)|} 2V dV \right)$$

- **Theorem 1.1** *Suppose* $|b_n| \le 1$, (t_r) *is* 1*-separated in* $[0, T]$ *and* $\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$ *for all* $r \le R$. *Then*

$$R \ll_\varepsilon T^\varepsilon \big(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4} \big).$$

- **Theorem 1.1** *Suppose* $|b_n| \le 1$, (t_r) *is* 1*-separated in* $[0, T]$ *and* $\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$ *for all* $r \le R$. *Then*

$$R \ll_\varepsilon T^\varepsilon \big(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4} \big).$$

- $$\sum_{r=1}^{R} |D_N(it_r)|^2 \le \sum_{\substack{r=1 \\ |D_N(it_r)| \le D_0}}^{R} D_0^2 + \sum_{\substack{r=1 \\ |D_N(it_r)| > D_0}}^{R} \left(D_0^2 + \int_{D_0}^{|D_N(it_r)|} 2V dV \right)$$

$$\le R D_0^2 + \int_{D_0}^{CN} \sum_{\substack{r=1 \\ |D_N(it_r)| > V}}^{R} 2V dV$$

- **Theorem 1.1** *Suppose* $|b_n| \le 1$, (t_r) *is* 1*-separated in* $[0, T]$ *and* $\left| \sum_{n=N}^{2N} b_n n^{it_r} \right| \ge V$ *for all* $r \le R$. *Then*

$$R \ll_\varepsilon T^\varepsilon \left(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4} \right).$$

$$\sum_{r=1}^{R} |D_N(it_r)|^2 \le R D_0^2 + \int_{D_0}^{CN} \sum_{\substack{r=1 \\ |D_N(it_r)|>V}}^{R} 2V dV$$

- **Theorem 1.1** *Suppose* $|b_n| \le 1$, (t_r) *is* 1*-separated in* $[0,T]$ *and* $\left|\sum_{n=N}^{2N} b_n n^{it_r}\right| \ge V$ *for all* $r \le R$*. Then*

$$R \ll_\varepsilon T^\varepsilon\big(N^2V^{-2} + N^{18/5}V^{-4} + TN^{12/5}V^{-4}\big).$$

$$\sum_{r=1}^{R} |D_N(it_r)|^2 \le RD_0^2 + \int_{D_0}^{CN} \sum_{\substack{r=1\\ |D_N(it_r)|>V}}^{R} 2VdV$$

-

$$\le RD_0^2 + \int_{D_0}^{CN} T^\varepsilon\big(N^2V^{-1} + (N^{18/5} + TN^{12/5})V^{-3}\big)dV$$

- **Theorem 1.1** *Suppose* $|b_n| \le 1$, (t_r) *is 1-separated in* $[0,T]$ *and* $\left|\sum_{n=N}^{2N} b_n n^{it_r}\right| \ge V$ *for all* $r \le R$. *Then*

$$R \ll_\varepsilon T^\varepsilon \big(N^2V^{-2} + N^{18/5}V^{-4} + TN^{12/5}V^{-4}\big).$$

$$\sum_{r=1}^{R} |D_N(it_r)|^2 \le RD_0^2 + \int_{D_0}^{CN} \sum_{\substack{r=1\\ |D_N(it_r)|>V}}^{R} 2VdV$$

- $$\le RD_0^2 + \int_{D_0}^{CN} T^\varepsilon\big(N^2V^{-1} + (N^{18/5} + TN^{12/5})V^{-3}\big)dV$$

- $\le RD_0^2 + T^\varepsilon N^2 \log(N/D_0) + T^\varepsilon(N^{18/5} + TN^{12/5})D_0^{-2}$

- $$\sum_{r=1}^{R} |D_N(it_r)|^2 \le RD_0^2 + T^{\varepsilon} N^2 \log(N/D_0) + T^{\varepsilon}(N^{18/5} + TN^{12/5})D_0^{-2}$$

- $$\sum_{r=1}^{R} |D_N(it_r)|^2 \le RD_0^2 + T^\varepsilon N^2 \log(N/D_0) + T^\varepsilon (N^{18/5} + TN^{12/5}) D_0^{-2}$$
- The choice $D_0 = R^{-1/4}(N^{18/5} + TN^{12/5})^{1/4}$ gives

$$\ll T^\varepsilon N^2 \log(N/D_0) + T^\varepsilon R^{1/2}(N^{18/5} + TN^{12/5})^{1/2}$$

- $$\sum_{r=1}^{R} |D_N(it_r)|^2$$
  $$\le RD_0^2 + T^\varepsilon N^2 \log(N/D_0) + T^\varepsilon(N^{18/5} + TN^{12/5})D_0^{-2}$$
- The choice $D_0 = R^{-1/4}(N^{18/5} + TN^{12/5})^{1/4}$ gives
  $$\ll T^\varepsilon N^2 \log(N/D_0) + T^\varepsilon R^{1/2}(N^{18/5} + TN^{12/5})^{1/2}$$
- and since $R \le T$ we have $D_0 > 1$ so that
  $$\sum_{r=1}^{R} |D_N(it_r)|^2 \ll T^\varepsilon N^2 \log N + T^\varepsilon R^{1/2}(N^{9/5} + T^{1/2}N^{6/5}).$$

- $\displaystyle\sum_{r=1}^{R} |D_N(it_r)|^2$
  $\le RD_0^2 + T^\varepsilon N^2 \log(N/D_0) + T^\varepsilon(N^{18/5} + TN^{12/5})D_0^{-2}$
- The choice $D_0 = R^{-1/4}(N^{18/5} + TN^{12/5})^{1/4}$ gives
  $$\ll T^\varepsilon N^2 \log(N/D_0) + T^\varepsilon R^{1/2}(N^{18/5} + TN^{12/5})^{1/2}$$
- and since $R \le T$ we have $D_0 > 1$ so that
  $$\sum_{r=1}^{R} |D_N(it_r)|^2 \ll T^\varepsilon N^2 \log N + T^\varepsilon R^{1/2}(N^{9/5} + T^{1/2}N^{6/5}).$$
- Note that apart from the $\log N$ the original theorem is easily recovered from this.

- $\sum_{r=1}^{R} |D_N(it_r)|^2$
$\le RD_0^2 + T^\varepsilon N^2 \log(N/D_0) + T^\varepsilon(N^{18/5} + TN^{12/5})D_0^{-2}$
- The choice $D_0 = R^{-1/4}(N^{18/5} + TN^{12/5})^{1/4}$ gives

$$\ll T^\varepsilon N^2 \log(N/D_0) + T^\varepsilon R^{1/2}(N^{18/5} + TN^{12/5})^{1/2}$$

- and since $R \le T$ we have $D_0 > 1$ so that

$$\sum_{r=1}^{R} |D_N(it_r)|^2 \ll T^\varepsilon N^2 \log N + T^\varepsilon R^{1/2}(N^{9/5} + T^{1/2}N^{6/5}).$$

- Note that apart from the log N the original theorem is easily recovered from this.
- Apropps my earlier remark about the largest eigenvalue I expect that for general b_n, $\sum_{r=1}^{R} |D_N(it_r)|^2 \ll$

$$T^\varepsilon\big(N \log N + R^{1/2}(N^{4/5} + T^{1/2}N^{1/5})\big)\|b\|^2.$$

- $$\sum_{r=1}^{R} |D_N(it_r)|^2 \ll T^{\varepsilon}\Big(N\log N + R^{1/2}(N^{4/5} + T^{1/2}N^{1/5})\Big).$$

- $$\sum_{r=1}^{R} |D_N(it_r)|^2 \ll T^{\varepsilon}\big(N \log N + R^{1/2}(N^{4/5} + T^{1/2}N^{1/5})\big).$$
- Now partial summation and Gallagher's argument gives rather routinely for $s_r = \sigma_r + it_r$ a set of complex numbers with $\sigma_r \geq \theta$ and t_r 1-separated,

$$\sum_{r=1}^{R} \left| \sum_{n=1}^{N} b_n n^{-s_r} \right|^2 \ll T^{\varepsilon} N^{2-2\theta} + T^{\varepsilon} R^{1/2}(N^{9/5-2\theta} + T^{1/2} N^{6/5-2\theta}).$$

- $$\sum_{r=1}^{R} |D_N(it_r)|^2 \ll T^{\varepsilon}\big(N\log N + R^{1/2}(N^{4/5} + T^{1/2}N^{1/5})\big).$$
- Now partial summation and Gallagher's argument gives rather routinely for $s_r = \sigma_r + it_r$ a set of complex numbers with $\sigma_r \ge \theta$ and t_r 1-separated,
$$\sum_{r=1}^{R}\left|\sum_{n=1}^{N} b_n n^{-s_r}\right|^2 \ll T^{\varepsilon}N^{2-2\theta} + T^{\varepsilon}R^{1/2}(N^{9/5-2\theta} + T^{1/2}N^{6/5-2\theta}).$$
- I am pretty sure that the method can be adapted to give for arbitrary a_n, and $\sigma_r \ge 0$, $\sum_{r=1}^{R}\left|\sum_{n=1}^{N} a_n n^{-s_r}\right|^2 \ll$
$$T^{\varepsilon}(N + R^{1/2}N^{4/5} + R^{1/2}T^{1/2}N^{1/5})\sum_{n=1}^{N}|a_n|^2$$

- Plugging this into the machine in Chapter 28 §28.4.2 of MNTIII we assume $7/10 \le \theta \le 8/10$.

- Plugging this into the machine in Chapter 28 §28.4.2 of MNTIII we assume $7/10 \le \theta \le 8/10$.
- For brevity let

$$A = T^{\frac{5/2}{3+5\theta}}.$$

and suppose for the time being that $K_j \le A$.

- Plugging this into the machine in Chapter 28 §28.4.2 of MNTIII we assume $7/10 \le \theta \le 8/10$.
- For brevity let
$$A = T^{\frac{5/2}{3+5\theta}}.$$
and suppose for the time being that $K_j \le A$.
- Choose k so that $A^2 \le K_j^k < A^3$. If
$$K_j^k < A^\lambda$$
where
$$\lambda = \frac{15(1-\theta)}{9-10\theta},$$
then use the bound given by the previous section.

- Plugging this into the machine in Chapter 28 §28.4.2 of MNTIII we assume $7/10 \le \theta \le 8/10$.
- For brevity let
$$A = T^{\frac{5/2}{3+5\theta}}.$$
and suppose for the time being that $K_j \le A$.
- Choose k so that $A^2 \le K_j^k < A^3$. If
$$K_j^k < A^\lambda$$
where
$$\lambda = \frac{15(1-\theta)}{9-10\theta},$$
then use the bound given by the previous section.
- If $A^\lambda \le K_j^k < A^3$, then use the classical bound
$$\sum_{r=1}^{R}\left|\sum_{n=1}^{N} b_n n^{-s_r}\right|^2 \ll T^\varepsilon N^{2-2\theta} + TN^{1-2\theta}.$$

- Plugging this into the machine in Chapter 28 §28.4.2 of MNTIII we assume $7/10 \le \theta \le 8/10$.
- For brevity let
$$A = T^{\frac{5/2}{3+5\theta}}.$$
and suppose for the time being that $K_j \le A$.
- Choose k so that $A^2 \le K_j^k < A^3$. If
$$K_j^k < A^\lambda$$
where
$$\lambda = \frac{15(1-\theta)}{9-10\theta},$$
then use the bound given by the previous section.
- If $A^\lambda \le K_j^k < A^3$, then use the classical bound
$$\sum_{r=1}^{R} \left| \sum_{n=1}^{N} b_n n^{-s_r} \right|^2 \ll T^\varepsilon N^{2-2\theta} + TN^{1-2\theta}.$$
- If instead $A < K_j$, then take $k = 2$ and again use the classical bound.