C:MeanLargeVals

A NOTE ON LEMMA 6.2

1. Lemma 6.2

A core ingredient in the treatment of S_2, leading to the use of Heath-Brown's theorem, is Lemma 6.2. However there are some obscurities in this which need to be cleared up. Lemma 6.2 as state is

Lemma 6.2. *For every t with $|t| \sim T_0 \geq T^\varepsilon$, we have*

$$\left|\sum_{m\neq 0} \hat{h}_t(mN)\right| \lesssim T_0^{-1/2} \int_{u \lessapprox 1} \left|\sum_{m \lessapprox T_0/N} m^{-i(t+u)}\right| du + O(T^{-100}).$$

There are a number of problems with this. The symbols $\lesssim$ and $\lessapprox$ are non-standard in analytic number theory, and they use $\sim$ in a non-standard way. Referring to §1.2,

$A \lesssim B$ means that $A \leq CB$ for an absolute constant C. They do not say that C must be positive, but it would be peculiar if they allowed it to be 0 or negative! Moreover I am lead to wondering if they really mean $|A| \leq CB$, in which case why not use Vinogradov's notation! In the context of the lemma, where the quantities on each side are non-negative why eschew the use of Vinogradov which is standard and normal?

$A \sim B$ means $B < A \leq 2B$ which is contrary to the normal usage that $A/B \to 1$ in some limiting process. Also one would expect this to be symmetric, namely $A \sim B$ means the same as $B \sim A$, but here it does not. A more commonly used notation for this $A \approx B$.

$A \lessapprox B$ means that for every $\varepsilon > 0$ there is a positive constant $C(\varepsilon)$ such that $A \leq C(\varepsilon)T^\varepsilon B$. I am again left wondering if they really mean $|A| \leq C(\varepsilon)T^\varepsilon B$, and why they don't just adapt Vinogradov as in $|A| \ll_\varepsilon T^\varepsilon B$.

Something else that puzzled me about the statement of the lemma was whether one was to integrate over all real numbers $u \leq C(\varepsilon)T^\varepsilon$. It turns out that there is a restatement of a form of the lemma at the bottom of page 16 which makes it clear that $|u| \leq C(\varepsilon)T^\varepsilon$ is intended.

Yet another source of puzzlement is, what happens in the lemma when $T_0 < N$ since the sum is then empty and the integral would be 0. It seems that the lemma indeed remains true. Nevertheless I would have expected some explanation!

Oh, and four other peculiarities in the statement of the lemma are

1. why is the general term in the sum $m^{-i(t+u)}$ and not $m^{i(t+u)}$.

2. what is the relationship of T to the other parameters. The only places T appears are in the error term and the hypothesis $T_0 \geq T^\varepsilon$! I am guessing that even if the proof does not require it, there is at least a tacit assumption that $|t| \leq T$.

3. Since there is an implicit constant in the $\lesssim$ symbols why does one have a $O(...)$?

4. The symbol ε appears explicitly in the hypothesis and implicitly twice in $\lessapprox$ symbols. Is it the same ε in each instance or, could it be different in each instance - in analytic number theory there is often an assumption that formulae containing an (implicit) ε hold for all $\varepsilon > 0$ and this can lead to complications on combining formulae, so one needs to be careful.

Anyway, in view of this I think we should prove the Lemma first before proceeding to deal with S_2. Also, let me restate the lemma in a form which is actually needed for the treatment of S_2.

Oh, and before we get in to the proof of the lemma I should also point out that the claim in the second sentence of the proof that $W(s)$ is entire and satisfies $|W(s)| \lesssim |s|^{-j}$ is false since W would have to be uniformly bound in a neighbourhood of 0 and hence in $\mathbb{C}$ and so by Liouville's theorem would a constant, which it plainly is not. Recall Lemma 4.3 only deals with an object of the kind $h_t(u) = W(it)$. Thus the bound for $W(s)$ should be (because the support of w is on $[1, 2]$)

$$W(s) \ll_j 2^{\operatorname{Re} s} |s|^{-j},$$

i.e. the function can grow exponentially along the positive real axis. Fortunately it is bounded when one moves to the left.

Lemma 6.2. *Suppose that $\varepsilon > 0$, T is sufficiently large in terms of ε, and N is a positive integer with $T^\varepsilon \leq N$. Suppose also that M is a positive real number with $T^\varepsilon/(2N) < M < 2T/N$ and $MN < |t| \leq 2MN$. Then*

$$\left| \sum_{m \neq 0} \hat{h}_t(mN) \right| \ll_\varepsilon (MN)^{-1/2} \int_{|u| \leq T^\varepsilon} \left| \sum_{1 \leq m \leq T^\varepsilon M} m^{i(t+u)} \right| du + T^{-100}.$$

Proof. First, let us dispose of the possibility that the sum in the integral is empty. Then we have $T^\varepsilon M < 1$, so that $MN < NT^{-\varepsilon}$ and so

$$|t| \leq 2NT^{-\varepsilon}.$$

By Lemma 4.3 for any fixed j we have

$$\hat{h}_t(mN) \ll \frac{1 + |t|^j}{|m|^j N^j} \ll \frac{1}{|m|^j N^j} + \frac{2^j}{T^{\varepsilon j} |m|^j}$$

and so on taking j large enough the sum on the left is $\ll_\varepsilon T^{-100}$.

Thus we may suppose that $T^\varepsilon M \geq 1$. At this point it is useful to add in the term $m = 0$ on the left. By Lemma 4.3, since $|t| > T^\varepsilon$,

$$\hat{h}_t(0) \ll_j \frac{1}{|t|^j} \ll_\varepsilon T^{-100}$$

on taking j large enough.

We have , after a change of variable in the Fourier transform,

$$\hat{h}_t(mN) = \int_{\mathbb{R}} N^{-1-it} w(u/N)^2 u^{it} e(-um)du = \hat{g}_t(m; N)$$

where

$$g_t(u; N) = N^{-1-it} w(u/N)^2 u^{it}.$$

Hence, by the Poisson summation formula,

$$\sum_{m \in \mathbb{Z}} \hat{h}_t(mN) = \sum_{m \in \mathbb{Z}} g_t(m; N)$$

so that

$$N^{1+it} \sum_{m \in \mathbb{Z}} \hat{h}_t(mN) = \sum_{m \in \mathbb{Z}} w(m/N)^2 m^{it}.$$

Since the $\operatorname{supp} w \in [1, 2]$ we have

$$N^{1+it} \sum_{m \in \mathbb{Z}} \hat{h}_t(mN) = \sum_{m=1}^{\infty} w(m/N)^2 m^{it}.$$

The sum on the right is now treated in the same kind of way that we used for the approximate functional equation for $\zeta(s)$. Recall we started out by writing a sum like the one on the right in the form

$$\frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(z - it) W(z) N^z dz$$

where $\theta > 1$, for some W. We used a rather special pair w and W. Here things are a little different but the principal is the same. We define W by the Mellin transform

$$W(z) = \int_{\mathbb{R}} w(u)^2 u^{z-1} du.$$

Because of the nature of w this is an entire function of z, and because w is in $\mathcal{C}^\infty$ and has its support on $[1, 2]$ repeated integration by parts gives

$$W(z) \ll_j 2^{\operatorname{Re} z} |z|^{-j}.$$

The inverse of the Mellin transform is the Perron transform

$$w(u)^2 = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} W(z) u^{-z} dz$$

which here will hold for any fixed $\theta > 0$.

As an aside, analytic number theorists are more familiar with these transforms in the context of Dirichlet series, when they are usually put in to the forms

$$\alpha(s)s^{-1} = \int_{\mathbb{R}} A(x)x^{-s-1}dx \text{ and } A(x) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \alpha(s)s^{-1}x^s ds$$

with

$$A(x) = \sum_{n\le x} a_n \text{ and } \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

The translation can be accomplished by taking $B(x) = A(1/x)$, $\beta(s) = \alpha(s)/s$.

Anyway, taking $\theta = 2$, say, we have

$$\begin{aligned} N^{1+it}\sum_{m\in\mathbb{Z}} \hat{h}_t(mN) &= \sum_{m=1}^{\infty} m^{it}\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} W(z)(N/m)^z dz \\ &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \zeta(z-it)W(z)N^z dz. \end{aligned}$$

Now the procedure is standard. We move to the line $\operatorname{Re} z = -1$, picking up the residue

$$W(1+it)N^{1+it}$$

at the pole $z = 1+it$. This contributes

$$W(1+it)N^{1+it} \ll_j N|t|^{-j} \ll T^{-100}$$

on choosing j large enough.

On the -1 line apply the functional equation

$$\zeta(z-it) = G(z-it)\zeta(1+it-z)$$

where G is the gamma factor

$$G(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}.$$

Note that G is an entire function and $G(s) \ll_\sigma (1+|t|)^{\frac{1}{2}-\sigma}$. Thus

$$\begin{aligned} &N^{1+it}\sum_{m\in\mathbb{Z}} \hat{h}_t(mN) = \\ &\qquad \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} G(z-it)\zeta(1+it-z)W(z)N^z dz + O(T^{-100}). \end{aligned}$$

Since the ζ here is in a region of absolute convergence we can write

$$\zeta(1+it-z) = Z_1(1+it-z) + Z_2(1+it-z)$$

where

$$Z_1(1+it-z) = \sum_{1\le m\le P} m^{z-1-it},\ Z_2(1+it-z) = \sum_{m>P} m^{z-1-it}$$

and $P = MT^{\epsilon}$ where ϵ is a positive number at our disposal. We move the part with Z_1 to the line $\operatorname{Re} z = 1$ and the part with Z_2 to the line $\operatorname{Re} z = -2k$. Thus

$$N^{1+it}\sum_{m\in\mathbb{Z}} \hat{h}_t(mN) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} G(z-it)Z_1(1+it-z)W(z)N^z dz$$
$$+\frac{1}{2\pi i}\int_{-2k-i\infty}^{-2k+i\infty} G(z-it)Z_2(1+it-z)W(z)N^z dz + O(T^{-100}).$$

The integrand in the second integral is

$$\ll_{j,k} (1+|\operatorname{Im} z - t|)^{\frac{1}{2}+2k}\sum_{m>P} m^{-2k-1}\min(1,|\operatorname{Im} z|^{-j})N^{-2k}$$
$$\ll_{j,k} (1+|\operatorname{Im} z - t|)^{\frac{1}{2}+2k}\min(1,|\operatorname{Im} z|^{-j})(PN)^{-2k}.$$

Taking j large enough we find that the contribution to the integral from the z with $|\operatorname{Im} z|\ge T^{\varepsilon}$ is $\ll_{\varepsilon} T^{-100}$. The contribution from the z with $|\operatorname{Im} z|\le T^{\varepsilon}$ is

$$\ll_{j,k}\int_{-T^{\varepsilon}}^{T^{\varepsilon}}(1+|t|)^{\frac{1}{2}+2k}(T^{\epsilon}MN)^{-2k}\ll_{j,k} T^{\frac{1}{2}}T^{-2k\epsilon}$$

since on hypothesis we have $|t|\le 2MN\le 4T$. Thus choosing k large enough we again obtain the bound $\ll T^{-100}$. Thus

$$\frac{1}{2\pi i}\int_{-2k-i\infty}^{-2k+i\infty} G(z-it)Z_2(1+it-z)W(z)N^z dz \ll T^{-100}.$$

Now consider the first integral

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} G(z-it)Z_1(1+it-z)W(z)N^z dz$$

Again the rapid decay of the function $W(z)$ means we obtain

$$\frac{1}{2\pi}\int_{|y|\le T^{\varepsilon}} G(1+iy-it)Z_1(it-iy)W(1+iy)N^{1+iy}dy + O(T^{-100}).$$

By the standard bound for G the integral here is

$$\ll N\int_{|y|\le T^{\varepsilon}}(1+|y-t|)^{-1/2}|Z_1(it-iy)|\min(1,|y|^{-2})dy$$
$$\ll (N/M)^{1/2}\int_{|y|\le T^{\varepsilon}}|Z_1(it-iy)|dy$$

as on hypothesis we have $MN < |t| \leq 2MN$.

Dividing through by N^{1+it} and recalling the bound for $\hat{h}_t(0)$ we obtain

$$\sum_{m \in \mathbb{Z} \setminus \{0\}} \hat{h}_t(mN) \ll_\varepsilon (MN)^{-1/2} \int_{|u| \leq T^\varepsilon} \left| \sum_{1 \leq m \leq T^\varepsilon M} m^{i(t+u)} \right| du + T^{-100}.$$

as required.