C:Sthree

THE THIRD SUM

1. Recap

Recall

$$w \in \mathcal{C}^\infty(\mathbb{R}),\ \operatorname{supp} w \in [1,2],\ w(x) = 1\,(x \in [6/5, 9/5]),$$

$$h_t(u) = w(u)^2 u^{it}$$

and its Fourier transform

$$\hat{h}_t(v) = \int_{\mathbb{R}} h_t(u) e(-uv) du.$$

Suppose that $\mathcal{W}$ is T^ε separated. Then

$$\sum_{t \in \mathcal{W}} |D_n(t)|^2 \ll_\varepsilon \left(N + \Big(\sum_{\mathbf{m} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}} I_{\mathbf{m}} \Big)^{1/3} \right) \sum_{n=1N+1}^{2N} |b_n|^2. \tag{1.1}$$ eq:one1

where

$$D_N(s) = \sum_{n=N+1}^{2N} b_n n^s \tag{1.2}$$ eq:one2

and

$$I_{\mathbf{m}} = N^3 \sum_{\mathbf{t} \in \mathcal{W}^3} \hat{h}_{t_1 - t_2}(m_1 N) \hat{h}_{t_2 - t_3}(m_2 N) \hat{h}_{t_3 - t_1}(m_3 N). \tag{1.3}$$ eq:one3

Theorem 4.6.[Proposition 4.6] *Suppose that $\mathcal{W}$ is T^ε separated and $|D_N(t)| > V$ for each $t \in \mathcal{W}$. Then*

$$\operatorname{card} W \ll_\varepsilon \left(N + \Big(\sum_{\mathbf{m} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}} I_{\mathbf{m}} \Big)^{1/3} \right) \|\mathbf{b}\|^2 V^{-2}.$$

Thus we have reduced the investigation to bounding

$$\sum_{\mathbf{m} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}} I_{\mathbf{m}} = S_1 + S_2 + S_3.$$

where S_j denotes the sum over $\mathbf{m}$ with exactly j of the m_j being non-zero. We already saw that
Theorem 5.1(Proposition 5.1, S_1 bound). Suppose that $\mathcal{W}$ is T^ε separated. Then

$$S_1 \ll_\varepsilon T^{-10}.$$

Theorem 6.1(Proposition 6.1, S_2 bound). Suppose that $\mathcal{W}$ is T^ε separated. Then for any $k \in \mathbb{N}$ we have

$$S_2 \ll_\varepsilon T^{7\varepsilon}\left(TNW^{2-\frac{1}{k}} + N^2W^2 + N^2W^{2-\frac{3}{4k}}T^{\frac{1}{2k}}\right)$$

2. The reduction of S_3

Now is the turn of S_3 where

$$S_3 = \sum_{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^3} I_{\mathbf{m}}$$

and

$$I_{\mathbf{m}} = N^3 \sum_{\mathbf{t}\in\mathcal{W}} \hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_2-t_3}(m_2N)\hat{h}_{t_3-t_1}(m_3N). \tag{2.1}$$ eq:two1

An immediate observation is that if we permute m_1, m_2, m_3, then by relabeling the t_1, t_2, t_3 to correspond we find that $I_{\mathbf{m}}$ is invariant. The number of permutations will vary according as the m_1, m_2, m_3 are distinct or not. However we are ultimately only concerned with the size of $I_{\mathbf{m}}$. Hence

$$|S_3| \le 6 \sum_{\substack{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^3 \\ |m_1|\le|m_2|\le|m_3|}} |I_{\mathbf{m}}| \tag{2.2}$$ eq:two2

Since $\hat{h}_t$ is a Fourier transform,

$$\hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_2-t_3}(m_2N)\hat{h}_{t_3-t_1}(m_3N) = \int_{\mathbb{R}^3} w_1(\mathbf{u})^2 u_1^{i(t_1-t_2)}u_2^{i(t_2-t_3)}u_3^{i(t_3-t_1)}e(-N\mathbf{m}.\mathbf{u})d\mathbf{u}. \tag{2.3}$$ eq:two3

where

$$w_1(\mathbf{u}) = w(u_1)w(u_2)w(u_3). \tag{2.4}$$ eq:two4

The crucial thing is that $N\mathbf{m}.\mathbf{u} = N(m_1u_1 + m_2u_2 + m_3u_3)$ is linear in $\mathbf{u}$. They speak of "stationary phase" and "non-stationary phase", but the usual terminology in analytic number theory would be "the saddle point method". In an integral of the form

$$\int_I g(x)e\big(f(x)\big)dx$$

the standard technique, going back to at least Dirichlet and probably to the Bernoullis, is to integrate by parts after first writing the integral as

$$\int_I \frac{g(x)}{2\pi i f'(x)} 2\pi i f'(x) e\big(f(x)\big)dx.$$

This is fine as long as the derivative is not close to 0. This leads to a bound of the form

$$\ll \sup_I |g(x)/f'(x)|.$$

If $f'(x) = 0$ (a saddle point), then in a neighbourhood of such a point one can use a simple argument based on the second derivative to show that under fairly mild conditions one has a bound of the form

$$\ll \frac{\sup_I |g(x)|}{\inf_I |f''(x)|^{1/2}}$$

and if this fails one can go on to higher derivatives. Anyway this is not relevant to our situation. Here $N(m_1u_1 + m_2u_2 + m_3u_3)$ is linear in each of the variables u_j. If we pick one of them, say u_3 and work with that as G&M do, then the derivative will be a “constant”, and we are in good shape as long as the constant is not too small. It is useful first to get rid of the large m_j.

An immediate observation is that Lemma 4.3 gives, for any non-negative integer j,

$$\hat{h}_t(mN) \ll_j \frac{1 + |t|^j}{|m|^j N^j}$$

(G & M have this the wrong way round - see the top of page 20!) and indeed we can use this to discard the large m_j. Thus if $\max_k |m_k| > T^{1+\varepsilon}N^{-1}$, then for any $j \geq 4$ we have

$$\sum_{\substack{\mathbf{m} \\ \max_k |m_k| > T^{1+\varepsilon}N^{-1}}} \hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_2-t_3}(m_2N)\hat{h}_{t_3-t_1}(m_3N)$$

$$\ll_j \sum_{\substack{0<|m_1|\leq|m_2|\leq|m_3| \\ |m_3|>T^{1+\varepsilon}N^{-1}}} \frac{T^j}{|m_3|^j N^j}$$

$$\ll_j \sum_{m_3 > T^{1+\varepsilon}N^{-1}} \frac{m_3^2 T^j}{m_3^j N^j}$$

$$\ll_j T^3 N^{-3} T^{(3-j)\varepsilon}.$$

Thus for some $j = j_0(\varepsilon)$, by (2.2) eq:two2,

$$|S_3| \leq 6 \sum_{\substack{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^3 \\ |m_1|\leq|m_2|\leq|m_3|\leq T^{1+\varepsilon}N^{-1}}} |I_{\mathbf{m}}| + O_\varepsilon(T^{-100}). \tag{2.5}$$

eq:two5

Suppose $|m_1| \leq |m_2| \leq |m_3| \leq T^{1+\varepsilon}N^{-1}$. By (2.1) eq:two1 and (2.2) eq:two2

$$I_{\mathbf{m}} = N^3 \int_{\mathbb{R}^3} w_1(\mathbf{u})^2 R(u_1/u_3)R(u_2/u_1)R(u_3/u_2)e(-N\mathbf{m}.\mathbf{u})d\mathbf{u}$$

where

$$R(u) = \sum_{t \in \mathcal{W}} u^{it}.$$

Make the change of variables

$$u_1, u_2 \to v_1, v_2 : u_1 = u_3 v_1, u_2 = u_3 v_2.$$

One can do this one variable at a time, or the more sophisticated way by computing the Jacobian. Either way note that the support for each u_j is in $[1, 2]$ so there are no complications with negative u_3. Moreover the support for v_1 and v_2 will lie in $[1/u_3, 2/u_3] \subseteq [1/2, 2]$. Thus

$$I_{\mathbf{m}} = N^3 \int_{[1/2,2]^2} R(v) R\Big(\frac{v_2}{v_1}\Big) R\Big(\frac{1}{v_2}\Big) J(\mathbf{v}, \mathbf{m}) d\mathbf{v}. \tag{2.6}$$

eq:two6

where

$$J(\mathbf{v}, \mathbf{m}) = \int_{\mathbb{R}} w(v_1 u)^2 w(v_2 u)^2 w(u)^2 u^2 e\big(- N u (m_1 v_1 + m_2 v_2 + m_3)\big) du.$$

Here we have an integral of the form

$$\int_{\mathbb{R}} F(u) e(\lambda u) du$$

where $F \in \mathcal{C}^\infty(\mathbb{R})$ and $F^{(j)}(u) = 0$ when $u \notin [1/2, 2]$. Thus integrating by parts j times we have

$$\int_{\mathbb{R}} F(u) e(\lambda u) du = \frac{(-1)^j}{(2\pi i \lambda)^j} \int_{\mathbb{R}} F^{(j)}(u) e(\lambda u) du.$$

Clearly when $\mathbf{v} \in [1/2, 2]^2$, $F(u) = w(v_1 u)^2 w(v_2 u)^2 w(u)^2 u^2$ and j is a non-negative integer we have

$$F^{(j)}(u) \ll_j 1.$$

Thus, when $|m_1 v_1 + m_2 v_2 + m_3| \geq T^\varepsilon / N$ we have

$$J(\mathbf{v}, \mathbf{m}) \ll_j T^{-j\varepsilon}$$

and so

$$\int_{\substack{\mathbf{v} \in [1/2,2]^2 \\ |m_1 v_1 + m_2 v_2 + m_3| \geq T^\varepsilon / N}} R(v_1) R\Big(\frac{v_2}{v_1}\Big) R\Big(\frac{1}{v_2}\Big) J(\mathbf{v}, \mathbf{m}) d\mathbf{v}$$
$$\ll_j T^6 T^{-j\varepsilon}.$$

Hence, by (2.6) eq:two6

$$I_m = N^3 \int_{\substack{\mathbf{v} \in [1/2,2]^2 \\ |m_1 v_1 + m_2 v_2 + m_3| \leq \frac{T^\varepsilon}{N}}} R(v_1) R\Big(\frac{v_2}{v_1}\Big) R\Big(\frac{1}{v_2}\Big) J(\mathbf{v}, \mathbf{m}) d\mathbf{v} + O(T^{-200}).$$

Hence, noting also that $|R(1/v_2)| = |R(v_2)|$,

$$I_{\mathbf{m}} \ll_\varepsilon N^3 \int_{\substack{\mathbf{v}\in[1/2,2]^2 \\ |m_1v_1+m_2v_2+m_3|\le \frac{T^\varepsilon}{N}}} |R(v_1)R\Big(\frac{v_2}{v_1}\Big)R(v_2)|J(\mathbf{v},\mathbf{m})d\mathbf{v} + T^{-200}.$$

Observe that $4|m_2| \ge |m_1v_1 + m_2v_2| \ge |m_3| - |m_1v_1 + m_2v_2 + m_3|$ so the integral will be 0 unless $4|m_2| \ge |m_3| - T^\varepsilon/N > 4|m_3|/5$. We also have $J(\mathbf{v},\mathbf{m}) \ll 1$. Thus, by (2.4)

Theorem 7.1(Proposition 7.1). *Suppose that $\mathcal{W}$ is T^ε separated. Then*

$$\begin{aligned} S_3 &\ll_\varepsilon \sum_{\substack{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^3 \\ |m_1|\le|m_2|\le|m_3|\le T^{1+\varepsilon}N^{-1}}} |I_{\mathbf{m}}| \\ &\ll_\varepsilon \sum_{\substack{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^3 \\ |m_1|\le|m_2|\le|m_3|\le T^{1+\varepsilon}N^{-1}}} N^3 \int_{\substack{\mathbf{v}\in[1/2,2]^2 \\ |m_1v_1+m_2v_2+m_3|\le \frac{T^\varepsilon}{N}}} |R(v_1)R\Big(\frac{v_2}{v_1}\Big)R(v_2)|d\mathbf{v} \\ &\qquad + T^{-200} \end{aligned}$$

and, when $|m_1| \le |m_2| \le |m_3|$, the integral here is 0 *unless $|m_2| \asymp |m_3|$.*

3. Commentary

At this point I want to deviate a little from the narrative to point something out. Let me jump ahead a little and consider the L^2 mean of R.

Lemma 8.2(L^2 bound). *Suppose that $\mathcal{W}$ is a T^ε separated set lying in an interval of length T. Then*

$$\int_{[1/4,4]} |R(v)|^2 dv \ll_\varepsilon \operatorname{card}\mathcal{W}.$$

Proof. We may certainly suppose that $\operatorname{card}\mathcal{W} \ge 1$.

Let $\psi(u) \in \mathcal{C}^\infty([0,\infty))$ have compact support and be $\psi(u) \gg 1$ for $u \in [1/4, 4]$. Then

$$\begin{aligned} \int_{[1/4,4]} |R(u)|^2 du \ll \int_0^\infty \psi(u)|R(u)|^2 \\ &= \sum_{\mathbf{t}\in\mathcal{W}^2} \int_0^\infty \psi(u)u^{it_1-it_2}du \\ &= \operatorname{card}\mathcal{W} \int_0^\infty \psi(u)du + \sum_{\substack{\mathbf{t}\in\mathcal{W}^2 \\ t_1\ne t_2}} \int_0^\infty \psi(u)u^{it_1-it_2}du. \end{aligned}$$

As previously, we may apply the saddle point method and obtain

$$\int_0^\infty \psi(u) u^{it_1 - it_2} du = \int_{\mathbb{R}} e^v \psi(e^v) e^{i(t_1-t_2)v} dv \ll_\varepsilon T^{-2}$$

when $t_1 \neq t_2$. □

Thus we have square root cancellation. We can speculate as to what happens if this continues to hold for the integrand in Theorem 7.1. For brevity write $R = \operatorname{card} \mathcal{W}$ and suppose $R \geq 1$. We would then have

$$\begin{aligned}
S_3 &\ll_\varepsilon \sum_{\substack{\mathbf{m} \in (\mathbb{Z}\backslash\{0\})^3 \\ |m_1| \leq |m_2| \leq |m_3| \leq T^{1+\varepsilon} N^{-1}}} N^3 \int_{\substack{\mathbf{v} \in [1/2,2]^2 \\ \left| v_2 + \frac{m_1 v_1 + m_3}{m_2} \right| \leq \frac{T^\varepsilon}{N|m_2|}}} R^{3/2} d\mathbf{v} + T^{-200} \\
&\ll_\varepsilon \sum_{\substack{\mathbf{m} \in (\mathbb{Z}\backslash\{0\})^3 \\ |m_1| \leq |m_2| \leq |m_3| \leq T^{1+\varepsilon} N^{-1}}} N^3 \frac{T^\varepsilon}{N|m_2|} R^{3/2} + T^{-200} \\
&\ll_\varepsilon N^2 R^{3/2+\varepsilon} (T^{1+\varepsilon} N^{-1})^2 \\
&\ll_\varepsilon T^{2+3\varepsilon} R^{3/2}
\end{aligned}$$

Now recall that

$$\sum_{\mathbf{t} \in \mathcal{W}} |D_N(t)|^2 \ll (N + \Theta^{1/3}) \|\mathbf{b}\|^2$$

where

$$\Theta = \sum_{\mathbf{m} \in \mathbb{Z}\backslash\{0\}} I_{\mathbf{m}}.$$

Putting in the bounds for the S_j gives

$$\Theta \ll T^{7\varepsilon} \left(TNR^{2-\frac{1}{k}} + N^2 R^2 + N^2 R^{2-\frac{3}{4k}} T^{\frac{1}{2k}} \right) + T^{2+3\varepsilon} R^{3/2}$$

This is probably good enough to establish the density hypothesis.

OK, so how real is this? Well in

$$\int_{\substack{\mathbf{v} \in [1/2,2]^2 \\ |m_1 v_1 + m_2 v_2 + m_3| \leq \frac{T^\varepsilon}{N}}} |R(v_1) R\left(\frac{v_2}{v_1}\right) R(v_2)| d\mathbf{v}$$

we would like, if possible to "save" an amount

$$\frac{T^\varepsilon}{N|m_2|}$$

when we integrate over v_2, or similarly over v_1. That means we realistically need to understand

$$\int_{[1/4,4]} |R(u)|^3 du.$$

But third moments are troublesome, and the normal approach is to estimate the fourth moment and use Cauchy-Schwarz.

Lemma 8.3* *Suppose that $\mathcal{W}$ is a T^ε separated set lying in an interval of length T. Then*

$$\int_{[1/4,4]} |R(v)|^4 dv \ll_\varepsilon T^{2\varepsilon}(\operatorname{card}\mathcal{W})^3.$$

More precisely

$$\int_{[1/4,4]} |R(v)|^4 dv \ll_\varepsilon 1 + T^\varepsilon E(\mathcal{W})$$

where

$$E(\mathcal{W}) = \operatorname{card}\{\mathbf{t} \in \mathcal{W}^4 : |t_1 - t_2 + t_3 - t_4| < 1\},$$

the, so-called, additive energy of $\mathcal{W}$. We also have

$$E(\mathcal{W}) \gg \max\big((\operatorname{card}\mathcal{W})^4 T^{-1}, (\operatorname{card}\mathcal{W})^2\big)$$

The last bound is probably close to best possible in most cases. However, the first bound is not very good for us in general but is achieved when the elements of $\mathcal{W}$ are close to being in arithmetic progression.

For example consider $\mathcal{W} = \{t_r : t_r = rT/R, 0 \le r \le R-1\}$. Put $v = \exp\left(\frac{2\pi Ru}{T}\right)$. Then $\int_{[1/4,4]} |R(v)|^4 dv$

$$= \frac{2\pi R}{T} \int_{|u| \le \frac{T}{R}\log 4} \left| \sum_{r=0}^{R-1} e(ur) \right|^4 \exp\left(\frac{2\pi Ru}{T}\right) du$$

$$\gg \int_0^1 \left| \sum_{r=0}^{R-1} e(ur) \right|^4 du = \int_0^1 \left| \sum_{|h| \le R} (R - |h|) e(uh) \right|^2 du$$

$$= \sum_{h=-R}^{R} (R - |h|)^2 = \frac{2R^3 + R}{3}.$$

This may seem a problem but if $\mathcal{W}$ were like this, then

$$\sum_{t \in \mathcal{W}} |D_N(it)|^2 \ll (R + RN/T) \sum_{n=1}^{N} |b_n|^2$$

and I think the density hypothesis would follow.

If we replace each t in $\mathcal{W}$ by, say, the nearest integer to t, and call the new set $\mathcal{N}$, then $E(W)$ is bounded by the number of solutions of $|n_1 - n_2 + n_3 - n_4| < 3$ and so it suffices to bound the number $E^*(\mathcal{N})$ of solutions of $n_1 - n_2 + n_3 - n_4 = 0$.

There ought to be theorems in additive combinatorics that tell us what happens in general when $E^*(\mathcal{N})$ is large. This will be large when $n_1 - n_2 + n_3$ has excessive overlap with $\mathcal{N}$.

The work in this area, such as Freiman's theorem, is concerned mostly with what happens when expressions such as $n + n'$ have excessive overlap with $\mathcal{N}$, and Freiman's theorem tells us that in that case the elements of $\mathcal{N}$ are in arithmetic progression.

Is that what is happening here? Understanding this might be the key to the density hypothesis.

Proof. We may certainly suppose that card $\mathcal{W} \geq 1$.

Let $\psi(u) \in \mathcal{C}^\infty([0,\infty))$ have compact support and be $\psi(u) \gg 1$ for $u \in [1/4, 4]$. Then

$$\int_{[1/4,4]} |R(u)|^4 du \ll \int_0^\infty \psi(u)|R(u)|^4 = \sum_{\mathbf{t}\in\mathcal{W}^4} \int_0^\infty \psi(u) u^{it_1 - it_2 + it_3 - it_4} du$$

As previously, we may apply the saddle point method when $|t_1 - t_2 + t_3 - t_4| > T^\varepsilon$ and obtain a total contribution $\ll_\varepsilon 1$. Thus

$$\int_{[1/4,4]} |R(u)|^2 du \ll_\varepsilon 1 + \sum_{\substack{\mathbf{t}\in\mathcal{W}^4 \\ |t_1-t_2+t_3-t_4|\leq T^\varepsilon}} 1.$$

Now, given any $t_1, t_2, t_3 \in \mathcal{W}$ there can be $\ll 1$ $t_4 \in \mathcal{W}$ with $|t_1 - t_2 + t_3 - t_4| \leq T^\varepsilon$. Now there is a general principle that the number of solutions of an inequality such as (or an equation for that matter) $|t_1 - t_2 + t_3 - t_4 - \alpha| < 1$ for some real number alpha is bounded by the number of solutions with $\alpha = 0$. Thus we can just divide the interval $-T^\varepsilon, T^\varepsilon$ into $\ll T^\varepsilon$ intervals of length, say, $1/2$. To prove the principle in our case observe that

$$\int_{\mathbb{R}} \frac{(\sin \pi u)^2}{\pi^2 u^2} e(-vu) du = \max(0, 1 - |v|)$$

and so

$$\begin{aligned}
\sum_{\substack{\mathbf{t}\in\mathcal{W}^4 \\ |t_1-t_2+t_3-t_4-\alpha|<1/2}} 1 &\leq 2 \int_{\mathbb{R}} \frac{(\sin \pi u)^2}{\pi^2 u^2} |R(e^{2\pi u})|^2 e(-\alpha u) du \\
&\leq 2 \int_{\mathbb{R}} \frac{(\sin \pi u)^2}{\pi^2 u^2} |R(e^{2\pi u})|^2 du \\
&\leq 2E(\mathcal{W})
\end{aligned}$$

which gives the second bound.

Summing over a set of $\gg T$ α spaced $1/2$ apart picks up every possible t_j and proves part of the last bound. The trivial observation that there are $(\operatorname{card}\mathcal{W})^2$ solutions to $|t_1 - t_2 + t_3 - t_4| < 1$ with $t_1 = t_2$, $t_3 = t_4$ gives the other part. $\square$

Corollary. *We have*

$$\int_{[1/4,4]} |R(u)|^3 du \ll_\varepsilon T^\varepsilon (\operatorname{card}\mathcal{W})^2.$$

This is not good enough. The set $\mathcal{W}$ in practice is quite "thin". That is the cardinality is small compared with T. For example the numbers $t_1 - t_2 + t_3 - t_4$ are spread, presumably, between $-2T$ and $2T$, but there are only $(\operatorname{card}\mathcal{W})^4$ of them so one can expect at best, in a general interval of length 1 that there are only about $(\operatorname{card}\mathcal{W})^4 T^{-1}$ of the $t_1 - t_2 + t_3 - t_4$ and this will of necessity be much smaller than $(\operatorname{card}\mathcal{W})^3$.

4. Back to Reality

Theorem 8.1.(Proposition 8.1) *Suppose that $\mathcal{W}$ is a T^ε separated set in an interval of length T. Then*

$$S_3 \ll T^{2+\varepsilon} (\operatorname{card}\mathcal{W})^{1/2} E(\mathcal{W})^{1/2}.$$

Proof. By Lemmas 8,2 and 8.3 and Schwarz's inequality we have

$$\int_{[1/4,4]} |R(u)|^3 du \ll_\varepsilon T^\varepsilon (\operatorname{card}\mathcal{W})^{1/2} E(\mathcal{W})^{1/2}.$$

Also

$$|R(v_1)R(v_2/v_1)R(v_2)| \le |R(v_1)|^3 + |R(v_2/v_1)|^3 + |R(v_2)|^3.$$

Thus

$$\int_{\substack{\mathbf{v}\in[1/2,2]^2 \\ |m_1v_1+m_2v_2+m_3| \le \frac{T^\varepsilon}{N}}} |R(v_1)|^3 d\mathbf{v} \ll T^\varepsilon (\operatorname{card}\mathcal{W})^{1/2} E(\mathcal{W})^{1/2} \frac{T^\varepsilon}{N|m_2|}.$$

Hence, summing over $\mathbf{m}$ we have

$$\sum_{\substack{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^3 \\ |m_1|\le|m_2|\le|m_3|\le T^{1+\varepsilon}N^{-1}}} N^3 \int_{\substack{\mathbf{v}\in[1/2,2]^2 \\ |m_1v_1+m_2v_2+m_3| \le \frac{T^\varepsilon}{N}}} |R(v_1)|^3 d\mathbf{v}$$
$$\ll T^{2+4\varepsilon} (\operatorname{card}\mathcal{W})^{1/2} E(\mathcal{W})^{1/2}.$$

A similar argument holds for $|R(v_2)|^3$. For the term $|R(v_2/v_1)|^3$ we make the substitute $v_2 = wv_1$ and note that then $w \in [1/(2v_1), 2/v_1] \subseteq$

$[1/4,4]$ and the contribution becomes

$$\sum_{\substack{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^3\\ |m_1|\le|m_2|\le|m_3|\le T^{1+\varepsilon}N^{-1}}} N^3\int_{\substack{\mathbf{v}\in[1/2,2]^2\\ |m_1v_1+m_2v_1w+m_3|\le\frac{T^\varepsilon}{N}}} |R(v_2/v_1)|^3 d\mathbf{v} =$$

$$N^3\int_{[1/4,4]}|R(w)|^3 \sum_{\substack{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^3\\ |m_1|\le|m_2|\le|m_3|\le T^{1+\varepsilon}N^{-1}}} \int_{\substack{v\in[1/2,2]\cap[1/(2w),2/w]\\ |v(m_1+m_2w)+m_3|\le\frac{T^\varepsilon}{N}}} v dv dw$$

The condition $|v(m_1+m_2w)+m_3|\le\frac{T^\varepsilon}{N}$ together with $m_3\ne0$ implies that $|v(m_1+m_2w)|>\frac12$ and so $|m_1+m_2w|>\frac{1}{16}$. Moreover the integral over v is

$$\ll\frac{T^\varepsilon}{N|m_1+m_2w|}.$$

and again we can proceed as for $|R(v_1)|^3$. Thus, by Theorem 7.1,

$$S_3\ll T^{2+4\varepsilon}(\operatorname{card}\mathcal{W})^{1/2}E(\mathcal{W})^{1/2}$$

as required.

Let me go back to §7. I have the feeling that §§9 onwards are over-complicated and this starts with Proposition 7.2. I would like to find a simpler approach, perhaps based on the observation that as far as $E\mathcal{W})$ and related objects are concerned one can suppose that $\mathcal{W}$ is a subset of integers in $[0,T]$ and this ought to simplify much of the argument. Unfortunately I don't have the time to sort all that out, so am forced largely to follow G&M.

Recall that

$$R(u)=\sum_{t\in\mathcal{W}}u^{it}.$$

and Theorem 7.1 gives

$$S_3\ll_\varepsilon \sum_{\substack{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^3\\ |m_1|\le|m_2|\le|m_3|\le\frac{T^{1+\varepsilon}}{N}\\ |m_2|\asymp|m_3|}} N^3 \int_{\substack{\mathbf{v}\in[1/2,2]^2\\ |m_1v_1+m_2v_2+m_3|\le\frac{T^\varepsilon}{N}}} \left|R(v_1)R\Big(\frac{v_2}{v_1}\Big)R(v_2)\right|d\mathbf{v} + T^{-200}$$

By dividing the sum over $\mathbf{m}$ into dyadic ranges we find that for some pair M_1, M with $0<M_1\le M\le T^{1+\varepsilon}/N$ we have

$$S_3\ll_\varepsilon T^{-200}+N^3(\log T)^3 \sum_{\substack{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^3\\ |m_1|\le|m_2|\le|m_3|\le M\\ |m_1|\asymp M_1,|m_2|\asymp|m_3|\asymp M}} I'_{\mathbf{m}}$$

where

$$I'_{\mathbf{m}} = \int_{v_1\in[1/2,2]} |R(v_1)| \int_{\substack{v_2\in[1/(2),2] \\ |m_1v_1m_2^{-1}+v_2+m_3m_2^{-1}|\le \frac{T^\varepsilon}{MN}}} \Big|R\Big(\frac{v_2}{v_1}\Big)R(v_2)\Big| d\mathbf{v}$$

At this point the $R(v_2/v_1)$ and $R(v_2)$ are replaced by smooth versions but the details are mostly missing and the motivation is non-existent. By Cauchy-Schwarz and a change of variable $v = v_1 w$ we have

$$\int_{\substack{v\in[1/2,2] \\ |v-(m_1v_1+m_3)/(-m_2)|\le \frac{T^\varepsilon}{MN}}} \Big|R\Big(\frac{v}{v_1}\Big)R(v)\Big| dv \le \left(\int_{\substack{w\in[1/(2v_1),2/v_1] \\ |w-(m_1+m_3v_1^{-1})/(-m_2)|\le \frac{2T^\varepsilon}{MN}}} |R(w)|^2 dw\right)^{1/2} \times \left(\int_{\substack{v\in[1/2,2] \\ |v-(m_1v_1+m_3)/(-m_2)|\le \frac{T^\varepsilon}{MN}}} |R(v)|^2 dv\right)^{1/2}.$$

Now let $\tilde\psi(x)$ be another $\mathcal{C}^\infty$ function which majorizes the characteristic function of $[1/4,4]$ and is 0 outside $[-CT^\varepsilon, CT^\varepsilon]$, and also satisfies $\|\tilde\psi^{(j)}\|_\infty \ll_j T^\varepsilon$. Then

$$\int_{\substack{w\in[1/(2v_1),2/v_1] \\ |w-(m_1+m_3v_1^{-1})/(-m_2)|\le \frac{2T^\varepsilon}{MN}}} |R(w)|^2 dw$$

is bounded by

$$\frac{1}{MN}\tilde R\big((m_1+m_3v_1^{-1})/(-m_2)\big)^2$$

where

$$\tilde R(u) = \left(\int_{[1/4,4]} MN\tilde\psi\big(MN(u-w)\big)|R(w)|^2 dw\right)^{1/2}.$$

They have an extra factor $\tilde\psi(e^u)$ in their integrand, but I don't see why. I see that later in §10 it has disappeared! Also they do not include a domain of integration. But $[1/4,4]$ is reasonable and works.

We can treat the second integral the same way. Thus

$$I'_m \ll \frac{1}{MN}\int_{v\in[1/2,2]} |R(v)|\tilde R\left(\frac{m_1v+m_3}{-vm_2}\right)\tilde R\left(\frac{m_1v+m_3}{-m_2}\right) dv$$

Since the sums are symmetric in positive and negative values of m_2 we have just established

Theorem 7.2(Proposition 7.2) *For some* M_1, M *with* $0 < M_1 \le M \le T/N$ *we have*

$$S_3 \ll \frac{N^2}{M} \sum_{|m_1| \asymp M_1, |m_2| \asymp |m_3| \asymp M} \tilde{I}_{\mathbf{m}} + T^{-100}$$

where

$$\tilde{I}_{\mathbf{m}} = \int_{v \in [1/2,2]} |R(v)| \tilde{R}\left(\frac{m_1 v + m_3}{v m_2}\right) \tilde{R}\left(\frac{m_1 v + m_3}{m_2}\right) dv.$$

Having belatedly introduced $\tilde{R}$ we should now restate the second part of Lemma 8.3

Lemma 8.2†. *For any* M *we have*

$$\int_{v \asymp 1} |\tilde{R}(v)|^4 \ll T^{\varepsilon} E(W).$$

5. Affine Transformations

Here they obtain a rather general bound for objects of the kind

$$J(f) = \sup_{0 < M_j \le M} \int_{\mathbb{R}} \left(\sum_{m_j \asymp M_j} f\left(\frac{m_1 u + m_3}{m_2}\right)\right)^2 du$$

where f is smooth and compactly supported. They do not state the region of integration, but in practice it will certainly be contained in $[1/4, 4]$ and probably in $[1/2, 2]$.

Theorem 9.1.(Proposition 9.1.) *Suppose that* f *is non-negative and supported on* $[1/C, C]$ *for some constant* $C > 1$. *Suppose also that for all* $j \ge 0$

$$|\hat{f}(\xi)| \ll_j T^{\varepsilon - j} |\xi|^j.$$

(or do they mean $|\hat{f}(\xi)| \ll_j T^{\varepsilon + j} |\xi|^{-j}$*? See their assertion at the top of page 31.) Then*

$$J(f) \ll T^{\varepsilon} M^6 \left(\int_{\mathbb{R}} f(u) du\right)^2 + T^{\varepsilon} M^4 \int_{\mathbb{R}} f(u)^2 du.$$

I am going to skip the proof of this for the time being, The proof is not very illuminating. It is really analysis, not number theory, and I wonder if there should be a more succinct and illuminating proof. I am also puzzled by the hypothesis $|\hat{f}(\xi)| \ll_j T^{\varepsilon - j} |\xi|^j$. Later they ask for "rapid decay for $|\xi| > T$" which is not what is being asserted here. I suspect that they mean $|\hat{f}(\xi)| \ll_j T^{\varepsilon + j} |\xi|^{-j}$, and this is what I will assume henceforward.

6. A Further Bound for S_3

Theorem 10.1.(Proposition 10.1) *Suppose that $\mathcal{W}$ is a T^ε separated set contained in an interval of length T. Then*

$$S_3 \ll_\varepsilon T^{2+\varepsilon}(\operatorname{card}\mathcal{W})^{3/2} + T^{1+\varepsilon}N(\operatorname{card}\mathcal{W})^{1/2}E(\mathcal{W})^{1/2}.$$

This is a useful improvement compared with Theorem 8.1,

$$T^{2+\varepsilon}(\operatorname{card}\mathcal{W})^{1/2}E(\mathcal{W})^{1/2},$$

when $E(\mathcal{W})$ is between $(\operatorname{card}\mathcal{W})^2$ and $(\operatorname{card}\mathcal{W})^3$ in size. The proof is now fairly straightforward.

Proof. Theorem 7.2 states that for some M_1, M with $0 < M_1 \le M \le T/N$ we have

$$S_3 \ll \frac{N^2}{M} \sum_{\substack{|m_1|\asymp M_1,\\ |m_2|\asymp|m_3|\asymp M}} \tilde{I}_{\mathbf{m}} + T^{-100}$$

where

$$\tilde{I}_{\mathbf{m}} = \int_{v\in[1/2,2]} |R(v)|\tilde{R}\left(\frac{m_1v+m_3}{vm_2}\right)\tilde{R}\left(\frac{m_1v+m_3}{m_2}\right) dv.$$

By the Cauchy-Schwarz inquality

$$\sum_{\mathbf{m}}{}^{*}\, \tilde{R}\left(\frac{m_1v+m_3}{vm_2}\right)\tilde{R}\left(\frac{m_1v+m_3}{m_2}\right)$$
$$\le \left(\sum_{\mathbf{m}}{}^{*}\, \tilde{R}\left(\frac{m_1v+m_3}{vm_2}\right)^2\right)^{1/2}\left(\sum_{\mathbf{m}}{}^{*}\, \tilde{R}\left(\frac{m_1v+m_3}{m_2}\right)^2\right)^{1/2}$$

where $\sum^*$ denotes summation over $|m_1| \asymp M_1, |m_2| \asymp |m_3| \asymp M$. Hence by Hölder's inequality

$$S_3 \ll T^{-100} + N^2M^{-1}S_{3,1}^{1/2}S_{3,3}^{1/4}S_{3,4}^{1/4}$$

where

$$
\begin{aligned}
S_{3,1} &= \int_{v\in[1/2,2]} |R(v)|^2 dv, \\
S_{3,3} &= \int_{v\in[1/2,2]} \left(\sum_{\mathbf{m}}{}^{*} \tilde{R}\left(\frac{m_1 v + m_3}{v m_2}\right)^2 \right)^2 dv, \\
&= \int_{v\in[1/2,2]} \left(\sum_{\mathbf{m}}{}^{*} \tilde{R}\left(\frac{m_1 + m_3 v}{m_2}\right)^2 \right)^2 dv, \\
S_{3,4} &= \int_{v\in[1/2,2]} \left(\sum_{\mathbf{m}}{}^{*} \tilde{R}\left(\frac{m_1 v + m_3}{m_2}\right)^2 \right)^2 dv.
\end{aligned}
$$

Note that in $S_{3,3}$ we made the change of variable $v \to 1/v$.

Since

$$S_{3,1} \ll \operatorname{card} \mathcal{W}$$

we have

$$S_3 \ll T^{-100} + N^2 M^{-1} (\operatorname{card} \mathcal{W})^{1/2} S_{3,3}^{1/4} S_{3,4}^{1/4}$$

Recall that

$$\tilde{R}(u)^2 = \int_{[1/4,4]} MN\tilde{\psi}\big(MN(u-w)\big)|R(w)|^2 dw$$

and its Fourier transform is

$$
\begin{aligned}
&\int_{\mathbb{R}} \tilde{R}(u)^2 e(-u\xi) du \\
&\qquad = \int_{\mathbb{R}} \int_{[1/4,4]} MN\tilde{\psi}\big(MN(u-w)\big)|R(w)|^2 dw e(-u\xi) du \\
&\qquad = \int_{[1/4,4]} |R(w)|^2 \int_{\mathbb{R}} MN\tilde{\psi}\big(MN(u-w)\big) e(-u\xi) du dw.
\end{aligned}
$$

Also recall that it is assumed that $\|\tilde{\psi}^{(j)}\|_\infty \ll_j T^\varepsilon$. Hence standard integration by parts j times shows that this is

$$\ll_j (\operatorname{card} \mathcal{W})(MN)^j |\xi|^{-j} T^\varepsilon$$

which seems to be consistent with what they probably intended to assume in the hypothesis of Theorem 9.1. Thus we can apply that theorem to $S_{3,3}$ and $S_{3,4}$ with $C = 2$ and $f(u) = \tilde{R}(u)^2$. Then

$$\begin{aligned}\int_{\mathbb{R}} f(u)du &= \int_{[1/4,4]} \int_{\mathbb{R}} MN\tilde{\psi}\big(MN(u-w)\big)|R(w)|^2 dudw \\ &\ll \int_{[1/4,4]} |R(w)|^2 dw \\ &\ll \operatorname{card} \mathcal{W}.\end{aligned}$$

and

$$\int_{\mathbb{R}} f(u)^2 du =$$

$$\int_{[1/4,4]^2} \int_{\mathbb{R}} M^2N^2\tilde{\psi}\big(MN(u-v_1)\big)\tilde{\psi}\big(MN(u-v_2)\big)du|R(v_1)R(v_2)|^2 d\mathbf{v}.$$

In the latter integrand write $|R(v_1)R(v_2)|^2 \le |R(v_1)|^4 + |R(v_2)|^4$ and then proceed as before. Thus

$$\int_{\mathbb{R}} f(u)^2 du \ll \int_{[1/4,4]} |R(w)|^4 dw \ll_\varepsilon T^\varepsilon E(\mathcal{W})$$

and we obtain

$$\max(S_{3,3}, S_{3,4}) \ll M^6(\operatorname{card} \mathcal{W})^2 + M^4 T^\varepsilon E(\mathcal{W}).$$

Recalling that

$$S_3 \ll T^{-100} + N^2M^{-1}(\operatorname{card} \mathcal{W})^{1/2} S_{3,3}^{1/4} S_{3,4}^{1/4}$$

we have

$$S_3 \ll N^2M^2(\operatorname{card} \mathcal{W})^{3/2} + T^\varepsilon N^2 M(\operatorname{card} \mathcal{W})^{1/2} E(\mathcal{W})^{1/2}.$$

This completes the proof of Theorem 10.1.