C:MeanLargeVals

THE SECOND SUM

1. Recap

Recall

$$w \in \mathcal{C}^\infty(\mathbb{R}),\ \operatorname{supp} w \in [1,2],\ w(x) = 1\,(x \in [6/5, 9/5]),$$

$$h_t(u) = w(u)^2 u^{it}$$

and its Fourier transform

$$\hat{h}_t(v) = \int_{\mathbb{R}} h_t(u) e(-uv) du.$$

Theorem 4.6.[Proposition 4.6] *Suppose that $\mathcal{W}$ is T^ε separated and $|D_N(t)| > V$ for each $t \in \mathcal{W}$. Then*

$$\operatorname{card} W \ll_\varepsilon \left(N + \Big(\sum_{\mathbf{m} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}} I_{\mathbf{m}} \Big)^{1/3} \right) \|\mathbf{b}\|^2 V^{-2}.$$

where

$$I_{\mathbf{m}} = N^3 \sum_{\mathbf{t} \in \mathcal{W}^3} \hat{h}_{t_1-t_2}(m_1 N) \hat{h}_{t_2-t_3}(m_2 N) \hat{h}_{t_3-t_1}(m_3 N).$$

Thus we have reduced the investigation to bounding

$$\sum_{\mathbf{m} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}} I_{\mathbf{m}} = S_1 + S_2 + S_3.$$

where S_j denotes the sum over $\mathbf{m}$ with exactly j of the m_j being non-zero. We already saw that

Theorem 5.1.(Proposition 5.1) We have

$$S_1 \ll_\varepsilon T^{-10}.$$

Now is the turn of S_2 where

$$\begin{aligned} S_2 = N^3 \sum_{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^2} \sum_{\mathbf{t}\in\mathcal{W}^3} & \hat{h}_{t_1-t_2}(0)\hat{h}_{t_2-t_3}(m_1N)\hat{h}_{t_3-t_1}(m_2N) \\ & + \hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_2-t_3}(0)\hat{h}_{t_3-t_1}(m_2N) \\ & \qquad + \hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_2-t_3}(m_2N)\hat{h}_{t_3-t_1}(0) \\ = 3N^3 \sum_{\mathbf{m}\in(\mathbb{Z}\setminus\{0\})^2} \sum_{\mathbf{t}\in\mathcal{W}^3} & \hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_2-t_3}(m_2N)\hat{h}_{t_3-t_1}(0) \\ = 3N^3 \sum_{\mathbf{t}\in\mathcal{W}^3} \hat{h}_{t_3-t_1}(0) & \sum_{m_1\in\mathbb{Z}\setminus\{0\}} \hat{h}_{t_1-t_2}(m_1N) \sum_{m_2\in\mathbb{Z}\setminus\{0\}} \hat{h}_{t_2-t_3}(m_2N). \end{aligned}$$

This is where Heath-Brown's theorem comes into play.

First we require some preparation. When $t_1 \neq t_3$ we have, by Lemma 4.3,

$$\hat{h}_{t_3-t_1}(0) \ll |t_3 - t_1|^j \ll T^{-100}$$

and for $m \neq 0$

$$\hat{h}_t(mN) \ll \frac{1+|t|^2}{m^2N^2}.$$

Thus the total contribution from the terms with $t_1 \neq t_3$ is

$$\ll T^{-100}T^3(1+T^4)N^{-2} \ll T^{-10}.$$

The remaining terms contribute

$$\begin{aligned} &= 3N^3\hat{h}_0(0) \sum_{\mathbf{t}\in\mathcal{W}^2} \sum_{m_1\in\mathbb{Z}\setminus\{0\}} \hat{h}_{t_1-t_2}(m_1N) \sum_{m_2\in\mathbb{Z}\setminus\{0\}} \hat{h}_{t_2-t_1}(m_2N) \\ &= 3N^3\hat{h}_0(0) \sum_{\mathbf{t}\in\mathcal{W}^2} \sum_{m_1\in\mathbb{Z}\setminus\{0\}} \hat{h}_{t_1-t_2}(m_1N) \sum_{m_2\in\mathbb{Z}\setminus\{0\}} \hat{h}_{t_2-t_1}(-m_2N) \\ &= 3N^3\hat{h}_0(0) \sum_{\mathbf{t}\in\mathcal{W}^2} \left| \sum_{m\in\mathbb{Z}\setminus\{0\}} \hat{h}_{t_1-t_2}(mN) \right|^2 \end{aligned}$$

since

$$\hat{h}_{-t}(-v) = \int_{\mathbb{R}} w(u)^2u^{-it}e(vu)du = \overline{\hat{h}_t(v)}.$$

Thus to summarise the story so far

$$S_2 = 3N^3\hat{h}_0(0) \sum_{\mathbf{t}\in\mathcal{W}^2} \left| \sum_{m\in\mathbb{Z}\setminus\{0\}} \hat{h}_{t_1-t_2}(mN) \right|^2 + O(T^{-10}).$$

Since, by Lemma 4.3,

$$\hat{h}_0(mN) \ll |m|^{-j}N^{-j}$$

and

$$\hat{h}_0(0) = \int_{\mathbb{R}} w(u)^2 du \ll 1,$$

the contribution of the terms with $t_1 = t_2$ to the above sum is

$$\ll N^{3-2j} T \ll T^{-10}.$$

Thus

$$S_2 \ll N^3 \sum_{\substack{\mathbf{t} \in \mathcal{W}^2 \\ t_1 \neq t_2}} \left| \sum_{m \in \mathbb{Z} \setminus \{0\}} \hat{h}_{t_1 - t_2}(mN) \right|^2 + T^{-10}.$$

The sum over the pairs t_1, t_2 is now divided dyadically according to the size of $|t_1 - t_2|$. This quantity satisfies $T^\varepsilon < |t_1 - t_2| \leq T$ so $T^\varepsilon N^{-1} \leq |t_1 - t_2| N^{-1} \leq T N^{-1}$. Thus we can suppose that $2^j < |t_1 - t_2| N^{-1} \leq 2^{j+1}$ for some integer j with $T^\varepsilon (2N)^{-1} < 2^j \leq 2TN^{-1}$. There are $\ll_\varepsilon$ $logT$ choices for j. Hence by taking $M = 2^j$ we have

$$S_2 \ll T^\varepsilon N^3 \sup_{\substack{M \\ T^\varepsilon/(2N) \leq M \leq 2T/N}} \sum_{\substack{\mathbf{t} \in \mathcal{W}^2 \\ MN < |t_1 - t_2| \leq 2MN}} \left| \sum_{m \in \mathbb{Z} \setminus \{0\}} \hat{h}_{t_1 - t_2}(mN) \right|^2 + T^{-10}.$$

At this point we employ

Lemma 6.2. *Suppose that $\varepsilon > 0$, T is sufficiently large in terms of ε, and N is a positive integer with $T^\varepsilon \leq N$. Suppose also that M is a positive real number with $T^\varepsilon/(2N) < M < 2T/N$ and $MN < |t| \leq 2MN$. Then*

$$\left| \sum_{m \in \mathbb{Z} \setminus \{0\}} \hat{h}_t(mN) \right| \ll_\varepsilon (MN)^{-1/2} \int_{|u| \leq T^\varepsilon} \left| \sum_{1 \leq m \leq T^\varepsilon M} m^{i(t+u)} \right| du + T^{-100}.$$

We apply this with $t = t_1 - t_2$. Thus, by Schwarz' inequality

$$\left| \sum_{m \in \mathbb{Z} \setminus \{0\}} \hat{h}_{t_1 - t_2}(mN) \right|^2 \ll \frac{T^\varepsilon}{MN} \int_{|u| \leq T^\varepsilon} \left| \sum_{1 \leq m \leq T^\varepsilon M} m^{i(t_1 - t_2 + u)} \right|^2 du + T^{-200}$$

and so

$$S_2 \ll T^{2\varepsilon} \sup_{\substack{M \\ T^\varepsilon/(2N)\le M\le 2T/N}} \int_{|u|\le T^\varepsilon} \frac{N^2}{M} \sum_{\substack{\mathbf{t}\in\mathcal{W}^2 \\ MN<|t_1-t_2|\le 2MN}} \left| \sum_{1\le m\le T^\varepsilon M} m^{i(t_1-t_2+u)} \right|^2 du + T^{-10}$$

Put $a(m) = a(m,u) = m^{iu}$ Then

$$S_2 \ll T^{3\varepsilon} \sup_{\substack{M \\ T^\varepsilon/(2N)\le M\le 2T/N}} \sup_{|u|\le T^\varepsilon} \frac{N^2}{M} \sum_{\substack{\mathbf{t}\in\mathcal{W}^2 \\ MN<|t_1-t_2|\le 2MN}} \left| \sum_{1\le m\le T^\varepsilon M} a(m) m^{i(t_1-t_2)} \right|^2 + T^{-10}$$

Having worked really hard to get here we can now relax and throw some things away! We have

$$S_2 \ll T^{3\varepsilon} \sup_{\substack{M,u \\ M\le 2T/N \\ |u|\le T^\varepsilon}} \frac{N^2}{M} \sum_{\mathbf{t}\in\mathcal{W}^2} \left| \sum_{1\le m\le T^\varepsilon M} a(m) m^{i(t_1-t_2)} \right|^2 + T^{-10}.$$

We divide the sum over M into $\ll \log T$ dyadic ranges $K < m \le 2K$ where $K \le T^\varepsilon M$. Thus

$$S_2 \ll T^{4\varepsilon} \sup_{\substack{K,u \\ K\le T^{1+\varepsilon}/N \\ |u|\le T^\varepsilon}} \frac{N^2}{K} \sum_{\mathbf{t}\in\mathcal{W}^2} \left| \sum_{K<m\le 2K} a(m) m^{i(t_1-t_2)} \right|^2 + T^{-10}$$

We now pick extremal values for K and u. The only information about $a(m) = m^{iu}$ that we will use is that $|a(m)| = 1$. Thus for some $K \le T^{1+\varepsilon}N^{-1}$ and $|u| \le T^\varepsilon$

$$S_2 \ll T^{5\varepsilon} \frac{N^2}{K} \sum_{\mathbf{t}\in\mathcal{W}^2} \left| \sum_{K<m\le 2K} a(m) m^{i(t_1-t_2)} \right|^2 + T^{-10}$$

Thus we can just concentrate on

$$\Psi = \sum_{\mathbf{t}\in\mathcal{W}^2} \left| \sum_{K<m\le 2K} a(m) m^{i(t_1-t_2)} \right|^2.$$

By Hölder's inequality for any $k \in \mathbb{N}$ we have

$$\Psi^k \ll (\operatorname{card}\mathcal{W})^{2k-2} \sum_{\mathbf{t}\in\mathcal{W}^2} \left| \sum_{K^k < m \leq (2K)^k} b(m) m^{i(t_1-t_2)} \right|^2$$

where

$$b(m) = \sum_{\substack{\mathbf{m}\in(K,2K]^k \\ m_1\dots m_k = m}} a(m_1)\dots a(m_k)$$

satisfies

$$|b(m)| \leq d_k(m)$$

This is exactly the kind of expression considered by Heath-Brown.

Theorem 1.6(Heath-Brown). *Let $\mathcal{T}$ be a 1-separated set of real numbers contained in an interval of length $[-T,T]$ and suppose $R = \operatorname{card}\mathcal{T}$ and $N \geq 1$, and that the c_n are complex numbers. Then there is a positive constant c such that*

$$\sum_{\mathbf{t}\in\mathcal{T}^2} \left| \sum_{n=1}^{N} c(n) n^{i(t_1-t_2)} \right|^2 \ll$$

$$\exp\left(\frac{c\log T}{\log\log T}\right)\left(NR + R^2 + R^{\frac{5}{4}}T^{\frac{1}{2}}\right)\max_{n\leq N} n|c(n)|^2$$

We apply this to Ψ^k. For brevity write $W = \operatorname{card} W$. Then

$$\Psi^k \ll_\varepsilon W^{2k-2}T^{\varepsilon/2}\left(K^{2k}W + K^kW^2 + K^kW^{\frac{5}{4}}T^{\frac{1}{2}}\right)\max_{K^k<m\leq(2K)^k} d_k(m)^2$$

so that

$$\Psi \ll_\varepsilon T^\varepsilon\left(K^2W^{2-\frac{1}{k}} + KW^2 + KW^{2-\frac{3}{4k}}T^{\frac{1}{2k}}\right)$$

Thus we have established

Theorem 6.1(Proposition 6.1, S_2 bound). For any $k \in \mathbb{N}$ we have

$$S_2 \ll_\varepsilon T^{7\varepsilon}\left(TNW^{2-\frac{1}{k}} + N^2W^2 + N^2W^{2-\frac{3}{4k}}T^{\frac{1}{2k}}\right)$$

2. Comment

One thing that worries me a little bit here is that Guth and Maynard do not keep track of the powers of T^ε that arise in their arguments. They blithely use the notation $A \lessapprox B$ to just mean $A \leq C(\varepsilon)T^\varepsilon B$. Normally one can get round any build up of powers of T^ε with the observation that the bounds hold for all small ε and one can replace ε by ε/k for a suitable k. However here ε is fixed by the assumption that $\mathcal{W}$ is ε separated. If one keeps track of the powers of T^ε as is done above one has the get out of gaol card at the end of the proof by observing that any 1-separated set can be divided into at most T^ε separated sets, so one has a bound in the 1-separated case, of course dependent on ε, but which holds for any fixed $\varepsilon > 0$. Thus I think it would be more satisfactory to state the final conclusions for 1-separated sets.