C:Sthree

SECTION 11

1. Recap

Recall

$$w \in \mathcal{C}^\infty(\mathbb{R}),\ \operatorname{supp} w \in [1,2],\ w(x) = 1\,(x \in [6/5, 9/5]),$$

$$h_t(u) = w(u)^2 u^{it}$$

and its Fourier transform

$$\hat{h}_t(v) = \int_{\mathbb{R}} h_t(u) e(-uv) du.$$

Suppose that $\mathcal{W}$ is T^ε separated. Then

$$\sum_{t\in\mathcal{W}} |D_n(it)|^2 \ll_\varepsilon \left(N + \Big(\sum_{\mathbf{m}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}} I_{\mathbf{m}}\Big)^{1/3}\right) \sum_{n=N+1}^{2N} |b_n|^2. \qquad (1.1)$$ eq:one1

where

$$D_N(s) = \sum_{n=N+1}^{2N} b_n n^s \qquad (1.2)$$ eq:one2

and

$$I_{\mathbf{m}} = N^3 \sum_{\mathbf{t}\in\mathcal{W}^3} \hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_2-t_3}(m_2N)\hat{h}_{t_3-t_1}(m_3N). \qquad (1.3)$$ eq:one3

Theorem 4.6.(Proposition 4.6) *Suppose that $\mathcal{W}$ is T^ε separated and $|D_N(t)| > V$ for each $t \in \mathcal{W}$. Then*

$$\operatorname{card} W \ll_\varepsilon \left(N + \Big(\sum_{\mathbf{m}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}} I_{\mathbf{m}}\Big)^{1/3}\right) \|\mathbf{b}\|^2 V^{-2}.$$

Thus we have reduced the investigation to bounding

$$\sum_{\mathbf{m}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}} I_{\mathbf{m}} = S_1 + S_2 + S_3.$$

where S_j denotes the sum over $\mathbf{m}$ with exactly j of the m_j being non-zero. We already saw that
Theorem 5.1(Proposition 5.1, S_1 bound) *Suppose that $\mathcal{W}$ is T^ε separated. Then*

$$S_1 \ll_\varepsilon T^{-10}.$$

Theorem 6.1(Proposition 6.1, S_2 bound). *Suppose that $\mathcal{W}$ is T^ε separated. Then for any $k \in \mathbb{N}$ we have*

$$S_2 \ll_\varepsilon T^{7\varepsilon}\left(TNW^{2-\frac{1}{k}} + N^2W^2 + N^2W^{2-\frac{3}{4k}}T^{\frac{1}{2k}}\right)$$

Theorem 8.1.(Proposition 8.1) *Suppose that $\mathcal{W}$ is a T^ε separated set in an interval of length T. Then*

$$S_3 \ll T^{2+\varepsilon}(\operatorname{card}\mathcal{W})^{1/2}E(\mathcal{W})^{1/2}$$

where

$$E(\mathcal{W}) = \operatorname{card}\{\mathbf{t} \in \mathcal{W}^4 : |t_1 - t_2 + t_3 - t_4| < 1\}.$$

Theorem 10.1.(Proposition 10.1) *Suppose that $\mathcal{W}$ is a T^ε separated set contained in an interval of length T. Then*

$$S_3 \ll_\varepsilon T^{2+\varepsilon}(\operatorname{card}\mathcal{W})^{3/2} + T^{1+\varepsilon}N(\operatorname{card}\mathcal{W})^{1/2}E(\mathcal{W})^{1/2}.$$

This is a useful improvement compared with Theorem 8.1 when $E(\mathcal{W})$ is between $(\operatorname{card}\mathcal{W})^2$ and $(\operatorname{card}\mathcal{W})^3$ in size.

2. Energy Bound

Here they are concerned with the interrelationship between Dirichlet polynomials and $E(\mathcal{W})$ when either is large. As usual we suppose that

$$D_N(s) = \sum_{n=N+1}^{2N} b_n n^s$$

with $|b_n| \le 1$.

Theorem 11.2 (Proposition 11.2.) *Suppose that $\mathcal{W}$ is a 1-separated set contained in an interval of length T, that $T \ge N^{3/4}$ and that $\sigma \in [0,1]$ is such that $D_N(it) \ge N^\sigma$ for $t \in \mathcal{W}$. Let $R = \operatorname{card}\mathcal{W}$. Then*

$$S_3 \ll T^{2+\varepsilon}R^{3/2} + TRN^{3-2\sigma} + T^{9/8}R^{29/16}N^{3/2-\sigma}.$$

This depends on inserting the following in Theorem 10.1.

Theorem 11.1 (Proposition 11.1.) *Suppose that $\mathcal{W}$ is a 1-separated set contained in an interval of length T, that $N^{3/4} \le T \le N$ and that $\sigma \in [0,1]$ is such that $D_N(it) \ge N^\sigma$ for $t \in \mathcal{W}$. Let $R = \operatorname{card}\mathcal{W}$. Then*

$$E(\mathcal{W}) \ll T^\varepsilon RN^{4-4\sigma} + T^{\frac{1}{4}+\varepsilon}R^{21/8}N^{1-2\sigma} + T^\varepsilon R^3N^{1-2\sigma}.$$

The proof of Theorem 11.1 is quite complex and is divided into lemmas.

Lemma 11.3 *We have*

$$D_N(it) \ll_A \int_{t-T^\varepsilon}^{t+T^\varepsilon} |D_N(iu)|du + T^{-A}.$$

The proof has some more nonsense in it, including a spurious factor $w(n/N)$!

Proof. Let $\psi(x)$ be a $\mathcal{C}^\infty$ with the properties that $\psi(x) = 1$ when $\log N \le 2\pi x \le \log(2N)$, that there is a constant $C > 1$ such that $\psi(x) = 0$ when $2\pi x \le \log(N/C)$ and when $2\pi x \ge \log(NC)$, and that for every $j \in \mathbb{N}$ we have $\hat{\psi}(\xi) \ll_j |\xi|^{-j}$. Then

$$\begin{aligned} D_N(it) &= \sum_{n=N+1}^{2N} b_n n^{it} \psi\left(\frac{\log n}{2\pi}\right) \\ &= \sum_{n=N+1}^{2N} b_n n^{it} \int_{\mathbb{R}} \hat{\psi}(\xi) e\left(\frac{\xi(\log n)}{2\pi}\right) d\xi. \end{aligned}$$

Since ψ is real the above is

$$\begin{aligned} &= \sum_{n=N+1}^{2N} b_n n^{it} \int_{\mathbb{R}} \hat{\psi}(\xi) e\left(\frac{-\xi(\log n)}{2\pi}\right) d\xi \\ &= \int_{\mathbb{R}} \hat{\psi}(\xi) D_N(it - i\xi) d\xi. \end{aligned}$$

The usual integration by parts argument gives the conclusion. □

This is used in the proof of the next lemma.

Lemma 11.4. *Recall that*

$$R(u) = \sum_{t \in \mathcal{W}} u^{it}$$

and we are assuming $\sigma \in [0,1]$ is such that $D_N(it) \ge N^\sigma$ for $t \in \mathcal{W}$. Then

$$E(\mathcal{W}) \ll T^\varepsilon N^{-2\sigma} \sum_{n_1, n_2 \sim N} |R(n_1/n_2)|^3.$$

The is more nonsense in the proof with spurious factors of $w(n/N)$! They also say that $\mathcal{W}$ is T^ε-separated when the hypothesis of the theorem says 1-separated.

Proof. We have, by Lemma 11.3,

$$\begin{aligned} E(\mathcal{W}) &\le \sum_{\substack{\mathbf{t} \in \mathcal{W}^4 \\ |t_1 - t_2 + t_3 - t_4| \le 1}} N^{-2\sigma} |D_N(it_4)|^2 \\ &\ll_A T^{-A} + N^{-2\sigma} \sum_{\substack{\mathbf{t} \in \mathcal{W}^4 \\ |t_1 - t_2 + t_3 - t_4| \le 1}} \int_{|u - t_4| \le T^\varepsilon} |D_N(iu)|^2 du \end{aligned}$$

Since the next bound is trivial when $E(\mathcal{W}) = 0$ we may suppose $E(\mathcal{W}) \geq 1$ and then suppress the T^{-A}. Also, since $\mathcal{W}$ is 1-separated, given t_1, t_2, t_3 there are $\ll 1$ choices for t_4. Thus $E(\mathcal{W})$

$$\begin{aligned}
&\ll N^{-2\sigma} \sum_{\mathbf{t}\in\mathcal{W}^3} \int_{|v|\leq 2T^\varepsilon} |D_N(iv+it_1-it_2+it_3)|^2 dv \\
&= N^{-2\sigma} \sum_{N<n_1,n_2\leq 2N} a_{n_1}\overline{a_{n_2}} \int_{|v|\leq 2T^\varepsilon} \sum_{\mathbf{t}\in\mathcal{W}^3} (n_1/n_2)^{iv+it_1-it_2+it_3} dv \\
&= N^{-2\sigma} \sum_{N<n_1,n_2\leq 2N} a_{n_1}\overline{a_{n_2}} \int_{|v|\leq 2T^\varepsilon} (n_1/n_2)^{iv} R(n_1/n_2)^2 R(n_2/n_1) dv \\
&\ll N^{-2\sigma} \sum_{N<n_1,n_2\leq 2N} |R(n_1/n_2)|^3.
\end{aligned}$$

□

At this point they use Heath-Brown's result to bound the second moment of R. Recall that this says that if $\mathcal{W}$ is a 1-separated subset of an interval of length T, and $|b_n| \ll T^\varepsilon$, then

$$\sum_{\mathbf{t}\in\mathcal{W}^2} |D_N(it_1-it_2)|^2 \ll T^\varepsilon(\mathrm{card}(\mathcal{W})^2 N + \mathrm{card}(\mathcal{W}) N^2 + \mathrm{card}(\mathcal{W})^{5/4} T^{1/2} N).$$

Lemma 11.5. *For any $M \geq 1$ we have*

$$\sum_{M\leq n_1,n_2\leq 2M} |R(n_1/n_2)|^2 \ll T^\varepsilon \left(\mathrm{card}(\mathcal{W}) M^2 + \mathrm{card}(\mathcal{W})^2 M + \mathrm{card}(\mathcal{W})^{5/4} T^{1/2} M\right).$$

Proof. Note that

$$\begin{aligned}
\sum_{M<n_1,n_2\leq 2M} |R(n_1/n_2)|^2 &= \sum_{M<n_1,n_2\leq 2M} \sum_{\mathbf{t}\in\mathcal{W}^2} (n_1/n_2)^{it_1-it_2} \\
&= \sum_{\mathbf{t}\in\mathcal{W}^2} \sum_{M<n_1,n_2\leq 2M} (n_1/n_2)^{it_1-it_2} \\
&= \sum_{\mathbf{t}\in\mathcal{W}^2} |D_N(it_1-it_2)|^2
\end{aligned}$$

□

Corollary. (Lemma 1.7) *We are assuming $\sigma \in [0,1]$ is such that $D_N(it) \geq N^\sigma$ for $t \in \mathcal{W}$. Suppose also that $N \geq T^{2/3}$. Then*

$$E(\mathcal{W}) \ll T^\varepsilon \left(\mathrm{card}(\mathcal{W})^2 N^{2-2\sigma} + \mathrm{card}(\mathcal{W})^3 N^{1-2\sigma}\right).$$

Proof. Lemma 11.4 and the crude bound $R(n_1/n_2)| \le \operatorname{card}(\mathcal{W})$ gives the above and an extra term

$$\operatorname{card}(\mathcal{W})^{9/4}T^{1/2}N^{1-2\sigma} \le \operatorname{card}(\mathcal{W})^{9/4}N^{7/4-2\sigma} = \big(\operatorname{card}(\mathcal{W})^2N^{2-2\sigma}\big)^{1/4}\big(\operatorname{card}(\mathcal{W})^3N^{1-2\sigma}\big)^{3/4}$$

□

It seems that this is good enough to break the Huxley barrier, but they go on to make an improvement when $E(\mathcal{W})$ is very large.

The above is wasteful in that it uses the crude estimate $R(n_1/n_2)| \le \operatorname{card}(\mathcal{W})$. To avoid this one needs to invoke the fourth moment.

Lemma 11.6. *For any $M \ge 1$ we have*

$$\sum_{\mathbf{n}\in[M,2M]^2} |R(n_1/n_2)|^4 \ll T^{\varepsilon}\big(M^2E(\mathcal{W}) + M\operatorname{card}(\mathcal{W})^4 + MT^{1/2}E(\mathcal{W})^{3/4}\operatorname{card}(\mathcal{W})\big).$$

I think much of this material can be streamlined by using the observation that we made earlier, namely that in bounding $E(\mathcal{W})$ it suffices to do so when the elements t of $\mathcal{W}$ are integers. Another observation which comes home to roost later is that in dealing with objects like $R(n_1/n_2)$ one should really reduce (as quickly as possible) to the case $(n_1,n_2)=1$. A further observation is that then the points n_1/n_2 are discrete and well spaced and one can use the technology that was developed to treat the large sieve.

Proof. For $u\in\mathbb{Z}$ let $r(u)$ denote the number of pairs $\mathbf{t}\in\mathcal{W}^2$ such that $\lfloor t_1-t_2\rfloor = u$ and then let $\mathcal{U}_B$ denote the set of $u\in\mathbb{Z}$ such that $B\le r(u)\le 2B$. Since for $\mathcal{U}_B\ne\emptyset$ we must have for each $u\in\mathcal{U}_B$ that $1\le r(u)\le\operatorname{card}(\mathcal{W})$ it follows that $\frac12 B\le\operatorname{card}(\mathcal{W})=R$, say. Also every u for which $r(u)\ne 0$ will be in some set $\mathcal{U}_B$ with $B=2^j$ for some j with $-1\le j\ll\log\operatorname{card}(\mathcal{W})$. Thus

$$|R(x)|^2 = \sum_{B=2^j}\sum_{u\in\mathcal{U}_B}\sum_{\substack{\mathbf{t}\in\mathcal{W}^2\\ \lfloor t_1-t_2\rfloor=u}} x^{it_1-it_2}.$$

Therefore by the Cauchy-Schwarz inequality

$$|R(x)|^4 \ll (\log T)\sum_{B=2^j}\left|\sum_{u\in\mathcal{U}_B}\sum_{\substack{\mathbf{t}\in\mathcal{W}^2\\ \lfloor t_1-t_2\rfloor=u}} x^{it_1-it_2}\right|^2.$$

Hence

$$\sum_{\mathbf{n}\in[M,2M]^2} |R(n_1/n_2)|^4$$
$$\ll (\log T) \sum_{B=2^j} \sum_{\mathbf{n}\in[M,2M]^2} \left| \sum_{u\in\mathcal{U}_B} \sum_{\substack{\mathbf{t}\in\mathcal{W}^2\\ \lfloor t_1-t_2\rfloor=u}} (n_1/n_2)^{it_1-it_2} \right|^2$$

Opening out the inside multiple sum on the right gives for each given $B = 2^j$

$$\sum_{\mathbf{u}\in\mathcal{U}_B^2} \sum_{\substack{\mathbf{t}\in\mathcal{W}^4\\ \lfloor t_1-t_2\rfloor=u_1\\ \lfloor t_3-t_4\rfloor=u_2}} \sum_{n_1\in[M,2M]} n_1^{it_1-it_2+it_3-it_4} \sum_{n_2\in[M,2M]} n_2^{it_2-it_1+it_4-it_3}.$$

At this point there are some circumlocutions because the t_j are not necessarily integers. For the purposes of this exposition I will assume they are, since our earlier observation means that one can always reduce to that case anyway. If you want to see the unnecessary details feel free to look at the ArXiv paper. Thus assuming integrality the above becomes

$$\sum_{\mathbf{u}\in\mathcal{U}_B^2} r(u_1)r(u_2) \left| \sum_{n\in[M,2M]} n^{iu_1-iu_2} \right|^2.$$

Hence

$$\sum_{\mathbf{n}\in[M,2M]^2} |R(n_1/n_2)|^4 \ll (\log T) \sum_{B=2^j} B^2 \sum_{\mathbf{u}\in\mathcal{U}_B^2} \left| \sum_{n\in[M,2M]} n^{iu_1-iu_2} \right|^2.$$

Once more we can apply Heath-Brown's theorem and obtain

$$\sum_{\mathbf{n}\in[M,2M]^2} |R(n_1/n_2)|^4 \ll (\log T) \sum_{B=2^j} B^2 \big(M^2 \operatorname{card}(\mathcal{U}_B)$$
$$+ M \operatorname{card}(\mathcal{U}_B)^2 + T^{1/2} M \operatorname{card}(\mathcal{U}_B)^{5/4} \big).$$

The lemma then follows from the observations that $B \operatorname{card}(\mathcal{U}_B) \ll \operatorname{card}(\mathcal{W})^2$ and $B^2 \operatorname{card}(\mathcal{U}_B) \ll E(\mathcal{W})$. □

At this point they observe that there is some advantage in looking at $d = \gcd(n_1, n_2)$. This will mean that n can be replaced by N/d in the bounds and if there is a term in the bound of the form $(N/d)^\theta$ with $\theta > 1$ then there are further savings to be had from summing over d separately. Also putting $m_j = n_j/d$ since now we have $\gcd(m_1, m_2) = 1$ the fractions m_1/m_2 are well spaced and one can use the technology of the large sieve. Since time is short and the technology is well understood I will just state the results.

Lemma 11.8. *We have*

$$\sum_{\substack{\mathbf{n}\in[N,2N]^2\\ \gcd(n_1,n_2)\le\Delta}} |R(n_1/n_2)|^3 \ll (N^2+T\Delta)E(\mathcal{W})^{1/2}\big(\operatorname{card}(\mathcal{W})\big)^{1/2}.$$

Lemma 11.9 *Suppose that $N\ge T^{3/4}$. Then*

$$\sum_{\substack{\mathbf{n}\in[N,2N]^2\\ \gcd(n_1,n_2)\ge N^2/T}} |R(n_1/n_2)|^3 \ll N\big(\operatorname{card}(\mathcal{W})\big)^3 +$$

$$NT^{1/4}\big(\operatorname{card}(\mathcal{W})\big)^{21/8} + E(\mathcal{W})^{1/2}\big(\operatorname{card}(\mathcal{W})\big)^{1/2}N^2.$$

Now we can complete the proof of Theorem 11.1. Lemma 11.4 gives

$$E(\mathcal{W}) \ll T^{\varepsilon}N^{-2\sigma}\sum_{\mathbf{n}\in[N,2N]^2}|R(n_1/n_2)|^3.$$

Then we apply Lemma 11.8 when $\gcd(n_1,n_2)\le N^2/T$ and Lemma 11.9 in the contrary case. Of course N^2/T has been chosen as the optimal choice, i.e. the bounds are the same. This gives

Theorem 11.1 (Proposition 11.1.) *Suppose that $\mathcal{W}$ is a 1-separated set contained in an interval of length T, that $N^{3/4}\le T\le N$ and that $\sigma\in[0,1]$ is such that $D_N(it)\ge N^{\sigma}$ for $t\in\mathcal{W}$. Let $R=\operatorname{card}\mathcal{W}$. Then*

$$E(\mathcal{W}) \ll T^{\varepsilon}RN^{4-4\sigma} + T^{\frac14+\varepsilon}R^{21/8}N^{1-2\sigma} + T^{\varepsilon}R^3N^{1-2\sigma}.$$