C:MeanLargeVals

# LARGE VALUES OF DIRICHLET POLYNOMIALS

## 1. Introduction and Notation

This is a brief overview with proofs of the classical large values bound for Dirichlet polynomials

$$D(s) = \sum_{n=1}^{N} a_n n^{-s} \tag{1.1}$$ eq:one1

that is quoted in Guth-Maynard. Here the $a_n$ $n = 1, \dots N$ are complex numbers, where $N \in \mathbb{N}$, and as usual with Dirichlet series and polynomials $s = \sigma + it$ is a complex number with real and imaginary parts $\sigma$ and $t$. We will suppose that $T$ is a real number with

$$T \geq 1, \tag{1.2}$$ eq:one2

$R \in \mathbb{N}$ and the $s_r = \sigma_r + it_r$ with $1 \leq r \leq R$ are $R$ complex numbers which satisfy

$$0 \leq \sigma_0 \leq \sigma_r \leq 1 \text{ and } |t_q - t_r| \geq 1 \quad (1 \leq q < r \leq R) \tag{1.3}$$ eq:one3

for some $\sigma_0 \in [0, 1]$.

We note that $R - 1 \leq \max_r t_r - \min_r t_r \leq T$, so that

$$R \leq T + 1. \tag{1.4}$$ eq:one4

We also introduce the peculiar notation

$$G = \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma_0}. \tag{1.5}$$ eq:one5

We are in particular concerned with bounding the number

$$R(V) = \text{card}\{1 \leq r \leq R : |D(s_r)| \geq V\} \tag{1.6}$$ eq:one6

where $V$ is a positive parameter at our disposal, in terms of $N, T, C, V$. Such estimates are largely equivalent to bounds for

$$\sum_{r=1}^{R} |D(s_r)|^2$$

as can be seen as follows. We have

$$R(V) \leq V^{-2} \sum_{r=1}^{R} |D(s_r)|^2 \tag{1.7}$$ eq:one7

and

$$\sum_{r=1}^{R} |D(s_r)|^2 \leq RV^2 + \int_V^{\mathcal{D}} 2XR(X)dx \tag{1.8}$$

eq:one8

where $\mathcal{D} = \max_r |D(s_r)|$.

## 2. The Special Case

To simplify matters we suppose to begin with that $\sigma_r = \sigma_0 = 0$.

thm:two1

**Theorem 2.1.** *Assume the above notation and that* $\sigma_r = 0$ $(r = 1, \ldots, R)$. *Then*

$$\sum_{r=1}^{R} |D(it_r)|^2 \ll (T+N)G\log(2N), \tag{2.1}$$

eq:two1

$$\sum_{r=1}^{R} |D(it_r)|^2 \ll \big(RT^{\frac{1}{2}}(\log(2T))^2 + N\big)G, \tag{2.2}$$

eq:two2

$$\sum_{r=1}^{R} |D(it_r)|^2 \ll \big(R^{\frac{2}{3}}T^{\frac{1}{3}}N^{\frac{1}{3}}(\log(2T))^{\frac{4}{3}} + N\big)G, \tag{2.3}$$

eq:two3

We remark that by working a bit harder some of the logarithmic powers can be reduced.

*Proof.* To prove (2.1) [eq:two1] we use a method introduced by Gallager. We have

$$D(it)^2 - D(iu)^2 = \int_u^t 2iD(iv)D'(iv)dv$$

so that

$$|D(it_r)|^2 \leq \int_{t_r-\frac{1}{2}}^{t_r+\frac{1}{2}} |D(iu)|^2 du + \int_{t_r-\frac{1}{2}}^{t_r+\frac{1}{2}} |D(iv)D'(iv)|dv$$

and

$$\sum_{r=1}^{R} |D(it_r)|^2 \leq \int_{-\frac{1}{2}}^{T+\frac{1}{2}} |D(iu)|^2 du + \int_{-\frac{1}{2}}^{T+\frac{1}{2}} |D(iv)D'(iv)|dv.$$

We also have

$$\int_{-\frac{1}{2}}^{T+\frac{1}{2}} |D(iu)|^2 du = \sum_{n=1}^{N} |a_n|^2 (T+1) \sum_{m=1}^{N} \sum_{\substack{n=1\\ n\neq m}}^{N} \frac{b_m\overline{b}_n - c_m\overline{c}_n}{-i\log(m/n)}$$

where

$$b_n = a_n n^{-i(T+1/2)},\ c_n = a_n n^{i/2}.$$

Hence, by Hilbert's inequality

$$\int_{-\frac{1}{2}}^{T+\frac{1}{2}} |D(iu)|^2 du \ll \sum_{n=1}^{N} |a_n|^2 (T+n) \ll G(T+N).$$

Similarly

$$\int_{-\frac{1}{2}}^{T+\frac{1}{2}} |D'(iv)|^2 dv \ll \sum_{n=1}^{N} |a_n|^2 (\log n)^2 (T+n) \ll G(T+N)(\log N)^2$$

so (eq:two1 2.1) follows from the Cauchy-Schwarx inequality. To prove (eq:two2 2.2) we invoke the duality lemma.

lem:two1 **Lemma 2.2** (Duality)**.** *Let $A = [c_{mn}]$ be a fixed $M \times N$ matrix with complex entries. The following three assertions concerning the non-negative constant $\lambda$ are equivalent.*
*(i) For any $\boldsymbol{z} \in \mathbb{C}^N$,*

$$\sum_{m=1}^{M} \Big| \sum_{n=1}^{N} c_{mn} z_n \Big|^2 \le \lambda^2 \sum_{n=1}^{N} |z_n|^2;$$

*(ii) For any $\boldsymbol{z} \in \mathbb{C}^N$ and any $\boldsymbol{w} \in \mathbb{C}^M$,*

$$\Big| \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} z_n w_m \Big| \le \Delta \Big( \sum_{n=1}^{N} |z_n|^2 \Big)^{1/2} \Big( \sum_{m=1}^{M} |w_m|^2 \Big)^{1/2};$$

*(iii)*
*For any $\boldsymbol{w} \in \mathbb{C}^M$,*

$$\sum_{n=1}^{N} \Big| \sum_{m=1}^{M} c_{mn} w_m \Big|^2 \le \lambda^2 \sum_{m=1}^{M} |z_m|^2.$$

*Proof of Lemma* lem:two1 *2.2.* We show that (i) and (ii) are equivalent. Then by interchanging the roles of $m$ and $n$ it is clear that (ii) and (iii) are equivalent.

(i) $\Longrightarrow$ (ii). By Cauchy's inequality

$$\Big| \sum_m \Big( \sum_n c_{mn} x_n \Big) y_m \Big| \le \Big( \sum_m \Big| \sum_n c_{mn} x_n \Big|^2 \Big)^{1/2} \Big( \sum_m |y_m|^2 \Big)^{1/2}.$$

In the first factor on the right we insert the bound provided by (i), and we obtain (ii).

(ii) $\Longrightarrow$ (i). Set

$$w_m = \sum_{n=1}^{N} c_{mn} z_n,$$

and let $S$ denote the left and side of (i). Then $S = \sum_n c_{mn} z_n \overline{w_m}$, and by (ii) we see that

$$S \le \Delta\Big(\sum_{n=1}^{N} |z_n|^2\Big)^{1/2}\Big(\sum_{m=1}^{M} |w_m|^2\Big)^{1/2} = \lambda\Big(\sum_{n=1}^{N} |z_n|^2\Big)^{1/2} S^{1/2}.$$

If $S = 0$, then (ii) is obviously satisfied. Otherwise $S > 0$, and we may square both sides above and divide by $S$ to obtain (i). □

We now return to the proof of (eq:two2 2.2). By the duality lemma it suffices to show that

$$\sum_{n=1}^{N}\Big|\sum_{r=1} b_r n^{-it_r}\Big|^2 \ll \big(RT^{\frac{1}{2}}(\log(2T))^2 + N\big)\sum_{r=1}^{R} |b_r|^2. \qquad (2.4)$$ eq:two4

It is convenient to insert the smooth weights $2\big(1 - n/(2N)\big)$ on the left and extend the summation to $2N$. Then we treat the left hand side by multiplying out and inverting the order. Hence we have

$$\sum_{n=1}^{N}\Big|\sum_{r=1} b_r n^{-it_r}\Big|^2 \le 2\sum_{q=1}^{R}\sum_{r=1}^{R} b_q \bar{b}_r S(t_r - t_s). \qquad (2.5)$$ eq:two5

where

$$S(t) = \sum_{n=1}^{2N}\big(1 - n/(2N)\big)n^{it}.$$

The terms with $q = r$ contribute

$$\sum_{r=1}^{R} |b_r|^2\left(N - \frac{1}{2}\right).$$

For the remaining terms we observe that

$$S(t) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \zeta(w - it)\frac{(2N)^w}{w(w+1)}dw.$$

We assume $|t| \le T$ and let $\theta = c/\log(2+|t|)$ where $c$ is a small constant, sufficiently small to ensure that $\theta \le \frac{1}{4}$. Then we move the vertical path to the line $\operatorname{Re} w = -\theta$, picking up residues from $w = 1 + it$ and $w = 0$. From known estimates for the zeta function we obtain

$$S(t) = \frac{(2N)^{1+it}}{(1+it)(2+it)} + \zeta(-it) + \frac{1}{2\pi i}\int_{-\theta-i\infty}^{-\theta+i\infty} \zeta(w - it)\frac{(2N)^w}{w(w+1)}.$$

We have

$$\zeta(-it) \ll (1 + |t|)^{\frac{1}{2}}\log(2 + |t|).$$

and by the functional equation and the bound

$$\zeta(w - it) = \rho(w - it)\zeta(1 + \theta - iv - it) \ll (1 + |v| + |t|)^{\frac{1}{2}+\theta}\theta^{-1}$$

we obtain

$$S(t) \ll \frac{N}{1 + |t|^2} + (1 + |t|)^{\frac{1}{2}} \log^2(1 + |t|).$$

Thus, by (2.5),

$$\sum_{n=1}^{N} \left| \sum_{r=1}^{R} b_r n^{-it_r} \right|^2 \ll \sum_{q=1}^{R} |b_q|^2 \Bigg( N + \sum_{\substack{r=1 \\ r\neq q}}^{R} \bigg( \frac{N}{1 + |t_q - t_r|^2} + T^{\frac{1}{2}} \log^2(2T) \bigg) \Bigg),$$

and therefore (2.4), and so (2.2), follow.

The inequality (2.3) will follow from (2.2) by a process of divide and rule! If we should have

$$N > (R^{\frac{2}{3}} T^{\frac{1}{3}} N^{\frac{1}{3}} (\log(2T))^{\frac{4}{3}}$$

then we would have

$$N > RT^{\frac{1}{2}} (\log(2T))^2$$

and the desired bound follows immediately from (2.2). Thus we may suppose that

$$N \le (R^{\frac{2}{3}} T^{\frac{1}{3}} N^{\frac{1}{3}} (\log(2T))^{\frac{4}{3}}.$$

Let

$$T_1 = \big(\log(2T)\big)^{-\frac{4}{3}} (NT/R)^{2/3}$$

and divide the interval $[0, T]$ into $\lceil T/T_1 \rceil$ intervals of length $\le T_1$. By the assumption on $N$ we have

$$T_1 \le T.$$

Note also that if we denote the $j$-th interval by $I_j = [u_j, v_j]$ we can replace the $a_n$ by $a_n n^{-iu_j}$ and the $t_r$ in the interval by $t_r - u_j$ and then apply (2.2) to the $j$-th interval. Let $R_j$ denote the number of $t_r \in I_j$.

Then

$$\begin{aligned}\sum_{r=1}^{R}|D(it_r)|^2 &\le \sum_{j=1}^{\lceil T/T_1\rceil}\sum_{\substack{r\\ t_r\in I_j}}|D(it_r)|^2\\ &\ll \sum_{j=1}^{\lceil T/T_1\rceil}\big(R_jT_1^{\frac12}(\log(2T))^2+N\big)G\\ &\ll (RT_1^{1/2}(\log T)^2+NT/T_1)G\\ &\ll R^{\frac23}T^{\frac13}N^{\frac13}\big(\log(2T)\big)^{\frac43}G\end{aligned}$$

as required □

## 3. The Main Theorems

We now advert to a general $\sigma_0\in[0,1]$.

thm:three1 **Theorem 3.1.** *Assume that* $\sigma_0\in[0,1]$ *and* $\sigma_r\ge\sigma_0$ $(r=1,\dots,R)$. *Then*

$$\sum_{r=1}^{R}|D(it_r)|^2\ll(T+N)G\log^2(2N), \tag{3.1}$$ eq:three1

$$\sum_{r=1}^{R}|D(it_r)|^2\ll\big(RT^{\frac12}(\log(2T))^2+N\big)G\log(2N), \tag{3.2}$$ eq:three2

$$\sum_{r=1}^{R}|D(it_r)|^2\ll\big(R^{\frac23}T^{\frac13}N^{\frac13}(\log(2T))^{\frac43}+N\big)G\log(2N), \tag{3.3}$$ eq:three3

*Proof.* It suffices to prove the case $\sigma_0=0$ because we can reduce to this case by replacing $a_n$ by $a_nn^{-\sigma_0}$ and $\sigma_r$ by $\sigma_r-\sigma_0$.

We proceed by adapting Gallagher's idea to this situation. We have

$$n^{-\sigma_r}=1-\int_0^{\sigma_r}n^{-v}(\log n)dv$$

and so

$$D(s_r)=D(it_r)-\int_0^{\sigma_r}D'(v+it)dv.$$

Hence

$$|D(s_r)|\le|D(it_r)|+\int_0^1|D'(v+it)|dv$$

and therefore by the Cauchy-Schwarz' inequality

$$\sum_{r=1}^{R}|D(s_r)|^2\ll\sum_{r=1}^{R}|D(it_r)|^2+\int_0^1\sum_{r=1}^{R}|D'(v+it_r)|^2dv.$$

Assuming any one of the bounds (2.1), (2.2), (2.3) with $a_n$ replaced if necessary by $a_n n^{-v}(\log n)$ we obtain

$$\sum_{r=1}^{R} |D(s_r|^2 \ll \lambda \sum_{n=1}^{N} |a_n|^2 \left(1 + \int_0^1 |a_n|^2 n^{-2v} (\log n)^2 dv\right)$$

where $\lambda$ is the appropriate factor on the right of (2.1), (2.2) or (2.3). □

We can now obtain explicit large values theorems.

**Theorem 3.2.** *With the notation of* (1.1),...,(1.6) *in addition to the universal bound* (1.4), $R \le T+1$, *we have*

$$R(V) \ll (T+N)GV^{-2}\log(2N)$$

*when*

$$\frac{T+N}{T} G\log(2N) < V^2 \le \left(\frac{TN}{T+N}\right)^{1/2} G(\log(2T))^2\log(2N),$$

$$R(V) \ll \frac{TNG^3\big(\log(2T)\big)^4\big(\log(2N)\big)^3}{V^6}$$

*when*

$$\left(\frac{TN}{T+N}\right)^{1/2} G(\log(2T))^2\log(2N) < V^2 \le T^{1/2}G\big(\log(2T)\big)^2\log(2N),$$

*and*

$$R(V) \ll NGV^{-2}\log(2N)$$

*when*

$$T^{1/2}G\big(\log(2T)\big)^2\log(2N) < V^2.$$

*Proof.* The bound (3.1) gives the first estimate at once. The estimate (3.3) implies

$$R(V) \ll TNG^3V^{-6}\big(\log(2T)\big)^4\big(\log(2N)\big)^3 + NGV^{-2}\log(2N)$$

and this implies the second and third bounds. □