Mean Values of Dirichlet polynomials and fourth moment of Zeta function

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$$\sum_{n=1}^{N} |a_n|^2 T + \sum_{\substack{1\le m,n\le N\\ m\ne n}} a_m \overline{a}_n \frac{(m/n)^{iT} - 1}{i\log(m/n)}.$$

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- Thus when $m \ne n$, $|\log(m/n)| \ge \frac{1}{\max(m,n)+\frac{1}{2}}$.

- The following generalization of Hilbert's inequality is useful

Theorem 1. Suppose that $\delta_1, \ldots, \delta_R$ are R positive numbers and $x_1, \ldots x_R$ are R real numbers satisfying

$$\min_{s \neq r} |x_s - x_r| \geq \delta_r \quad (r = 1, \ldots, R),$$

and suppose further that $a_1, \ldots a_N$ are N complex numbers. Then

$$\sum_{\substack{1 \leq r,s \leq R \\ r \neq s}} \frac{a_r \overline{a}_s}{x_r - x_s} \ll \sum_{r=1}^{R} |a_r|^2 \delta_r^{-1}.$$

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- **Corollary 1.1** We have

$$\int_0^T \left| \sum_{n=1}^{N} a_n n^{it} \right|^2 dt = \sum_{n=1}^{N} |a_n|^2 \big(T + O(n)\big).$$

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$$\int_0^T \left|\sum_{n=1}^{N} a_n n^{it}\right|^2 dt = \sum_{n=1}^{N} |a_n|^2\big(T + O(n)\big).$$

- Note $a_m\overline{a}_n \dfrac{(m/n)^{iT} - 1}{\log(m/n)} = \dfrac{a'_m\overline{a}'_n - a_m\overline{a}_n}{\log m - \log n}$.

- Schur obtained the best possible bound

$$\Big| \sum_{\substack{1\le r,s\le R\\ r\ne s}} \frac{a_r\overline{a}_s}{r-s} \Big| \le \pi \sum_{r=1}^{R} |a_r|^2.$$

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- The $R\times R$ matrix $\mathcal{M}$ with entries 0 when $r=s$

$$c_{rs} = \frac{\delta_r^{1/2}\delta_s^{1/2}}{x_r - x_s} \text{ when } r\ne s$$

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- If we take $b_r=a_r\delta_r^{-1/2}$, then

$$\Big|\sum_{\substack{1\le r,s\le R\\ r\ne s}}\frac{a_r\overline{a}_s}{x_r-x_s}\Big|=|\mathbf{b}\mathcal{M}\mathbf{b}^*|\le\lambda\mathbf{b}.\mathbf{b}^*=\lambda\sum_{r=1}^R|a_r|^2\delta_r^{-1}.$$

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- We want to bound λ. Let $\mathbf{b}$ be an eigenvector corresponding to $i\lambda$ so that

$$\sum_{\substack{r=1 \\ r \ne s}}^{R} \frac{b_r \delta_r^{1/2} \delta_s^{1/2}}{x_r - x_s} = i\lambda b_s, \sum_{\substack{r=1 \\ r \ne s}}^{R} \frac{a_r}{x_r - x_s} = i\lambda a_s \delta_s^{-1}.$$

and normalised so that $\sum_{r=1}^{R} |b_r|^2 = 1$.

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- Choosing a_r as above we have $\sum_{r=1}^R |a_r|^2\delta_r^{-1} = 1$ and, by Cauchy-Schwarz,

$$\lambda^2 = \Big|\sum_{\substack{1\le r,s\le R\\ r\ne s}} \frac{a_r\bar{a}_s}{x_r-x_s}\Big|^2 \le \sum_{r=1}^R \delta_r\Big|\sum_{\substack{s=1\\ s\ne r}}^R \frac{\bar{a}_s}{x_r-x_s}\Big|^2.$$

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- Squaring out and interchanging the order this is

$$\sum_{s=1}^{R} \sum_{t=1}^{R} \overline{a}_s a_t \sum_{\substack{r=1 \\ r \ne s, r \ne t}}^{R} \frac{\delta_r}{(x_r - x_s)(x_r - x_t)}$$

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$$\sum_{s=1}^{R} |a_s|^2 \sum_{\substack{r=1\\ r\ne s}}^{R} \frac{\delta_r}{(x_r - x_s)^2}$$

- When $s \ne t$ we use the identity

$$\frac{1}{(x_r - x_s)(x_r - x_t)} = \frac{1}{x_s - x_t} \left(\frac{1}{x_r - x_s} - \frac{1}{x_r - x_t} \right).$$

- $$\sum_{\substack{r=1\\r\neq s}}^{R} \frac{a_r}{x_r - x_s} = i\lambda a_s \delta_s^{-1}$$

$$\lambda^2 \le \sum_{s=1}^{R} |a_s|^2 \sum_{\substack{r=1\\r\neq s}}^{R} \frac{\delta_r}{(x_r - x_s)^2} +$$

$$\sum_{s=1}^{R} \sum_{\substack{t=1\\t\neq s}}^{R} \frac{\overline{a}_s a_t}{x_s - x_t} \sum_{\substack{r=1\\r\neq s, r\neq t}}^{R} \delta_r \left(\frac{1}{x_r - x_s} - \frac{1}{x_r - x_t} \right).$$

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- We divide the triple sum here into two parts.

$$\sum_{s=1}^{R}\sum_{\substack{r=1\\r\neq s}}^{R}\frac{\overline{a}_s\delta_r}{x_r-x_s}\sum_{\substack{t=1\\t\neq r,t\neq s}}^{R}\frac{a_t}{x_s-x_t}-$$

$$\sum_{t=1}^{R}\sum_{\substack{r=1\\r\neq t}}^{R}\frac{a_t\delta_r}{x_r-x_t}\sum_{\substack{s=1\\s\neq r,s\neq t}}^{R}\frac{\overline{a}_s}{x_s-x_t}.$$

- $$\sum_{\substack{r=1\\r\neq s}}^{R}\frac{a_r}{x_r-x_s}=i\lambda a_s\delta_s^{-1}, \sum_{s=1}^{R}\sum_{\substack{r=1\\r\neq s}}^{R}\frac{\overline{a}_s\delta_r}{x_r-x_s}\sum_{\substack{t=1\\t\neq r,t\neq s}}^{R}\frac{a_t}{x_s-x_t}$$
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- We apply the eigenvalue property to the inner sums, but there are terms missing which we need to add back in.

$$\sum_{\substack{t=1\\t\neq r,t\neq s}}^R \frac{a_t}{x_s-x_t} = -i\lambda a_s\delta_s^{-1} - \frac{a_r}{x_s-x_r},$$

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- In each case the first terms cancel.

- We are left with

$$\sum_{s=1}^{R}\sum_{\substack{r=1\\r\neq s}}^{R}\frac{\overline{a}_s a_r\delta_r}{(x_r-x_s)^2}+\sum_{t=1}^{R}\sum_{\substack{r=1\\r\neq t}}^{R}\frac{a_t\overline{a}_r\delta_r}{(x_r-x_t)^2}$$

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- Putting it together

$$\lambda^2\leq\sum_{s=1}^{R}\sum_{\substack{r=1\\r\neq s}}^{R}\frac{(|a_s|^2+a_s\overline{a}_r+\overline{a}_s a_r)\delta_r}{(x_r-x_s)^2}$$

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- The first of the three terms is fairly straightforward.
Lemma 0. We have

$$\sum_{\substack{r=1\\r\neq s}}^{R}\frac{\delta_r}{(x_r-x_s)^2}\ll\delta_s^{-1}.$$

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$$\sum_{\substack{r=1\\r\neq s}}^{R}\frac{\delta_r}{(x_r-x_s)^2}\ll\delta_s^{-1}.$$

- Proof. Since x^{-2} is convex we have

$$\frac{\delta_r}{(x_r-x_s)^2}\le\int_{x_r-\delta_r/2}^{x_r+\delta_r/2}(x-x_s)^{-2}dx.$$

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- That leaves

$$T=\sum_{r=1}^{R}\sum_{\substack{s=1\\s\neq r}}^{R} \frac{|a_r a_s|(\delta_r+\delta_s)}{(x_r-x_s)^2}$$

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$$\frac{T^2}{4} = \Big(\sum_{r=1}^{R} \sum_{\substack{s=1 \\ s \neq r}}^{R} \frac{|a_r a_s| \delta_s}{(x_r - x_s)^2} \Big)^2 \le \sum_{r=1}^{R} \delta_r \Big(\sum_{\substack{s=1 \\ s \neq r}}^{R} \frac{|a_s| \delta_s}{(x_r - x_s)^2} \Big)^2$$

$$= \sum_{s=1}^{R} \sum_{t=1}^{R} |a_s a_t| \delta_s \delta_t \sum_{\substack{r=1 \\ r \neq s, r \neq t}}^{R} \frac{\delta_r}{(x_r - x_s)^2 (x_r - x_t)^2}.$$

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$$= \sum_{s=1}^{R} \sum_{t=1}^{R} |a_s a_t|\delta_s \delta_t \sum_{\substack{r=1 \\ r\neq s, r\neq t}}^{R} \frac{\delta_r}{(x_r - x_s)^2 (x_r - x_t)^2}.$$

- When $s = t$, in a similar way to Lemma 0, the inner sum is $\ll \delta_s^{-3}$ and so

$$T^2 \ll 1 + \sum_{s=1}^{R} \sum_{\substack{t=1 \\ t\neq s}}^{R} |a_s a_t|\delta_s \delta_t \sum_{\substack{r=1 \\ r\neq s, r\neq t}}^{R} \frac{\delta_r}{(x_r - x_s)^2 (x_r - x_t)^2}.$$

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$$= \frac{1}{(x_s - x_t)^2} \left(\frac{1}{x_r - x_s} - \frac{1}{x_r - x_t} \right)^2$$

$$\ll \frac{1}{(x_s - x_t)^2} \left(\frac{1}{(x_r - x_s)^2} + \frac{1}{(x_r - x_t)^2} \right)$$

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- Now we can apply Lemma 0 once more to obtain

$$T^2 \ll 1 + \sum_{s=1}^{R} \sum_{\substack{t=1 \\ t\neq s}}^{R} |a_s a_t| \frac{\delta_t + \delta_t}{(x_s - x_t)^2} \ll 1 + T$$

so $T \ll 1$.

Schur's Original Proof

- Assume $\sum_{r=-R}^{R} |a_r|^2 = \sum_{s=-R}^{R} |b_s|^2 = 1$ and $N \geq R$. By C-S

$$\Big| \sum_{\substack{-R\leq r,s\leq R\\ r\neq s}} \frac{a_r\overline{b}_s}{r-s}\Big|^2 \leq \sum_{r=-N}^{N} \Big| \sum_{\substack{s=-R\\ s\neq r}}^{R} \frac{\overline{b}_s}{r-s}\Big|^2.$$

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- The terms with $s = t$ contribute

$$\sum_{s=-R}^{R} |b_s|^2 \sum_{\substack{r=-R \\ r \neq s}}^{R} \frac{1}{(r-s)^2} < \frac{\pi^2}{3} \sum_{s=-R}^{R} |b_s|^2.$$

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- When $s \neq t$ use $\displaystyle\frac{1}{(r-s)(r-t)} = \frac{1}{s-t}\left(\frac{1}{r-s} - \frac{1}{r-t}\right)$.

- Thus $\Big|\sum_{\substack{-R\le r,s\le R\\ r\ne s}} \frac{a_r\overline{b}_s}{r-s}\Big|^2 \le$

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- Likewise for the other sum.

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- Likewise for the other sum.
- Hence we obtain the upper bound

$$\sum_{s=-R}^{R} \sum_{\substack{t=-R\\ t\ne s}}^{R} \frac{2\,\mathrm{Re}\,\overline{b}_s b_t}{(s-t)^2} \le 2\sum_{s=-R}^{R} |b_s|^2 \sum_{\substack{t=-R\\ t\ne s}}^{R} \frac{1}{(s-t)^2} < \frac{2\pi^2}{3}.$$

Slick Proof of Hilbert's Inequality

- Note $\displaystyle\int_0^1 e(\alpha h)\left(\alpha-\frac{1}{2}\right)d\alpha=\begin{cases}0 & h=0,\\ \frac{1}{2\pi i h} & h\in\mathbb{Z}\setminus\{0\},\end{cases}$

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- Thus $\displaystyle\sum_{r=-R}^{R}\sum_{\substack{s=-R\\ s\neq r}}^{R}\frac{a_r\overline{b}_s}{2\pi i(r-s)}=$

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- Note $\displaystyle\int_0^1 e(\alpha h)\left(\alpha-\frac12\right)d\alpha=\begin{cases}0 & h=0,\\ \frac{1}{2\pi ih} & h\in\mathbb{Z}\setminus\{0\},\end{cases}$
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- so that $\displaystyle\left|\sum_{r=-R}^{R}\sum_{\substack{s=-R\\ s\ne r}}^{R}\frac{a_r\overline{b}_s}{r-s}\right|\le$

$$\pi\left(\int_0^1\left|\sum_{r=-R}^{R}a_r e(\alpha r)\right|^2 d\alpha\int_0^1\left|\sum_{s=-R}^{R}\overline{b}_s e(-\beta s)\right|^2 d\beta\right)^{1/2}$$

- Thus

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- One can also show that the π is best possible.
- Let $\eta>0$ and $f(\alpha)=\begin{cases}0 & 0\le\alpha<1-\eta\\ \eta^{-1/2} & 1-\eta\le\alpha<1.\end{cases}$ and consider its Fourier series $f(\alpha)=\sum_{r=-\infty}^{\infty}a_re(\alpha r)$.

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- Then $\sum_{r=-\infty}^{\infty}\sum_{\substack{s=-\infty\\ s\neq r}}^{\infty}\frac{a_r\overline{b}_s}{r-s}=2\pi i\int_0^1 f(\alpha)^2\left(\alpha-\frac{1}{2}\right)d\alpha=$
$$2\pi i\eta^{-1}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{1}{2}-\eta\right)^2\right)=\pi i(1-\eta)$$
and $\sum_{r=-\infty}^{\infty}|a_r|^2=\int_0^1 f(\alpha)^2d\alpha=1$.

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- This has an analytic continuation to $\mathbb{C} \setminus \{1\}$ with a pole of order k at $s = 1$.
- One can also show that

$$\sum_{n \le x} d_k(n) = x\mathcal{P}_k(\log x) + O(x^{1-1/k+\varepsilon})$$

where $\mathcal{P}_k$ is a polynomial of degree $k-1$ and

$$\sum_{n \le x} d_k(n)^2 \sim x\mathcal{Q}_k(\log x)$$

where $\mathcal{Q}_k(x)$ is a polynomial of degree $k^2 - 1$.

- From above

$$\int_0^T \left| \sum_{n\le N} d_k(n) n^{-\frac12 - it} \right|^2 dt$$

$$= \sum_{n\le N} d_k(n)^2 n^{-1}\bigl(T + O(n)\bigr)$$

$$= TP_k(\log N) + O\bigl(N(\log N)^{k^2-1}\bigr)$$

where $P_k(n)$ is a polynomial of degree k^2 with leading coefficient

$$\frac{1}{(k^2)!}\prod\left(\left(1-\frac1p\right)^{k^2}\sum_{h=0}^{\infty}\binom{k+h-1}{h}^2 p^{-h}\right).$$

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- So we want to approximate to $\zeta(\frac12 + it)^k$ by a Dirichlet polynomial of length N at most about T

- By the functional equation for ζ we have the functional equation

$$\zeta(s)^k = \gamma(s)^k \zeta(1-s)^k$$

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$$\gamma(s) = \pi^{s-\frac{1}{2}} \left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \right)$$
- Note that $\gamma(s)$ is analytic for all s except s with $\frac{s+1}{2} \in \mathbb{N}$ and in particular whenever $\operatorname{Re} s < 1$.
- Thus, by Stirling's formula for complex argument.
Lemma 1 Suppose that $\sigma_0, \sigma_1 \in \mathbb{R}$, $\sigma_0 < \sigma_1$, and $s = \sigma + it$. Then
$$\gamma(s) \ll |t|^{\frac{1}{2}-\sigma}$$
uniformly in the region $\{s : |t| \geq 1, \sigma_0 \leq \sigma \leq \sigma_1\}$. In particular γ is bounded on the $\frac{1}{2}$-line.

- **Lemma 2** Suppose that σ_0, $\sigma_1 \in \mathbb{R}$, $\sigma_0 < \sigma_1$, and $s = \sigma + it$. Then

$$\zeta(s)^k \ll |t|^{k\mu(\sigma)+\varepsilon}$$

uniformly in the region $\{s : |t| \geq 1, \sigma_0 \leq \sigma \leq \sigma_1\}$, where

$$\mu(\sigma) = \begin{cases} \frac{1}{2} - \sigma & (\sigma \leq 0), \\ \frac{1-\sigma}{2} & (0 < \sigma \leq 1), \\ 0 & (\sigma > 1), \end{cases}$$

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- This is trivial when $\sigma \ge 1 + \delta$ and δ is any sufficiently small (in terms of k and ε) positive real number, and is immediate from the functional equation when $\sigma \le -\delta$.
- In the range $-\delta < \sigma < 1 - \delta$ it follows from a general convexity principle for Dirichlet series, see Titchmarsh, The Theory of Functions, second edition, Oxford University Press, 1939, §§5.65, 9.41.

- There are "approximate functional equations" for $\zeta(s)$ and $\zeta(s)^2$ which go back to a famous paper of Hardy and Littlewood, Acta Mathemtica, 1917. of the form

$$\zeta(s)^k = \sum_{n\le x} d_k(n)n^{-s} + \gamma(s)^k \sum_{n\le y} d_k(n)n^{s-1} + E(x,y)$$

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- These are tricky to prove and use and are not useful when $k > 2$.
- This can be simplified by using weights, but the classical literature seems to have overlooked this.

- An approximate functional equation.
  Lemma 3 Suppose that $0 < \sigma < 1$, $-1 < \phi < -\sigma$, $0 < \psi < 1 - \sigma$, $j \ge \frac{3}{2}k + 1$, $x \ge 1$, $y \ge 1$. Then $\zeta(s)^k =$

$$\sum_{n \le x} d_k(n) n^{-s} (1 - n/x)^j$$

$$+ \gamma(s)^k \sum_{n \le y} d_k(n) n^{s-1} - U(s) - V(s) - R(s)$$

$$\lambda_j(x, z) = \frac{x^z j!}{z(z+1)\dots(z+j)}, \quad U(s) =$$

$$\frac{1}{2\pi i} \int_{\phi - i\infty}^{\phi + i\infty} \gamma(s+w)^k \sum_{n > y} d_k(n) n^{s+w-1} \lambda_j(x, w) dw,$$

$$R(s) = \operatorname{Res}\big(\zeta(z)^k \lambda_j(x, z - s)\big)_{z=1}, \quad V(s) =$$

$$\frac{1}{2\pi i} \int_{\psi - i\infty}^{\psi + i\infty} \gamma(s+w)^k \sum_{n \le y} d_k(n) n^{s+w-1} \lambda_j(x, w) dw$$

- Moving the path to $\operatorname{Re} w = -\infty$ when $u \geq 1$ and $\operatorname{Re} w = +\infty$ when $u < 1$ gives for $j \geq 1$, $u > 0$, $\theta > 1$,

$$\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \lambda_j(u,w)dw = \begin{cases} (1-1/u)^j & (u \geq 1), \\ 0 & (u < 0). \end{cases}$$

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- By the absolute convergence of this and the uniform convergence of $\zeta(s+w)^k$ when $\operatorname{Re} w = \theta$

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- By Lemma 2, when $\operatorname{Re}(s+w) > -1$,

$$\zeta(s+w)^k - \mathcal{P}_k\left(\frac{1}{s+w-1}\right) \ll (1+|\operatorname{Im}(s+w)|)^{\frac{3}{2}k+\varepsilon}.$$

Hence, as $j+1 \ge \frac{3}{2}k+1$, we may move the vertical path to the line $\operatorname{Re} w = \phi$, picking up the residues of the integrand at $w = 1-s$ and $w = 0$.

- so that $\sum_{n\le x}\frac{d_k(n)}{n^s}(1-n/x)^j = R(s)+\zeta(s)^k+$

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$$\frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty}\gamma(s+w)^k\zeta(1-s-w)^k\lambda_j(u,w)dw.$$
- As $-\operatorname{Re}(s+w)=-\sigma-\phi>0$ we may write
$$\zeta(1-s-w)^k=\sum_{n\le y}d_k(n)n^{s+w-1}+\sum_{n>y}d_k(n)n^{s+w-1}.$$

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- The second series here gives rise to $U(s)$,

- so that $\sum_{n\le x}\frac{d_k(n)}{n^s}(1-n/x)^j = R(s)+\zeta(s)^k+$

$$\frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty}\gamma(s+w)^k\zeta(1-s-w)^k\lambda_j(u,w)dw.$$

- As $-\operatorname{Re}(s+w)=-\sigma-\phi>0$ we may write

$$\zeta(1-s-w)^k=\sum_{n\le y}d_k(n)n^{s+w-1}+\sum_{n>y}d_k(n)n^{s+w-1}.$$

- The second series here gives rise to $U(s)$,
- and we treat the part arising from the first, namely

$$\frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty}\gamma(s+w)^k\sum_{n\le y}d_k(n)n^{s+w-1}\lambda_j(u,w)dw,$$

by moving the path of integration to the line $\operatorname{Re}w=\psi$, picking up a further residue at $w=0$, obtaining

$$-\gamma(s)^k\sum_{n\le y}f(n)n^{s-1}+V(s).$$

This completes the proof of the lemma.

- **Lemma 3** Suppose that $0<\sigma<1$, $-1<\phi<-\sigma$, $0<\psi<1-\sigma$, $j\ge\frac{3}{2}k+1$, $x\ge1$, $y\ge1$. Then $\zeta(s)^k=$

$$\sum_{n\le x} d_k(n)n^{-s}(1-n/x)^j + \gamma(s)^k\sum_{n\le y} d_k(n)n^{s-1} - U(s) - V(s) - R(s)$$

$$\lambda_j(x,z)=\frac{x^z j!}{z(z+1)\dots(z+j)},\quad U(s)= \frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty}\gamma(s+w)^k\sum_{n>y} d_k(n)n^{s+w-1}\lambda_j(x,w)dw,$$

$$R(s)=\operatorname{Res}\big(\zeta(z)^k\lambda_j(x,z-s)\big)_{z=1},\quad V(s)= \frac{1}{2\pi i}\int_{\psi-i\infty}^{\psi+i\infty}\gamma(s+w)^k\sum_{n\le y} d_k(n)n^{s+w-1}\lambda_j(x,w)dw$$

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- **Theorem 2.** Suppose that $1 - 1/k < \sigma < 1$ and $\delta = \frac{k}{4}(\sigma - 1 + \frac{1}{k})$. Then

$$\int_0^T |\zeta(\sigma + it)|^{2k} = T \sum_{n=1}^{\infty} d_k(n)^2 n^{-2\sigma} + O(T^{1-\delta}).$$

Also

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- The latter is more interesting for us.
- Consider Lemma 3 when $k = 2$ and $s = \frac{1}{2} + it$.

- **Lemma 3** with $k = 2$. Suppose that $0 < \sigma < 1$, $-1 < \phi < -\frac{1}{2}$, $0 < \psi < \frac{1}{2}$, $j \geq 4$, $x \geq 1$, $y \geq 1$. Then

$$\zeta(\tfrac{1}{2} + it)^2 = S(t) + S^*(t) - U(t) - V(t) - R(t)$$

where $\lambda_j(x, z) = \frac{x^z j!}{z(z+1)\dots(z+j)}$,

$$S(t) = \sum_{n\leq x} \frac{d(n)}{n^{\frac{1}{2}+it}}(1 - \tfrac{n}{x})^j, \; S^*(t) = \gamma(\tfrac{1}{2} + it)^2 \sum_{n\leq y} \frac{d(n)}{n^{\frac{1}{2}-it}},$$

$$U(t) =$$

$$\frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty} \gamma(\tfrac{1}{2} + it + w)^2 \sum_{n>y} d(n) n^{it+w-\frac{1}{2}} \lambda_j(x, w) dw,$$

$$V(t) =$$

$$\frac{1}{2\pi i}\int_{\psi-i\infty}^{\psi+i\infty} \gamma(\tfrac{1}{2} + it + w)^2 \sum_{n\leq y} d(n) n^{it+w-\frac{1}{2}} \lambda_j(x, w) dw,$$

$$R(t) = \operatorname{Res}\big(\zeta(z)^2 \lambda_j(x, z - \tfrac{1}{2} - it)\big)_{z=1},$$

- By making multiple uses of Cauchy-Schwarz and the heavy convergence of λ_j when, say, $j = 10$, bounding $\zeta(\frac{1}{2} + it)^4$ reduces to bounding

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$$S(t) = \sum_{n \le x} \frac{d(n)}{n^{\frac{1}{2}+it}} (1 - \tfrac{n}{x})^{10},$$

$$S^*(t) = \gamma(\tfrac{1}{2} + it)^2 \sum_{n \le y} \frac{d(n)}{n^{\frac{1}{2}-it}},$$

$$\gamma(\tfrac{1}{2} + it + \phi + iv)^2 \sum_{n > y} d(n) n^{it+\phi+iv-\frac{1}{2}},$$

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- For convenience we take $\phi = -\frac{3}{4}$, $\psi = \frac{1}{4}$.

- For $S(t)$

$$\int_T^{2T} \Big| \sum_{n\le x} \frac{d(n)}{n^{\frac12+it}} (1-\tfrac{n}{x})^{10} \Big|^2 dt \ll \sum_{n\le x} d(n)^2 n^{-1}(T+n)$$
$$\ll T(\log x)^4 + x(\log x)^3.$$

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- It already looks as though we should take $x = y = T$.

- $\phi = -\frac{3}{4}$, $\psi = \frac{1}{4}$, $\theta = \phi$ or ψ.

$$\gamma(\tfrac{1}{2} + it + \phi + iv)^2 \sum_{n>y} d(n) n^{it+\phi+iv-\frac{1}{2}}$$

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$$\gamma(\tfrac{1}{2} + it + \phi + iv)^2 \sum_{n>y} d(n) n^{it+\phi+iv-\frac{1}{2}}$$

- $\gamma(\frac{1}{2} + \theta + i\alpha) \ll (1+|\alpha|)^{-\theta}$, $\lambda_{10}(x, \theta + iv) \ll \frac{x^\theta}{(1+|v|)^{11}}$.

$$\int_T^{2T} \left| \gamma(\tfrac{1}{2} + it + \phi + iv)^2 \sum_{n>y} d(n) n^{it+\phi+iv-\frac{1}{2}} \right|^2 dt$$

$$\ll (1 + |T+v|)^{-4\phi} \sum_{n>y} d(n)^2 n^{2\phi-1} (T+n)$$

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- $$\int_T^{2T} |U(t)|^2 dt \ll (xy/T^2)^{2\phi} (T + y)(\log y)^3.$$

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$$\ll (1 + |T + v|)^{-4\psi}(T + y) y^{2\psi} (\log y)^3$$

- $$\int_T^{2T} |V(t)|^2 dt \ll (xy/T^2)^{2\psi}(T + y)(\log y)^3.$$