A USEFUL GENERAL PRINCIPLE

R. C. VAUGHAN

1. Statement of Principle

Suppose that F is a non-negative real function such that for some $L \geq 1$, $\gamma > 0$, $x \geq 1$, $y \geq 1$, whenever $|\alpha - b/r| \leq r^{-2}$ and $(r,b) = 1$ with $r \in \mathbb{N}$ and $b \in \mathbb{Z}$ we have

$$F(\alpha) \ll Lxr^{-\gamma} + Ly + Lx^{1-\gamma}r^{\gamma}.$$

Then whenever $|\alpha - a/q| \leq q^{-2}$ and $(q,a) = 1$ with $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ we have

$$F(\alpha) \ll \frac{Lx}{(q + x|\alpha q - a|)^{\gamma}} + Ly + Lx^{1-\gamma}(q + x|\alpha q - a|)^{\gamma}.$$

First observe that if $\alpha = a/q$, then the result is immediate on taking $b/r = a/q$. Thus we can suppose that $\alpha \neq a/q$. Now choose r, b so that

$$\left|\alpha - \frac{b}{r}\right| \leq \frac{|\alpha q - a|}{2r},\ r \leq \frac{2}{|\alpha q - a|}.$$

We cannot have $a/q = b/r$, for than we would have $|\alpha - a/q| = |\alpha - b/r| = |\alpha - a/q|/2$ and so $\alpha = a/q$ which is expressly excluded. Thus

$$\frac{1}{qr} \leq \left|\frac{b}{r} - \frac{a}{q}\right| \leq \left|\alpha - \frac{a}{q}\right| + \left|\alpha - \frac{b}{r}\right| \leq \left|\alpha - \frac{a}{q}\right| + \frac{q}{2r}\left|\alpha - \frac{a}{q}\right| \leq \left|\alpha - \frac{a}{q}\right| + \frac{1}{2rq}.$$

Hence

$$\frac{1}{2|\alpha q - a|} \leq r \leq \frac{2}{|\alpha q - a|}.$$

Thus by our assumption on F, now using both a/q and b/r, we have

$$\begin{aligned} F(\alpha) &\ll L\min\left\{xq^{-\gamma} + y + x^{1-\gamma}q^{\gamma}, x^{1-\gamma}(x|\alpha q - a|)^{\gamma} + y + x(x|\alpha q - a|)^{-\gamma}\right\} \\ &\ll L\min\left\{xq^{-\gamma}, x(x|\alpha q - a|)^{-\gamma}\right\} + Ly + Lx^{1-\gamma}q^{\gamma} + Lx^{1-\gamma}(x|\alpha q - a|)^{\gamma} \\ &\ll Lx(q + x|\alpha q - a|)^{-\gamma} + Ly + Lx^{1-\gamma}(q + x|\alpha q - a|)^{\gamma}. \end{aligned}$$

Whilst useful for pruning when α is on the major arcs the generality is really spurious.

RCV: Dept. of Mathematics, Penn. State University, University Park, PA 16802, USA.
Email address: rcv4@psu.edu