COMMENTARY ON AN EXERCISE IN "THE HARDY-LITTLEWOOD METHOD"

1. Description of the Problem

This is a brief commentary on Exercise 3.3.2 of Vaughan RV97 [1997]. This is stated as follows.

2. *Suppose that* $a_1, \ldots, a_4$ *are fixed non-zero integers with* a_1, a_2, a_3 *not all the same sign. Show that*

$$R(n) = \sum_{\substack{p_1 \le n \\ a_1p_1+a_2p_2+a_3p_3+a_4=0}} \sum_{p_2 \le n} \sum_{p_3 \le n} (\log p_1)(\log p_2)(\log p_3)$$

satisfies

$$R(n) = J(n)\mathfrak{S} + O\big(n^2(\log n)^{-A}\big)$$

where $J(n)$ *is the number of solutions of*

$$a_1m_1 + a_2m_2 + a_3m_3 + a_4 = 0$$

with $m_j \le n$ *and*

$$\mathfrak{S} = \sum_{q=1}^{\infty} \phi(q)^{-3} \prod_{j=1}^{4} c_q(a_j).$$

Here

$$c_q(a) = \frac{\phi(q)\mu\big(q/(q,a)\big)}{\phi\big(q/(q,a)\big)}$$

is Ramanujan's sum.

It is clear that if $\gcd(a_1, a_2, a_3) \nmid a_4$, then $R(n) = J(n) = 0$. Thus we may assume that $\gcd(a_1, a_2, a_3)|a_4$, and so

$$\gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, a_2, a_3). \tag{1.1}$$ eq:gcda

For convenience in what follows write

$$R(n; \mathbf{a}), J(n; \mathbf{a}), \mathfrak{S}(\mathbf{a})$$

for $R(n), J(n)$ and $\mathfrak{S}$ respectively. Also let $d = \gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, a_2, a_3)$ and $a'_j = a_j/d$. Clearly

$$R(n; \mathbf{a}) = R(n; \mathbf{a}'),\ J(n; \mathbf{a}) = J(n; \mathbf{a}'),$$

but what about $\mathfrak{S}$? The general term in the definition of $\mathfrak{S}$ is a multiplicative function of q and converges absolutely. Thus

$$\mathfrak{S}(\mathbf{a}) = \prod \left(1 + \sum_{k=1}^{\infty} \phi(p^k) \prod_{j=1}^{4} \frac{\mu\big(p^k/(p^k, a_j)\big)}{\phi\big(p^k/(p^k, a_j)\big)} \right).$$

Suppose that $b_j = b_j(p)$ is defined by $p^{b_j} \| a_j$. Then without loss of generality we may suppose that

$$0 \le b_1 \le b_2 \le b_3 \le b_4. \tag{1.2}$$

eq:oneb

Then the terms in the sum over k are 0 whenever $k \ge b_1 + 2$. Hence

$$\mathfrak{S}(\mathbf{a}) = \prod_p \left(1 + \sum_{k=1}^{b_1} \phi(p^k) + \phi(p^{b_1+1}) \frac{(-1)}{p-1} \prod_{j=2}^{4} \frac{\mu\big(p^{b_1+1}/(p^{b_1+1}, a_j)\big)}{\phi\big(p^{b_1+1}/(p^{b_1+1}, a_j)\big)} \right). \tag{1.3}$$

eq:FrakSa

If $b_2 \ge b_1 + 1$, the expression in the product is

$$p^{b_1} - p^{b_1} = 0,$$

$$\mathfrak{S}(\mathbf{a}) = 0$$

which is what we would expect since $(a_2, a_3, a_4) > 1$ but $(a_2, a_3, a_4) \nmid a_1$, which severely limits the number of solutions of our equation in. primes. Recalling our assumption (eq:oneb 1.2) the same will hold for any permutation of the $\mathbf{a}$.

Thus we may suppose that $b_2(p) = b_1(p)$ for every p. Then, by (eq:FrakSa 1.3)

$$\mathfrak{S}(\mathbf{a}) = \prod_p \left(1 + \sum_{k=1}^{b_1} \phi(p^k) + \frac{\phi(p^{b_1+1})}{(p-1)^2} \prod_{j=3}^{4} \frac{\mu\big(p^{b_1+1}/(p^{b_1+1}, a_j)\big)}{\phi\big(p^{b_1+1}/(p^{b_1+1}, a_j)\big)} \right).$$

When $b_3(p) \ge b_1(p) + 1$ the factor corresponding to p becomes

$$p^{b_1} + \phi(p^{b_1+1}) \frac{1}{(p-1)^2} = \frac{p^{b_1+1}}{p-1}.$$

When $b_3(p) = b_1(p) < b_4(p)$ it becomes

$$p^{b_1} - \phi(p^{b_1+1}) \frac{1}{(p-1)^3} = p^{b_1} \left(1 - \frac{1}{(p-1)^2} \right).$$

Finally, when $b_4(p) = b_1$ it becomes

$$p^{b_1} + \phi(p^{b_1+1}) \frac{1}{(p-1)^4} = p^{b_1} \left(1 + \frac{1}{(p-1)^3} \right).$$

In fact what we have just demonstrated is that

$$\mathfrak{S}(\mathbf{a}) = (a_1, a_2, a_3, a_4) \mathfrak{S}(\mathbf{a}'),$$

whence, by (eq:gcda 1.1),

$$\mathfrak{S}(\mathbf{a}) = (a_1, a_2, a_3)\mathfrak{S}(\mathbf{a}'),$$

In other words the Exercise is wrong when

$$(a_1, a_2, a_3) > 1 \text{ and } (a_1, a_2, a_3) | a_4.$$

So how to repair it? The simple solutions would be either to assume that $(a_1, a_2, a_3) = 1$, or add a factor $(a_1, a_2, a_3)^{-1}$ on the right, as the conclusion does hold in either case. However it is instructive to investigate further, and this can be done in two different directions.

2. First direction

We investigate the intended original solution, based on an analysis of

$$R(n); \mathbf{a}) = \int_0^1 e(\alpha a_4) \prod_{j=1}^{3} f(a_j \alpha) \mathrm{d}\alpha$$

where

$$f(\alpha) = \sum_{p \le n} (\log p) e(\alpha p).$$

Following the standard approach as described in §3.1 of Vaughan *ibid.* one reaches

$$R(n; \mathbf{a}) = \mathfrak{S}(\mathbf{a}) \int_{-P/n}^{P/n} e(a_4 \beta) \prod_{j=11}^{3} v(a_j \beta) \mathrm{d}\beta + O\big(b^2 (\log n)^{-A}\big)$$

where $P = (\log n)^B$ with $B \ge 2A$ and A is some suitably large but fixed real number, and where

$$v(\gamma) = \sum_{m=1}^{n} e(\gamma m).$$

The intent is to use the bound

$$v(\gamma) \ll \min\big(n, \|\gamma\|^{-1}\big) \tag{2.1}$$ eq:vbound

to replace the interval $[-P/n, P/n]$ by $\big[-\frac{1}{2}, \frac{1}{2}\big]$. This is where the problem occurs. Everything would be fine as long as $\|a_j \beta\| \ge P n^{-1}$ for at least one of the j. However, if for example $\beta = k/(a_1, a_2, a_3)$, then $\|a_j \beta\| = 0$ for every j. Thus one needs to investigate when one can have simultaneously, for $j = 1, 2, 3$, an inequality of the kind

$$|a_j \beta - l_j| \le \Delta$$

for non-zero integers l_j. Here we have written Δ for P/n. Since Δ is small it follows that

$$\frac{l_1}{a_1} = \frac{l_2}{a_2} = \frac{l_3}{a_3} = \frac{k}{(a_1,a_2,a_3)} = \frac{k}{d}$$

for some $k \neq 0$. To see this observe that we have

$$\frac{l_1}{a_1'} = \frac{l_2}{a_2'} = \frac{l_3}{a_3'}.$$

Write the common value as

$$\frac{k}{m}$$

with $(k,m)=1$. Then $l_jm = ka_j'$, whence $m|a_j'$ and so $m|(a_1',a_2',a_3') = 1$.

Thus we can at least show that

$$\int_{-P/n}^{P/n} e(a_4\beta)\prod_{j=1}^{3} v(a_j\beta)\mathrm{d}\beta$$
$$= \int_{-\frac{1}{2d}}^{\frac{1}{2d}} e(a_4\beta)\prod_{j=1}^{3} v(a_j\beta)\mathrm{d}\beta + O\big(n^2(\log n)^{-A}\big).$$

Now when $d|h$ we have

$$\int_{-\frac{1}{2d}}^{\frac{1}{2d}} e(h\beta)\beta = \begin{cases} \frac{1}{d} & \text{when } h=0 \\ 0 & \text{otherwise.}\end{cases}$$

Since $d|a_j$ for $j=1,\dots 4$, we do indeed have always $d|a_1m_1+a_2m_2+a_3m_3+a_4$. Hence we obtain

$$d^{-1}J(n)$$

for the main term. Thus we have the first form of the corrected exercise.

2. *Suppose that $a_1,\dots,a_4$ are fixed non-zero integers with a_1,a_2,a_3 not all the same sign. Show that*

$$R(n) = \sum_{\substack{p_1\le n\\ a_1p_1+a_2p_2+a_3p_3+a_4=0}}\sum_{p_2\le n}\sum_{p_3\le n}(\log p_1)(\log p_2)(\log p_3)$$

satisfies

$$R(n) = (a_1,a_2,a_3)^{-1}J(n)\mathfrak{S} + O\big(n^2(\log n)^{-A}\big)$$

where $J(n)$ is the number of solutions of

$$a_1m_1+a_2m_2+a_3m_3+a_4=0$$

with $m_j \le n$ *and*

$$\mathfrak{S} = \sum_{q=1}^{\infty} \phi(q)^{-3} \prod_{j=1}^{4} c_q(a_j).$$

3. Second Direction

CH98
Hooley [1998] requires an estimate for expressions on the kind $R(n;\mathbf{a})$ and it is clear that he is aware of the lacuna in Exercise 3.3.2. He therefore adopts the alternative approach, as briefly alluded to on page 15 of Vaughan *ibid.* of replacing the approximating sum $v(\beta)$ by an integral, in this case

$$w(\beta) = \int_0^n e(\beta\gamma)\mathrm{d}\gamma$$

which in this instance takes the simpler form

$$w(\beta) = \frac{e(\beta n) - 1}{2\pi\beta}.$$

eq:vbound
Crucially the bound (2.1) is replaced by

$$w(\beta) \ll \min\left(n, |\beta|^{-1}\right)$$

which decays all the way to infinity.

Now following in the footsteps of the first direction we reach

$$R(n;\mathbf{a}) = \mathfrak{S}(\mathbf{a})I(\mathbf{a}) + O\left(n^2(\log n)^{-A}\right)$$

where

$$I(\mathbf{a}) = \int_{\mathbb{R}} e(a_4\beta) \prod_{j=1}^{3} w(a_j\beta)\mathrm{d}\beta. \tag{3.1}$$ eq:Ia

This integral can be evaluated *via* the Fourier inversion formula to give

$$I(\mathbf{a}) = V(a_4; a_1, a_2, a_3) \tag{3.2}$$ eq:Va

where $V(y; a_1, a_2, a_3)$ is defined for $y \in \mathbb{R}$ to be the volume of the two dimensional region in $\mathbb{R}^3$ defined by $a_1x_1 + a_2x_2 + a_3x_3 = -y, 0 \le x_j \le n$. This follows readily on writing

$$\prod_{j=1}^{3} w(a_j\beta) = \int_{\mathbb{R}} e(-\beta y)V(y; a_1, a_2, a_3)\mathrm{d}y,$$

so that this is $\widehat{V}(\beta; a_1, a_2, a_3)$, whence

$$I(\mathbf{a}) = \int_{\mathbb{R}} e(a_4\beta)\widehat{V}(\beta; a_1, a_2, a_3)\mathrm{d}\beta = V(a_4; a_1, a_2, a_3).$$

The disadvantage of this approach is that there are some technical details entailed in verifying the conditions for the Fourier inversion. We nevertheless obtain a new version of the exercise.

2. *Suppose that* $a_1, \dots, a_4$ *are fixed non-zero integers with* a_1, a_2, a_3 *not all the same sign. Show that*

$$R(n) = \sum_{\substack{p_1 \le n \\ a_1p_1+a_2p_2+a_3p_3+a_4=0}} \sum_{p_2 \le n} \sum_{p_3 \le n} (\log p_1)(\log p_2)(\log p_3)$$

satisfies

$$R(n) = K(n)\mathfrak{S} + O\big(n^2(\log n)^{-A}\big)$$

where $K(n)$ *is the area of that part of the plane in* $\mathbb{R}^3$ *defined by* $a_1x_1 + a_2x_2 + a_3x_3 + a_4 = 0, 0 \le x_j \le n$, *and* $\mathfrak{S}$ *satisfies*

$$\mathfrak{S} = \sum_{q=1}^{\infty} \phi(q)^{-3} \prod_{j=1}^{4} c_q(a_j).$$

Now recall the lacuna. We have $R(n;\mathbf{a}) = R(n;\mathbf{a}')$ but $\mathfrak{S}(\mathbf{a}) = (a_1, a_2, a_3)\mathfrak{S}(\mathbf{a}')$. How does V behave? Adverting to (3.2) and (3.1) and applying the change of variable $\beta = d^{-1}\gamma$ we have

$$\begin{aligned} V(\mathbf{a}) &= \int_{\mathbb{R}} e(da_4'\beta) \prod_{j=1}^{3} w(da_j'\beta)\mathrm{d}\beta \\ &= d^{-1} \int_{\mathbb{R}} e(a_4'\gamma) \prod_{j=1}^{3} w(a_j'\gamma)\mathrm{d}\gamma \\ &= d^{-1}V(\mathbf{a}'), \end{aligned}$$

which fits.

References

CH98 [1998] C. Hooley, On the Barban-Davenport-Halberstam theorem VIII, J. reine ang. Math. 499 (1998), 1–46.

RV97 [1997] R. C. Vaughan, The Hardy–Littlewood method, second edition, Cambridge, Tract No. 125, 1997.