6. Inhomogeneous Approximation

We first show that inhomogeneous approximations cannot be localised in the manner of Dirichlet's thoerem for homogeneous approximations. The following was conjectured by Hardy and Littlewood, and proved by Khintchin.

Theorem 6.1 (Khintchin, 1926). *Suppose that $\phi(q)$ is a non-negative function tending to 0 as $q \to \infty$. Then there exist irrational α and β such that there are infinitely many Q such that the pair of inequalities $|q| \le Q$, $\|q\alpha - \beta\| < \phi(Q)$ have no solution.*

Proof. We put $\beta = \frac{1+\sqrt{5}}{2}$ and take $\alpha = [0; a_1, a_2, \dots]$, where the a_n are to be determined inductively as follows. Choose Q so large that $\phi(Q) < \frac{1}{6q_n^2}$. Then choose a_{n+1} so large that $q_{n+1} > 6q_nQ$. We now show that $\|q\alpha - \beta\| > \phi(Q)$ when $|q| \le Q$. Since $|\alpha - p_n/q_n| < 1/(q_nq_{n+1})$, it follows that $\|q\alpha - \beta\| \ge \|qp_nq_n^{-1} - \beta\| - |q|/(q_nq_{n+1}) > \|qp_nq_n^{-1} - \beta\| - 1/(6q_n^2)$. But $|y/x - \beta| > 1/(3x^2)$ for all x, y with $x \ne 0$. Thus the above is $> 1/(3q_n^2) - 1/(6q_n^2) = 1/(6q_n^2) > \phi(Q)$.

We now consider simultaneous inhomogeneous approximation. If real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ are given, then we can ask whether the n-tuples of fractional parts $k\alpha_1 - [k\alpha_1], k\alpha_2 - [k\alpha_2], \dots, k\alpha_n - [k\alpha_n]$ is dense in $[0,1)^n$. In other words, for any $\boldsymbol{\beta}$ and any $\varepsilon > 0$ does there exist an integer k such that $\|k\alpha_i - \beta_i\| < \varepsilon$ for $1 \le i \le n$. We can exclude one possibility quite easily. Suppose that there exist integers $a_1, a_2, \dots, a_n$ not all 0 such that $\sum_{i=1}^n a_i\alpha_i \in \mathbb{Z}$, and suppose there is an integer k with the property $\|k\alpha_i - \beta_i\| < \varepsilon$ for $1 \le i \le n$. Then

$$\|\sum_{i=1}^n a_i\beta_i\| = \|\sum_{i=1}^n a_i(\beta_i - k\alpha_i)\| \le \sum_{i=1}^n |a_i|\varepsilon.$$

Thus if the inequalities are to hold for every $\varepsilon > 0$, then $\sum_{i=1}^n a_i\beta_i \in \mathbb{Z}^n$. However, in general the point $\boldsymbol{\beta}$ will not satisfy such a relationship. Thus in order for $k\alpha_1 - [k\alpha_1], k\alpha_2 - [k\alpha_2], \dots, k\alpha_n - [k\alpha_n]$ to be dense in $[0,1)^n$ we have to exclude all possible relationships $\sum_{i=1}^n a_i\alpha_i \in \mathbb{Z}$ amongst the α_i. Thus the condition that $\alpha_1, \alpha_2, \dots, \alpha_n, 1$ be linearly independent over $\mathbb{Q}$ is necessary. Kronecker's theorem, in its simplest form is the assertion that this condition is sufficient.

Theorem 6.2 (Kronecker). *Suppose that $\alpha_1, \alpha_2, \dots, \alpha_n, 1$ are linearly independent over $\mathbb{Q}$. Then for each $\boldsymbol{\beta} \in \mathbb{R}^n$ and each $\varepsilon > 0$ there are arbitrarily large positive integers k such that $\|k\alpha_i - \beta_i\| < \varepsilon$ for $1 \le i \le n$.*

There are many proofs of Kronecker's theorem and we offer two of them. The first is a very simple application of Fourier series and echoes the theme developed in the previous chapter. We need a special Fourier series and the necessary properties can be obtained easily via the Féjer kernel.

We have

$$\int_{-\alpha}^{\alpha} F_H(\beta)d\beta = 2\alpha + \sum_{\substack{h=-H \\ h\ne 0}}^{H} \left(1 - \frac{|h|}{H}\right) \frac{e(\alpha h) - e(-\alpha h)}{2\pi i h}.$$

Thus when $0 \le \alpha \le \frac{1}{2}$ we have

$$1 - \int_{\alpha \le |\beta| \le \frac{1}{2}} F_H(\beta)d\beta = 2\alpha + 2\sum_{0<|h|\le H} \frac{e(\alpha h)}{2\pi i h} - 2\sum_{h=1}^{H} \frac{e(\alpha h) - e(-\alpha h)}{2\pi i H},$$

and so

$$\alpha - \frac{1}{2} = - \sum_{0<|h|\le H} \frac{e(\alpha h)}{2\pi i h} + O\left(\min\left(1, \frac{1}{H\|\alpha\|}\right)\right). \tag{6.1}$$

Now suppose $\frac{1}{2} < \alpha < 1$ and consider the above formula with α replaced by $1-\alpha$. By then replacing h by $-h$ we see that (6.1) holds generally for $0 \le \alpha < 1$. Thus we have the first part of the following lemma.

Lemma 6.1. *Suppose that α is a real number, a is a real number with $0 < a \le \frac{1}{2}$, and H is a positive integer, and let $f(\alpha) = \frac{1}{2}(\alpha - [\alpha])^2 - \frac{1}{2}(\alpha - [\alpha]) + \frac{1}{12}$. Then (i)*

$$\alpha - [\alpha] - \frac{1}{2} = - \sum_{0<|h|\le H} \frac{e(\alpha h)}{2\pi i h} + O\left(\min\left(1, \frac{1}{H\|\alpha\|}\right)\right),$$

(ii)

$$f(\alpha) = \sum_{\substack{h=-\infty \\ h\ne 0}}^{\infty} \frac{e(\alpha h)}{4\pi^2 h^2},$$

(iii)

$$\max\left(0, 1 - \frac{\|\alpha\|}{a}\right) = a + \frac{2f(\alpha) - f(\alpha - a) - f(\alpha + a)}{a}.$$

To prove (ii) we observe that we may suppose that $0 \le \alpha < 1$. We integrate both sides of (i) from 0 to α. The left hand side becomes $\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha$ and the error term is $O(H^{-1}\log(2H))$. The general term in the sum becomes $(e(\alpha h) - 1)/(2\pi i h)^2$. Letting $H \to \infty$ gives the desired conclusion.

To prove (iii) it is a simple matter to verify separately the cases $0 \le \alpha \le a$ and $a < \alpha \le 1/2$ and the general case follows by symmetry and periodicity.

First proof of Theorem 6.2. By the lemma

$$\max\left(0, 1 - \frac{\|\alpha\|}{a}\right) = a + \sum_{\substack{h=-\infty \\ h\ne 0}}^{\infty} b_h e(\alpha h)$$

with the b_h satisfying $|b_h| \ll a^{-1}(1+|h|)^{-2}$. Thus, provided that $0 < \varepsilon < \frac{1}{2}$, we have

$$\prod_{i=1}^{n} \max\left(0, 1 - \frac{\|\gamma_j\|}{\varepsilon}\right) = \varepsilon^n + \sum_{\mathbf{h}\ne\mathbf{0}} c(\mathbf{h}) e(\boldsymbol{\gamma}.\mathbf{h})$$

where

$$c(\mathbf{h}) \ll \varepsilon^{-n} \prod_{i=1}^{n} \frac{1}{(1+|h_i|)^2}.$$

Let L denote the number of $k \le K$ such that $\|\alpha_i k - \beta_i\| \le \varepsilon$ for $i = 1, \ldots, n$. Then

$$L \ge \varepsilon^n K + \sum_{\mathbf{h}\ne\mathbf{0}} c(\mathbf{h}) \sum_{k=1}^{K} e(\boldsymbol{\alpha}.\mathbf{h}k - \boldsymbol{\beta}.\mathbf{h}).$$

The terms in the infinite sum with $|h_i| > H$ for at least one i contribute $O(\varepsilon^{-n}K/H)$ and this will be small compared with the main term provided that $H = [C\varepsilon^{-2n}]$ for a suitable positive number C. In the remaining terms, by hypothesis, we have $\boldsymbol{\alpha}.\mathbf{h} \notin \mathbb{Z}$. Thus they contribute

$$\ll \sum_{\mathbf{h}}{}' \frac{1}{\|\boldsymbol{\alpha}.\mathbf{h}\|}$$

where the sum is over $\mathbf{h} \neq \mathbf{0}$ with $|h_i| \leq H$ for $1 \leq i \leq n$. For large K this is negligible by comparison with the main term, and so $L > 0$ as required.

For our second proof of Kronecker's theorem we will make a connection with a similar formulation of the result with integers k replaced by real numbers t.

Theorem 6.3 (Kronecker). *Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$. Then for each $\boldsymbol{\beta} \in \mathbb{R}^n$ and each $\varepsilon > 0$ there are arbitrarily large positive real numbers t such that $\|t\alpha_i - \beta_i\| < \varepsilon$ for $1 \leq i \leq n$.*

Proof of Theorems 6.2 and 6.3. The proof is due to Ka-Lam Kueh (198?). Let I_n denote the assertion of Theorem 6.2 and R_n denote the assertion of Theorem 6.3. We prove R_1, $R_n \implies I_n$, $I_n \implies R_{n+1}$.

"R_1" The hypothesis of R_1 ensures that $\alpha_1 \neq 0$. Consider t of the form $t = \frac{k+\beta_1}{\alpha_1}$ where $k \in \mathbb{Z}$. Then for such a t we have $\|t\alpha_1 - \beta_1\| = \|k\| = 0$. The set of such t forms an arithmetic progression and thus contains large positive members.

"$R_n \implies I_n$" By Theorem 3.13 (or the standard generalisations of Dirichlet's Theorem to simultaneous homogeneous approximation) there exist integers $q, q_1, q_2, \ldots, q_n$ with $q > 0$ and such that $|q\alpha_i - q_i| < \varepsilon/2$ for $1 \leq i \leq n$. Suppose that we have a linear relationship amongst the numbers $q\alpha_i - q_i$, say that $\sum_{i=1}^k a_i(q\alpha_i - q_i) = 0$. This can be rewritten as $\sum_{i=1}^n qa_i\alpha_i - (\sum_{i=1}^n a_iq_i) = 0$. But by the hypothesis of I_n we know that $\alpha_1, \ldots, \alpha_n, 1$ are linearly independent. Hence all the a_i vanish. In other words the $q\alpha_i - q_i$ are linearly independent over $\mathbb{Q}$. By R_n it follows that there exist arbitrarily large real numbers t such that $\|t(q\alpha_i - q_i) - \beta_i\| < \varepsilon/2$ for $1 \leq i \leq n$. Put $k = [t]q$. Then

$$\begin{aligned}\|k\alpha_i - \beta_i\| &= \|([t] - t)(q\alpha_i - q_i) + (t(q\alpha_i - q_i) - \beta_i) + [t]q_i\| \\ &\leq \|q\alpha_i - q_i\| + \|t(q\alpha_i - q_i) - \beta_i\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

"$I_n \implies R_{n+1}$" Since $\alpha_1, \ldots, \alpha_{n+1}$ are linearly indpendent over $\mathbb{Q}$, it follows that $\alpha_{n+1} \neq 0$ and that $\alpha_1/|\alpha_{n+1}|, \alpha_2/|\alpha_{n+1}|, \ldots, \alpha_n/|\alpha_{n+1}|, 1$ are linearly independent over $\mathbb{Q}$. Then by I_n with β_i replaced by $\beta_i' = \beta_i \mp \frac{\beta_{n+1}\alpha_i}{|\alpha_{n+1}|}$ we see that there exist arbitrarily large positive integers k such that $\|k\frac{\alpha_i}{|\alpha_{n+1}|} - \beta_i'\| < \varepsilon$ for $1 \leq i \leq n$. That is, $\|\frac{k\pm\beta_{n+1}}{|\alpha_{n+1}|}\alpha_i - \beta_i\| = \|t\alpha_i - \beta_i\| < \varepsilon$ for $i = 1, \ldots, n$ where $t = \frac{k\pm\beta_{n+1}}{|\alpha_{n+1}|}$. We choose the upper sign when $\alpha_{n+1} > 0$, the lower sign when $\alpha_{n+1} < 0$.