Math 597e Primes, Spring 2008, Problems 8

Due Tuesday 25th March

This is the previous homework continued. Let $f(\alpha)$ be as in the lectures, let $Q = N^{1/2}(\log N)^{-B}$ and $P = (\log N)^D$ where B and D will be fixed later. Further let $K(\alpha) = |\sum_{h=1}^{H} e(\alpha h)|^2 = \sum_{|h|\le H}(H-|h|)e(\alpha h)$, where we suppose that $H \ll \log N$. Define $\mathfrak{M}(q,a) = [a/q - P/N, a/q + P/N]$ and take $\mathfrak{M}$ to be the union of the $\mathfrak{M}(q,a)$ with $1 \le a \le q \le Q$ and $(a,q) = 1$. Also let $\mathfrak{U} = (P/N, 1 + P/N]$ and $R(N,h) = \sum_{p_1,p_1}(\log p_1)(\log p_2)$ where the sum is over primes p_1, p_2 with $p_j \le N$ and $p_2 - p_1 = h$. The story so far is, if $D \ge 2$, then $HN\log N + 2\sum_{h=1}^{H}(H-h)R(N,h) \ge I_5 + 2\Re I_4 + O(HN)$ where $I_5 = N\sum_{|h|\le H}(H-|h|)\mathfrak{S}(h,Q)$ and $I_4 \le \sum_{q\le Q}\mu(q)^2\phi(q)^{-1}\sum_{\substack{r=1\\(r,q)=1}}^{q}|S(r)|I(r)$ here $S(r)$ and $I(r)$ are defined in question 8.

9. In the notation of question 8, prove if $\delta(u,q,r) = \sum_{n\le u} c(n,q,r)$, then $\Delta(\beta,q,r) = e(\beta N)\delta(N,q,r) - 2\pi i\int_1^N e(\beta u)\delta(u,q,r)du$. Deduce that if $I_6 = \int_{-P/N}^{P/N}|\sum_{n=1}^{N} e(n\beta)|(1+N|\beta|)d\beta$, then

$$I_4 \ll I_6 \sum_{q\le Q}\mu(q)^2\phi(q)^{-1}\left(\max_{(r,q)=1}\sup_{u\le N}|\delta(u,q,r)|\right)\sum_{\substack{r=1\\(r,q)=1}}^{q}|S(r)|.$$

10. Prove that $I_6 \ll \log N + P$.

11. Prove that Ramanujan's sum $c_q(j) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e(aj/q)$ satisfies $|c_q(j)| \le (q,j)$. Deduce that $S(r) \le H\sum_{|h|\le H}(q, h-r)$, that $\sum_{\substack{r=1\\(r,q)=1}}^{q}|S(r)| \ll H^2qd(q)$ and that $I_4 \ll (\log N + P)H^2\sum_{q\le Q}\mu(q)^2\phi(q)^{-1}qd(q)\left(\max_{(r,q)=1}\sup_{u\le N}|\delta(u,q,r)|\right)$.

12. Observe that when $q \le N$, $\max_{(r,q)=1}\sup_{u\le N}|\delta(u,q,r) \ll q^{-1}N\log N$ and prove that $\sum_{q\le Q}\mu(q)^4\phi(q)^{-2}q^2d(q)^2\left(\max_{(r,q)=1}\sup_{u\le N}|\delta(u,q,r)|\right) \ll N(\log N)^5$. Deduce from Bombier's theorem that if $B > B_0(A,D)$, then $I_4 \ll N$ and hence $HN\log N + 2\sum_{h=1}^{H}(H-h)R(N,h) \ge NH\mathfrak{S}(0,Q) + 2N\sum_{h=1}^{H}(H-h)\mathfrak{S}(h,Q) + O(HN)$.

13. Observe that $\mathfrak{S}(0,Q) = \log Q + 0(1)$ and that $\phi(q)^{-2}\sum_{\substack{a=1\\(a,q)=1}}^{q} e(ah/q) \ll q^{-3/2}h$. Prove that $\sum_{1\le h\le H}(H-h)\sum_{R<q\le Q}\phi(q)^{-2}\sum_{\substack{a=1\\(a,q)=1}}^{q} e(ah/q) \ll H^3R^{-1/2} \ll H$ when $R = H^4$.

14. Observe that $K(a/q) - H \ll H + \min(H^2, \|a/q\|^{-2})$. Deduce that when $q > 1$, $\sum_{\substack{a=1\\(a,q)=1}}^{q}(K(a/q) - H) \ll qH$ and that when $R = H^4$, $\sum_{1<q\le R}\sum_{\substack{a=1\\(a,q)=1}}^{q}(K(a/q) - H) \ll H(\log H)^2$. Conclude that

$$\frac{1}{2}HN\log N + 2\sum_{h=1}^{H}(H-h)R(N,h) \ge H^2N + O(HN(\log\log N)^2).$$

15. Prove that if $H = (\frac{1}{2} + \varepsilon)\log N$ and $N > N_0(\varepsilon)$, then there is an $h \le H$ such that $R(N,h) \gg_\varepsilon N$. Show that there are primes p_1, p_2 with $p_2 > N/\log^3 N$ such that $p_2 - p_1 = h$. Conclude that $\liminf \frac{p_{n+1}-p_n}{\log p_n} \le \frac{1}{2}$.