Math 597e Primes, Spring 2008, Problems 7

Due Tuesday 18th March

This and the next homework prove that $\liminf \frac{p_{n+1}-p_n}{\log p_n} \le \frac{1}{2}$ (Bombieri & Davenport, 1965). Let $f(\alpha)$ be as in the lectures, let $Q = N^{1/2}(\log N)^{-B}$ and $P = (\log N)^D$ where B and D will be fixed later. Further let $K(\alpha) = |\sum_{h=1}^{H} e(\alpha h)|^2 = \sum_{|h|\le H}(H-|h|)e(\alpha h)$ be the Fejér kernel, where we suppose that $H \ll \log N$. We now define the major arc $\mathfrak{M}(q,a) = [a/q - P/N, a/q + P/N]$ and take $\mathfrak{M}$ to be the union of the $\mathfrak{M}(q,a)$ with $1 \le a \le q \le Q$ and $(a,q) = 1$. Also let $\mathfrak{U} = (P/N, 1 + P/N]$ and $R(N,h) = \sum_{p_1,p_1}(\log p_1)(\log p_2)$ where the sum is over primes p_1, p_2 with $p_j \le N$ and $p_2 - p_1 = h$. You may assume anything proved in class or in earlier homeworks.

1. Prove that $\int_{\mathfrak{U}} |f(\alpha)|^2 K(\alpha) d\alpha = \sum_{h=-H}^{H}(H-|h|)R(N,h)$.

2. Prove that $R(N,0) = N \log N + O(N)$.

3. Prove that $\int_{\mathfrak{U}} |f(\alpha)| K(\alpha) d\alpha \ll H^2 (N \log N)^{1/2}$.

4. Show that if $\alpha \in \mathfrak{M}(q,a)$, then $K(\alpha) = K(a/q) + O(PH^2N^{-1})$. Deduce that $HN\log N + 2\sum_{h=1}^{H}(H-h)R(N,h) \ge I_1 + O(HN)$ where

$$I_1 = \sum_{q\le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} K(a/q) \int_{\mathfrak{M}(q,a)} |f(\alpha)|^2 d\alpha.$$

5. Let $f_q(\alpha) = \sum_{\substack{p\le N \\ p\nmid q}} (\log p) e(\alpha p)$. Prove that $f(\alpha) - f_q(\alpha) \ll \log q$ and that $I_1 = I_2 + O\big(N^{1/2}(\log N)^4\big)$ where I_2 is as I_1 but with f replaced by f_q.

6. Define $V(\alpha)$ by $V(\alpha) = \mu(q)\phi(q)^{-1}\sum_{n=1}^{N} e\big((\alpha - a/q)\big)$ when $\alpha \in \mathfrak{M}(q,a)$ and by 0 when $\alpha \notin \mathfrak{M}$. Let $E(\alpha) = f_q(\alpha) - V(\alpha)$. Prove that $|f_q(\alpha)|^2 \ge |V(\alpha)|^2 + 2\Re\big(V(\alpha)\overline{E(\alpha)}\big)$. Deduce that $I_2 \ge I_3 + 2\Re I_4$ where I_3, I_4 are as I_1 but with $|f|^2$ replaced by $|V|^2$ and $V\overline{E}$ respectively.

7. Prove that $\int_{\mathfrak{M}(q,a)} |V(\alpha)|^2 = \mu(q)^2\phi(q)^{-2}\big(N + O(N/P)\big)$ and deduce that with $\mathfrak{S}(h,Q)$ as in the lectures $I_3 = I_5 + O\big(H^2(\log Q)NP^{-1}\big)$ where $I_5 = N\sum_{|h|\le H}(H-|h|)\mathfrak{S}(h,Q)$.

In summary, provided that $D \ge 3$,

$$HN\log N + 2\sum_{h=1}^{H}(H-h)R(N,h) \ge I_5 + 2\Re I_4 + O(HN).$$

8. Let $c(n,q,r) = \log n - 1/\phi(q)$ when n is prime and $n \equiv r \pmod q$, and $c(n,q,r) = -1/\phi(q)$ otherwise, and let $\Delta(\beta,q,r) = \sum_{n\le N} c(n,q,r)e(\beta n)$. Prove that if $\alpha \in \mathfrak{M}(q,a)$, $\alpha = a/q + \beta$, then $E(\alpha) = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \Delta(\beta,q,r)e(ar/q)$. Deduce that if $S(r) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} K(a/q)e(ar/q)$ and $I(r) = \int_{-P/N}^{P/N} |\sum_{n=1}^{N} e(n\beta)||\Delta(\beta,q,r)| d\beta$, then

$$|I_4| \le \sum_{q\le Q} \mu(q)^2\phi(q)^{-1} \sum_{\substack{r=1 \\ (r,q)=1}}^{q} |S(r)| I(r).$$