MATH 597E PRIMES, SPRING 2008, PROBLEMS 5

Due Tuesday 26th February

Let A be fixed and positive, N be large, $Q = (\log N)^B$ with $B = B(A)$, and $\mathfrak{M}(q,a)$, $\mathfrak{M}$, $\mathfrak{m}$, f, $\mathfrak{S}(h,Q)$, $\mathfrak{S}(h)$ be as in the lectures. Assume any result from the lectures that may be useful.

1. Prove that

$$\int_{\mathfrak{M}} |f(\alpha)|^2 e(h\alpha) d\alpha = (N - |h|)\mathfrak{S}(h,Q) + O\left(N(\log N)^{-A}\right)$$

when $|h| \leq N$.

2. Prove that

$$\sum_{h\in\mathbb{Z}} \left| \int_{\mathfrak{m}} |f(\alpha)|^2 e(\alpha h) d\alpha \right|^2 \ll N^3 (\log N)^{-A}.$$

3. (Lavrik 1961.) Prove that

$$\sum_{h=1}^{N} |R(N,h) - (N-h)\mathfrak{S}(h)|^2 \ll N^3 (\log N)^{-A}$$

where

$$R(N,h) = \sum_{\substack{p_1,p_2 \leq N \\ p_2 - p_1 = h}} \log p_1 \log p_2.$$