Math 597e Primes, Spring 2008, Problems 4

Due Tuesday 19th February

Throughout we suppose that for each prime p, $0 \le \omega(p) < p$ and that $F(Q) = \sum_{q\le Q} \mu(q)^2 \prod_{p|q} \frac{\omega(p)}{p-\omega(p)}$.

1. (Vaughan 1973). (a) By considering $\frac{\omega(p)}{p-\omega(p)} = \sum_{h=1}^{\infty} \left(\frac{\omega(p)}{p}\right)^h$, or otherwise, prove that

$$F(Q) = \sum_{\substack{r \\ s(r)\le Q}} \prod_{p^h\|r} \left(\frac{\omega(p)}{p}\right)^h = \sum_m \sum_{\substack{r,\Omega(r)=m \\ s(r)\le Q}} \prod_{p^h\|r} \left(\frac{\omega(p)}{p}\right)^h$$

where $s(r)$ is the squarefree kernel of r and $\Omega(m)$ is the total number of prime factors of m.
(b) Prove that

$$\sum_{\substack{r,\Omega(r)=m \\ s(r)\le Q}} \prod_{p^h\|r} \left(\frac{\omega(p)}{p}\right)^h \ge \frac{1}{m!}\left(\sum_{p\le Q^{1/m}} \frac{\omega(p)}{p}\right)^m.$$

(c) Suppose that $\sum_{p\le X} \frac{\omega(p)}{p} \ge f(X)$. Prove that $F(Q) \ge \max_m \exp\left(m \log\left(m^{-1} f(Q^{1/m})\right)\right)$.
2. ("Rankin's Trick" but Rankin said that it was shown to him by Ingham). Suppose that $Q \ge 1$, and $P \in \mathbb{N}$. Let

$$F(Q,P) = \sum_{q\le Q, q|P} \mu(q)^2 \prod_{p|q} \frac{\omega(p)}{p-\omega(p)}.$$

(a) Prove that, with the obvious abuse of notation, $F(\infty, P) = \prod_{p|P} \frac{p}{p-\omega(p)}$.
(b) Suppose that $\theta \ge 0$. Prove that $F(\infty,P) - F(Q,P) \le Q^{-\theta} \sum_{q|P} q^\theta \mu(q)^2 \prod_{p|q} \frac{\omega(p)}{p-\omega(p)}$ and

$$\frac{F(Q,P)}{F(\infty,P)} \ge 1 - Q^{-\theta} \prod_{p|P} \left(1 + (p^\theta - 1)\omega(p)/p\right).$$

(c) Suppose that f is a strictly increasing and continuous real function on $[1,\infty)$ and suppose that for $X \ge X_0$

$$\log X \le e f(X) \quad \text{and} \quad \sum_{p\le X} \omega(p) p^{-1} \log p \le f(X). \tag{1}$$

Show that $\sum_{p\le X} (p^\theta - 1) p^{-1} \omega(p) \le (X^\theta - 1)(\log X)^{-1} f(X)$ and that $-\theta \log Q + (X^\theta - 1)(\log X)^{-1} f(X)$ is minimised as a function of θ by the choice

$$\theta = \frac{1}{\log X} \log \frac{\log Q}{f(X)}.$$

(d) Suppose that $X \le Q$ and $P = \prod_{p\le X} p$. Deduce that $\frac{F(Q,P)}{F(\infty,P)} \ge 1 - \exp\left(-\frac{\log Q}{\log X} \log \frac{\log Q}{e f(X)} - f(X)(\log X)^{-1}\right)$ and show that if f satisfies

$$f(X)(\log X)^{-1} \to \infty \quad \text{as} \quad X \to \infty, \quad \text{and} \quad X = g(e^{-1} \log Q) \tag{2}$$

where g is the inverse function of f, and $Q > Q_0$, then $\frac{1}{2} F(\infty,P) < F(Q,P) \le F(\infty,P)$.
(e) Prove that $F(\infty,P) \ge \exp\left(\sum_{p\le X} \frac{\omega(p)}{p}\right)$ and that if there is a positive constant C such that

$$\sum_{p\le X} \omega(p) p^{-1} \log p \ge C f(X), \tag{3}$$

then $\sum_{p\le X} \omega(p) p^{-1} \ge C f(X) (\log X)^{-1}$.
(f) Deduce that if (1), (2), (3) hold, then for $Q > Q_1$, $F(Q) \ge F(Q,P) \gg \exp\left(\frac{C \log Q}{e \log(g(e^{-1} \log Q))}\right)$.

Surprisingly the fact that X is much smaller than Q does not lose too much. By using the Rankin trick in the form $F(Q) \le Q^\theta \sum_{q=1}^{\infty} q^{-\theta} \mu(q)^2 \prod_{p|q} \frac{\omega(p)}{p-\omega(p)}$ combined with a condition of the kind (1) it can be shown that, for $Q > Q_2$, $F(Q) \le \exp\left(C'(\log Q)/\log\left(g(e^{-1}\log Q)\right)\right)$.

Generally both methods lead to the same sort of conclusion. Thus when $\lambda > 1$, in Q.1 if $f(X) = C(\log X)^{\lambda-1}$ or if $f(X) = C(\log X)^\lambda$ in Q.2, then one gets $F(Q) > \exp(C'(\log Q)^{1-1/\lambda})$ and if $F(x) = C(\log\log X)^\lambda$ in Q.1 or $F(X) = (\log X)(\log\log X)^\lambda$ in Q.2, then $F(Q) > \exp(C'(\log\log Q)^\lambda)$.