MATH 597E PRIMES, SPRING 2008, PROBLEMS 3

Due Tuesday 12th February

1. Recall that

$$\sum_{x=1}^{q} \chi(x) e(xn/q) = \overline{\chi}(n)\tau(\chi)$$

holds for all χ modulo q when $(n,q) = 1$.
(a) Show that if $a_n = 0$ whenever $(n,q) > 1$ then

$$\sum_{\chi} |\tau(\chi)|^2 |S(\chi)|^2 = \varphi(q) \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |T(a/q)|^2.$$

(b) Suppose that $a_n = 0$ whenever n has a prime factor $\le Q$. Show that

$$\sum_{q\le Q} \frac{1}{\varphi(q)} \sum_{\chi \pmod q} |\tau(\chi)|^2 |S(\chi)|^2 \le \Lambda(N,Q) \sum_{n=M+1}^{M+N} |a_n|^2.$$

(c) Suppose that $a_n = 0$ whenever $(n,q) > 1$, and that the character $\chi \pmod q$ is induced by the primitive character $\chi^* \pmod d$. Show that $S(\chi) = S(\chi^*)$. Recall also that $\tau(\chi) = \tau(\chi^*)\mu(q/d)$ if $(q/d, d) = 1$ and that $\tau(\chi) = 0$ otherwise.
(d) Let ${\sum^*}_{\chi}$ denote summation over the primitive characters modulo q. Show that if the a_n are as in (b) then

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \Big(\sideset{}{^*}\sum_{\chi} |S(\chi)|^2 \Big) \Big(\sum_{\substack{k\le Q/q \\ (k,q)=1}} \frac{\mu(k)^2}{\varphi(k)} \Big) \le \Lambda(N,Q) \sum_{n=M+1}^{M+N} |a_n|^2.$$

(e) Show that if the a_n are as in (b) then

$$\sum_{q\le Q} \big(\log Q/q\big) \sideset{}{^*}\sum_{\chi} |S(\chi)|^2 \le \Lambda(N,Q) \sum_{n=M+1}^{M+N} |a_n|^2.$$

(f) Let $\mathcal{N}$ be the set of those integers $n \in [M+1, M+N]$ such that $(n,q) = 1$ for all $q \le Q$. Put $Z = \text{card}\mathcal{N}$. Show that

$$Z^2 \log Q + \sum_{1<q\le Q} \big(\log Q/q\big) \sideset{}{^*}\sum_{\chi} \Big| \sum_{n\in\mathcal{N}} \chi(n) \Big|^2 \le \Lambda(N,Q) Z.$$