MATH 597E PRIMES, SPRING 2008, PROBLEMS 2

Due Tuesday 5th February

1. Let $\mathcal{Q}$ be a set of pairwise coprime positive integers not exceeding Q, suppose that $T(x) = \sum_{n=M+1}^{M+N} c_n e(nx)$, and that $Z(q,h) = \sum_{n=M+1, n\equiv h \pmod q}^{M+N} c_n$.
(a) Show that

$$\sum_{q\in\mathcal{Q}} \sum_{a=1}^{q-1} |T(a/q)|^2 \le \Lambda(N, Q^2) \sum_{n=M+1}^{M+N} |c_n|^2.$$

(b) Show that

$$\sum_{q\in\mathcal{Q}} q \sum_{h=1}^{q} |Z(q,h) - Z/q|^2 \le \Lambda(N, Q) \sum_{n=M+!}^{M+N} |c_n|^2.$$

2. Let $x_1, x_2, \ldots, x_R$ be points in $\mathbf{T}$. For $\delta > 0$ let $N_\delta(x)$ denote the number r for which $\|x_r - x\| < \delta$.
(a) Show that

$$\sum_{\substack{1\le r\le R \\ \|x_r - x\| \le \delta/2}} \frac{1}{N_\delta(x_r)} \le 1$$

for all $x \in \mathbf{T}$.
(b) Show that if M and N are integers, $N \ge 1$, and $T(x)$ is as given in the lectures, then

$$\sum_{r=1}^{R} \frac{|T(x_r)|^2}{N_\delta(x_r)} \le \Big(\frac{1}{\delta} + \pi N\Big) \sum_{n=M+1}^{M+N} |c_n|^2$$

for all $\delta > 0$.