MATH 597B, SPRING 2015, PROBLEMS 13

Due Tuesday 28th April

1. Suppose that $k \ge 2$. Let $\mathcal{R}_k \subset [0,1]^k$ be defined by $\mathcal{R}_k = \{t : t_i \ge 0, t_1 + \cdots + t_k \le 1\}$, and let $m \in \mathbb{N}$ and $f(\mathbf{t}) = (1 - t_1 - \cdots - t_k)^m$. Given $t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_k \in [0,1]^{k-1}$ with $t_1 + \cdots + t_{j-1} + t_{j+1} + \cdots + t_k \le 1$ let A_j denote the interval $[0, 1 - t_1 - \cdots - t_{j-1} - t_{j+1} - \cdots - t_k]$ (and take it to be the empty set otherwise) and define

$$I_j(f) = \int_0^1 \cdots \int_0^1 \left(\int_{A_j} f(\mathbf{t}) dt_j \right)^2 dt_1 \ldots dt_{j-1} dt_{j+1} \ldots dt_k$$

and

$$J(f) = \int_{\mathcal{R}_k} f(\mathbf{t})^2 d\mathbf{t}.$$

(i) Prove that $\displaystyle\sum_{j=1}^{k} I_j(f) = \frac{k(2m+2)!}{(2m+1+k)!(m+1)^2}$ and $J(f) = \dfrac{(2m)!}{(2m+k)!}$.

(ii) Prove that $\dfrac{\sum_{j=1}^k I_j(f)}{J(f)} = 4\left(1 - \dfrac{1}{2m+2}\right)\left(1 - \dfrac{2m+1}{2m+1+k}\right)$.

(iii) (Goldston, Pintz, Yıldırım) Prove that if the level θ of distribution satisfies $\theta > \frac{1}{2}$, then there are infinitely many bounded gaps in the sequence of primes.

2. Let $\mathcal{R}_k$ be as in question 1. For $\mathbf{t} \in \mathcal{R}_k$ let $\alpha_k(\mathbf{t}) = t_1 + \cdots + t_k$ and $\beta_k(\mathbf{t}) = t_1^2 + \cdots + t_k^2$.

(i) Suppose that a and a_j are non–negative integers. Prove that

$$\int_{\mathcal{R}_k} (1 - \alpha_k(\mathbf{t}))^a \prod_{j=1}^{k} t_j^{a_j} d\mathbf{t} = \frac{a! \prod_{j=1}^k a_j!}{(k + a + \sum_{j=1}^k a_j)!}.$$

(ii) Suppose that a and b are non–negative integers. Prove that

$$\int_{\mathcal{R}_k} (1 - \alpha_k(\mathbf{t}))^a \beta_k(\mathbf{t})^b d\mathbf{t} = \frac{a!b!}{(k+a+2b)!} \sum_{\substack{\mathbf{b} \\ b_1 + \cdots + b_k = b}} \prod_{j=1}^{k} \frac{(2b_j)!}{b_j!}.$$

(The multinomial theorem applied to β_k^b is useful here.)

3. (Maynard) (i) Let $k = 5$. In the notation of question 1, when $\mathbf{t} \in \mathcal{R}_5$, let

$$f(\mathbf{t}) = (1 - \alpha_5(\mathbf{t}))\beta_5(\mathbf{t}) + \frac{7}{10}(1 - \alpha_5(\mathbf{t}))^2 + \frac{1}{14}\beta_5(\mathbf{t})^2 - \frac{3}{14}(1 - \alpha_5(\mathbf{t})).$$

Prove that

$$\frac{\sum_{j=1}^5 I_j(f)}{J(f)} = \frac{1417255}{708216}.$$

(ii) Prove that if the level of distribution θ is 1, then $\liminf_{n \to \infty} p_{n+1} - p_n \le 12$.