MATH 597B, SPRING 2015, PROBLEMS 12

Due Tuesday 21st April

1. As in homework 11 call a set $\mathbf{h}$ of distinct non-negative integers $h_1, \dots, h_k$ *sf–admissible* when there is no prime p such that every residue class modulo p^2 contains at least one of them. Let $S(x; \mathbf{h})$ denote the number of $n \le x$ such that $n + h_1, \dots, n + h_k$ are simultaneously squarefree. Given a k–tuple of positive integers $\mathbf{d} = d_1, \dots, d_k$ let $d = d_1 \dots d_k$ and given another one $\mathbf{r}$ we use $\mathbf{d}|\mathbf{r}$ to mean $d_j | r_j$ $(j = 1, \dots, k)$ and $\mathbf{d}^2$ to mean $d_1^2, \dots, d_k^2$. Write $n + \mathbf{h}$ for the k–tuple $n + h_1, \dots, n + h_k$. Let $\rho(\mathbf{d})$ denote the number of solutions of $\mathbf{d}^2 | n + \mathbf{h}$ in n modulo d^2 and let $\rho^*(\mathbf{d})$ denote the number of solutions of $\mathbf{d}^2 | n + \mathbf{h}$ in n modulo $\mathrm{lcm}[d_1, \dots, d_k]^2$. Let $\nu_p(\mathbf{h})$ denote the number of different residue classes modulo p^2 amongst the $h_1, \dots, h_k$.
(i) Prove that $\rho(\mathbf{d}) = d^2 \mathrm{lcm}[d_1, \dots, d_k]^{-2} \rho^*(\mathbf{d})$ and $\rho^*(\mathbf{d}) \le 1$.
(ii) Prove that

$$\sum_{\max(d_j) > y} \frac{\mu(d_1) \dots \mu(d_k)}{d^2} \rho(\mathbf{d}) \ll \sum_{\max(d_j) > y} \frac{\mu(d_1)^2 \dots \mu(d_k)^2}{[d_1, \dots, d_k]^2} \ll \sum_{m > y} \frac{2^{k\omega(m)}}{m^2} \ll y^{\varepsilon - 1}$$

and deduce that

$$T_k(x, y) = x \sum_{m=1}^{\infty} \frac{g(m)}{m^2} + O\left(x y^{\varepsilon - 1}\right)$$

where

$$g(m) = \sum_{\substack{\mathbf{d} \\ [d_1, \dots, d_k] = m}} \mu(d_1) \dots \mu(d^k) \rho^*(\mathbf{d}).$$

(iii) Prove that $\rho(\mathbf{d})$ is multiplicative, i.e. given $\mathbf{d}$, $\mathbf{e}$, define $\mathbf{de} = d_1 e_1, \dots, d_k e_k$ and deduce that if $(d, e) = 1$, then $\rho(\mathbf{de}) = \rho(\mathbf{d}) \rho(\mathbf{e})$.
(iv) Prove that $g(m)$ is multiplicative and has its support on the squarefree numbers.
(v) Deduce that

$$\sum_{m=1}^{\infty} \frac{g(m)}{m^2} = \prod_p \left(1 + g(p) p^{-2}\right).$$

(vi) Prove that $1 + g(p) p^{-2} = 1 - \nu_p(\mathbf{h}) p^{-2}$.
(vii) Prove that

$$S(x; h) = x \prod_p \left(1 - \frac{\nu_p(\mathbf{h})}{p^2}\right) + O(x^{1-\delta})$$

and hence that if $\mathbf{h}$ is sf–admissible, then there are infinitely many n such that $n + h_j$ are simultaneously square free for $j = 1, \dots, k$.

2. Find the minimal diameter of 20–tuples which are sf–sdmissible, i.e $\max h_j - h_i$ is minimal.