# MATH 597B, SPRING 2015, PROBLEMS 11

## Due Thursday 9th April

1. Suppose that $k \ge 2$ and the $1 < q_1 < q_2 < \cdots < q_k$. Prove that if for each $j$ we have $p|q_j \Rightarrow p > k$, then $q_1, \dots, q_k$ forms an admissible set.

2. Call a set $\mathbf{h}$ of distinct non-negative integers $h_1, \dots, h_k$ *sf–admissible* when there is no prime $p$ such that every residue class modulo $p^2$ contains at least one of them. Let $S(x;\mathbf{h})$ denote the number of $n \le x$ such that $n+h_1, \dots, n+h_k$ are simultaneously squarefree.
(i) Let $f(n)$ denote the characteristic function of the squarefree numbers. Prove that $S(x;\mathbf{h}) = \sum_{n \le x} f(n+h_1)\dots f(n+h_k)$ and $f(n) = \sum_{d^2|n} \mu(d)$.
(ii) Suppose that $0 < \delta < 1/(3k)$ and let $y = x^\delta$ and $f(n;y) = \sum_{\substack{d \le y \\ d^2|n}} \mu(d)$. Prove that for $j = 1, \dots, k$

$$S(x;\mathbf{h}) = T_j(x;y) + O(x^{1+\varepsilon}y^{-1})$$

where

$$T_j(x;y) = \sum_{n \le x} f(n+h_1;y)\dots f(n+h_j;y)f(n+h_{j+1})\dots f(n+h_k).$$

(iii) Given a $k$–tuple of positive integers $\mathbf{d} = d_1, \dots, d_k$ let $d = d_1 \dots d_k$ and given another one $\mathbf{r}$ we use $\mathbf{d}|\mathbf{r}$ to mean $d_j|r_j$ ($j = 1, \dots, k$)a and $\mathbf{d}^2$ to mean $d_1^2, \dots, d_k^2$. Write $n + \mathbf{h}$ for the $k$–tuple $n+h_1, \dots, n+h_k$. Let $\rho(\mathbf{d})$ denote the number of solutions of $\mathbf{d}^2|n+\mathbf{h}$ in $n$ modulo $d^2$. Prove that $\rho(\mathbf{d}) \le d^2$ and

$$T_k(x;y) = x \sum_{d_1 \le y, \dots, d_k \le y} \frac{\mu(d_1)\dots\mu(d_k)}{d^2}\rho(\mathbf{d}) + O(y^{3k}).$$

(iv) Let $\nu_p(\mathbf{h})$ denote the number of different residue classes modulo $p^2$ amongst the $h_1, \dots, h_k$. Suppose that $k = 2$. Prove that

$$S(x;h) = x\prod_p \left(1 - \frac{\nu_p(\mathbf{h})}{p^2}\right) + O(x^{1-\delta}).$$

With a little bit of work this can be generalised to all $k \ge 2$ and establishes the squarefree $k$–tuples theorem (R. R. Hall, Squarefree numbers on short intervals, Mathematika, 29, (1982), 717).