MATH 775, SPRING 2015, PROBLEMS 10

Due Tuesday 31st March

Let $c_q(h)$ denote Ramanujan's sum $\displaystyle\sum_{\substack{a=1\\(a,q)=1}}^{q} e(ah/q)$, and for $k\ge 2$ define $\displaystyle\mathfrak{S}_k(h)=\sum_{q=1}^{\infty}\frac{\mu(q)^k}{\phi(q)^k}c_q(h)$.

1. (i) Prove that if $k\ge 2$ and $h\in\mathbb{N}$, then $\mathfrak{S}_k(h)$ converges absolutely.

(ii) Suppose that $H\in\mathbb{N}$ and $q\ge 2$. Prove that $\displaystyle\sum_{h=1}^{H} c_q(h)\ll\sum_{a=1}^{q-1}\|a/q\|^{-1}\ll q\log q$.

(iii) Suppose that $k\ge 3$. Prove that $\displaystyle\sum_{h=1}^{H}\mathfrak{S}_k(h)=H+O(1)$.

(iv) Suppose that $Q>1$ and $h\in\mathbb{N}$. Prove that $\displaystyle\sum_{q>Q}\frac{\mu(q)^2}{\phi(q)^2}c_q(h)\ll Q^{-1}\sum_{l|h}\frac{l^2\mu(l)^2}{\phi(l)^2}$ and that

$$\sum_{h=1}^{H}\sum_{q>Q}\frac{\mu(q)^2}{\phi(q)^2}c_q(h)\ll Q^{-1}H\log 2H.$$

(v) Prove that $\displaystyle\sum_{h=1}^{H}\sum_{1<q\le Q}\frac{\mu(q)^2}{\phi(q)^2}c_q(h)\ll(\log Q)^2$.

(vi) Prove that $\displaystyle\sum_{h=1}^{H}\mathfrak{S}_2(h)=H+O(\log^2 H)$.

2. Let $H\in\mathbb{N}$ and $\mathcal{H}_k$ be the set of k–tuples $\mathbf{h}=h_1,\dots,h_k$ with $h_1,\dots,h_k$ distinct and $1\le h_j\le H$ for each j, and let $\nu_p(\mathbf{h})$ be the number of distinct residue classes modulo p amongst the $\mathbf{h}$. Then define, for $\mathbf{h}\in\mathcal{H}_k$, $\displaystyle\mathfrak{S}_k^*(\mathbf{h})=\prod_p\left(1-\frac{\nu_p(\mathbf{h})}{p}\right)\left(1-\frac{1}{p}\right)^{-k}$. Also, let $F(q;\mathbf{h})$ be the multiplicative function of q with

$$F(p;\mathbf{h})=\left(1-\frac{\nu_p(\mathbf{h})}{p}\right)\left(1-\frac{1}{p}\right)^{-k}-1,\quad F(p^j;\mathbf{h})=0\ (j>1).$$

(i) Prove that if $k\ge 2$ and $\mathbf{h}\in\mathcal{H}_k$, then $\mathfrak{S}_k^*(\mathbf{h})$ converges absolutely.

(ii) Prove that if $k\ge 2$ and $\mathbf{h}\in\mathcal{H}_k$, then $\displaystyle\mathfrak{S}_k^*(\mathbf{h})=\sum_{q=1}^{\infty}F(q;\mathbf{h})$.

(iii) Suppose that $k=2$ and $\mathbf{h}\in\mathcal{H}_2$. Prove that $F(q;\mathbf{h})=\frac{\mu(q)^2}{\phi(q)^2}c_q(h_2-h_1)$ and $\mathfrak{S}_2^*(\mathbf{h})=\mathfrak{S}_2(h_2-h_1)$.

(iv) Prove that $\operatorname{card}\mathcal{H}_k=H^k+O(H^{k-1})$.

(v) Prove that $\displaystyle\sum_{\mathbf{h}\in\mathcal{H}_2}\mathfrak{S}_2^*(\mathbf{h})=2\sum_{h=1}^{H}(H-h)\mathfrak{S}_2(h)=2\int_1^H\sum_{h\le x}\mathfrak{S}_2(h)dx=H^2+O(H\log^2 H)$.