MATH 597B, SPRING 2015, PROBLEMS 9

Due Tuesday 24th March

1. (H.-E. Richert, unpublished) (a) Show that

$$\sum_{x<n\le x+y}\Big(\sum_{d^2|n}\Lambda_d\Big)^2 = y\sum_{d,\,e}\frac{\Lambda_d\Lambda_e}{[d,e]^2} + O\bigg(\Big(\sum_d |\Lambda_d|\Big)^2\bigg).$$

(b) Let $f(n)=n^2\prod_{p|n}\big(1-p^{-2}\big)$. Show that $\sum_{d|n}f(d)=n^2$.
(c) For $1\le d\le z$ let Λ_d be real numbers such that $\Lambda_1=1$. Show that, subject to this condition, the minimum of $\sum_{d,\,e}\Lambda_d\Lambda_e/[d,e]^2$ is $1/L$ where $L=\sum_{n\le z}\mu(n)^2/f(n)$. Show also that $\Lambda_d\ll 1$ for the extremal Λ_d.
(d) Show that if $z\in\mathbb{N}$, then $\zeta(2)-1/z\le L\le\zeta(2)$.
(e) Let $Q(x)$ denote the number of squarefree numbers not exceeding x. Show that for $x\ge 0$, $y\ge 1$,

$$Q(x+y)-Q(x)\le\frac{y}{\zeta(2)}+O\big(y^{2/3}\big).$$