MATH 597B, SPRING 2015, PROBLEMS 8

Due Tuesday 17th March

1. Let $f(n)$ be an arithmetic function such that $f(1) = 1$. Show that f is multiplicative if and only if $f(m)f(n) = f((m,n))f([m,n])$ for all pairs of positive integers m, n.

2. (Hooley (1972), Montgomery & Vaughan (1979)) By lower and upper bound sifting functions we mean functions $\lambda^{\pm} : \mathbb{N} \to \mathbb{R}$ with the properties

$$\sum_{m|n} \lambda_m^- \le \sum_{m|n} \mu(m) \le \sum_{m|n} \lambda_m^+$$

respectively.

(i) Let λ_d^+ be an upper bound sifting function such that $\lambda_d^+ = 0$ for all $d > z$. Show that for any q,

$$0 \le \frac{\varphi(q)}{q} \sum_{\substack{d \\ (d,q)=1}} \frac{\lambda_d^+}{d} \le \sum_d \frac{\lambda_d^+}{d}.$$

(Hint: Multiply both sides by $P/\varphi(P) = \sum 1/m$ where m runs over all integers composed of the primes dividing P, and $P = \prod_{p \le z} p$.)

(ii) Let Λ_d be real with $\Lambda_d = 0$ for $d > z$. Show that for any q,

$$0 \le \frac{\varphi(q)}{q} \sum_{\substack{d,\, e \\ (de,q)=1}} \frac{\Lambda_d \Lambda_e}{[d,e]} \le \sum_{d,\, e} \frac{\Lambda_d \Lambda_e}{[d,e]}.$$

(iii) Let λ_d^- be a lower bound sifting function such that $\lambda_d^- = 0$ for $d > z$. Show that for any q,

$$\frac{\varphi(q)}{q} \sum_{\substack{d \\ (d,q)=1}} \frac{\lambda_d^-}{d} \ge \sum_d \frac{\lambda_d^-}{d}.$$