MATH 597B, SPRING 2015, PROBLEMS 7

Due Tuesday 3rd March

1. Suppose that $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and that R, $Q \in \mathbb{R}$ with $1 \leq R \leq Q$. Let

$$S(Q,R) = \sum_{R<q\leq Q} (\log(2Q/q))f(q)$$

and

$$T(y) = \sum_{R<q\leq y} qf(q) \quad (R \leq y \leq Q).$$

(i) Prove that

$$S(Q,R) = T(Q)\frac{\log 2}{Q} + \int_R^Q T(y)\frac{\log(2Q/y)+1}{y^2}dy.$$

(ii) Suppose that there are X, Y, $Z \in \mathbb{R}_{\geq 0}$ such that $T(y) \leq X + Yy + Zy^2$. Prove that

$$S(Q,R) \ll XR^{-1}\log Q + Y(\log Q)^2 + ZQ.$$

(iii) Deduce that Theorem 20.1 implies that when $x^{\frac{1}{2}}(\log x)^{-A} \leq Q \leq x^{\frac{1}{2}}$ we have

$$\sum_{\log^{A+1} x<q\leq Q} \frac{\log(2Q/q)}{\phi(q)} \sideset{}{^*}\sum_{\chi \pmod q} |\sup_{y\leq x} \psi(y;q)| \ll x^{\frac{1}{2}}Q(\log x)^3.$$

2. (S. Chowla (1932)) Let $f(n)$ be an arithmetic function, put

$$g(n) = \sum_{[d,e]=n} f(d)\overline{f(e)},$$

and let σ_c denote the abscissa of convergence of the Dirichlet series $\sum g(n)n^{-s}$.
(a) Show that if $\sigma > \max(1, \sigma_c)$ then

$$\zeta(s)\sum_{d,\,e}\frac{f(d)\overline{f(e)}}{[d,e]^s} = \sum_{n=1}^{\infty}\left|\sum_{d|n} f(d)\right|^2 n^{-s}.$$

(b) Show that

$$\sum_{d,\,e}\frac{\mu(d)\mu(e)}{[d,e]^2} = \frac{6}{\pi^2}.$$

(c) Show that

$$\sum_{\substack{d,\,e\\ [d,e]=n}} \mu(d)\mu(e) = \mu(n)$$

for all positive integers n.