MATH 597B, SPRING 2015, PROBLEMS 6

Due Tuesday 24th February

We recall some things from Math 568. Given a Dirichlet character χ modulo q we define

$$\tau(\chi;a)=\sum_{x=1}^{q}\chi(x)e(ax/q),\quad \tau(x)=\tau(x;1).$$

When χ is a primitive character modulo q, $|\tau(\chi)|=\sqrt{q}$ and $\tau(\chi;a)=\overline{\chi}(a)\tau(\chi)$ for every $a\in\mathbb{Z}$. For a general Dirichlet character modulo q there is an associated conductor q^* with $q^*|q$ and an associated primitive character χ^* modulo q^* such that $\chi(x)=\chi_0\chi^*(x)$ for every $x\in\mathbb{Z}$ where χ_0 is the principal character modulo q, and we say that χ^* induces χ. We suppose throughout that $M\in\mathbb{Z}$ and $N\in\mathbb{N}$.

1. Given a character χ modulo q show that

$$\sum_{x=M+1}^{M+N}\chi(x)=\frac{1}{q}\sum_{a=1}^{q}\tau(\chi;a)T(-a/q)$$

where $T(\alpha)=\sum_{x=M+1}^{M+N}e(x\alpha)$.

2. Suppose that $q>1$ and χ is a primitive character modulo q. Prove that

$$\left|\sum_{x=M+1}^{M+N}\chi(x)\right|\le q^{-\frac{1}{2}}\sum_{\substack{a=1\\(a,q)=1}}^{q-1}\frac{1}{2\|a/q\|}\le q^{\frac{1}{2}}\sum_{a\le(q-1)/2}\frac{1}{a}.$$

3. Prove that if $0\le\alpha\le1$, then $\alpha\le\log\frac{1+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}$.

4. Deduce that if $q>1$ and χ is a primitive character modulo q, then

$$\left|\sum_{x=M+1}^{M+N}\chi(x)\right|\le q^{\frac{1}{2}}\log q.$$

5. (Pólya–Vinogradov) Prove that if χ is a non–principal character modulo q induced by a primitive character χ^* with conductor q^*. then

$$\sum_{x=M+1}^{M+N}\chi(x)=\sum_{d|q/q^*}\mu(s)\chi^*(d)\sum_{M/d<y\le(M++N)/d}\chi^*(y)$$

and deduce that

$$\left|\sum_{x=M+1}^{M+N}\chi(x)\right|\le d(q/q^*)(q^*)^{\frac{1}{2}}\log q^*\ll q^{\frac{1}{2}}\log q.$$