MATH 597B, SPRING 2015, PROBLEMS 5

Due Tuesday 17th February

1. Suppose $\alpha \in \mathbb{R}$, x, $y \in \mathbb{R}$ with $x \geq 1$, $y \geq 1$ and that there is an $A \in \mathbb{R}$ such that for $m \leq y$ the complex numbers a_m satisfy $|a_m| \leq A$.
(i) Prove that

$$\sum_{m\leq y} a_m \sum_{n\leq x/m} e(\alpha mn) \ll A \sum_{m\leq y} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right).$$

Suppose further that there are $q \in \mathbb{N}$, $a \in \mathbb{Z}$ with $(q,a) = 1$ such that $|\alpha - a/q| \leq q^{-2}$. When $1 \leq m \leq y$ put $m = hq + j$ where $-q/2 < j \leq q/2$ so that $h \leq \frac{y}{q} + \frac{1}{2}$.
(ii) If $h = 0$, then prove that $\|\alpha m\| \geq \|aj/q\| - \frac{1}{2q} \geq \frac{1}{2}\|aj/q\|$. Deduce that

$$\sum_{m\leq \min(y,q/2)} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right) \ll \sum_{1\leq j\leq \min(y,q/2)} \|aj/q\|^{-1} \ll \sum_{l\leq \min(y,q)} \frac{q}{l}.$$

(iii) Suppose $h > 0$. Put $\beta = q^2(\alpha - a/q)$ and let k be a nearest integer to βh. Prove that if

$$\left\|\frac{ja+k}{q}\right\| \geq \frac{2}{q},$$

then

$$\|\alpha m\| \geq \frac{1}{2}\left\|\frac{ja+k}{q}\right\|.$$

Deduce that

$$\sum_{hq-q/2<m\leq hq+q/2} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right) \ll \frac{x}{hq} + \sum_{l=1}^{q-1} \frac{q}{l}.$$

(iv) Prove that

$$\sum_{m\leq y} a_m \sum_{n\leq x/m} e(\alpha mn) \ll \left(\frac{x}{q} + y + q\right) A \log 2x.$$

(v) Prove that

$$\sum_{m\leq y} a_m \sum_{n\leq x} e(\alpha mn) \ll \left(\frac{xy}{q} + y + q\right) A \log 2x.$$

2. Let $c_q(n) = \sum^{q}_{\substack{a=1\\(q,a)-1}} e(an/q)$, Ramanujan's sum.
(i) Prove that if n is fixed and q varies, then $c_q(n)$ is multiplicative.
(ii) Prove that if $(q,n) = 1$, then $c_q(n) = \mu(q)$.
(iii) Let $f(q) = \sum_{\substack{a=1\\(q,a)=1}} c_q(a)^k e(an/q)$. Prove that $f(q) = \mu(q)^k c_q(n)$.
(iv) Prove that if $k \geq 3$, then $\sum_{q=1}^{\infty} f(q)\phi(q)^{-k} = \prod_{p|n} \left(1 + \frac{(-1)^k}{(p-1)^{k-1}}\right) \prod_{p\nmid n} \left(1 - \frac{(-1)^k}{(p-1)^k}\right)$.